

**Semidefinite programming  
for polynomial optimization  
and robust control**

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## Main points

Convex optimisation can be used to solve numerically non-convex optimisation problems, specifically **polynomial matrix inequalities**

**Numerical linear algebra** is a key ingredient

Applications in many branches of engineering and applied maths, in particular **systems control**

## PMIs and LMIs

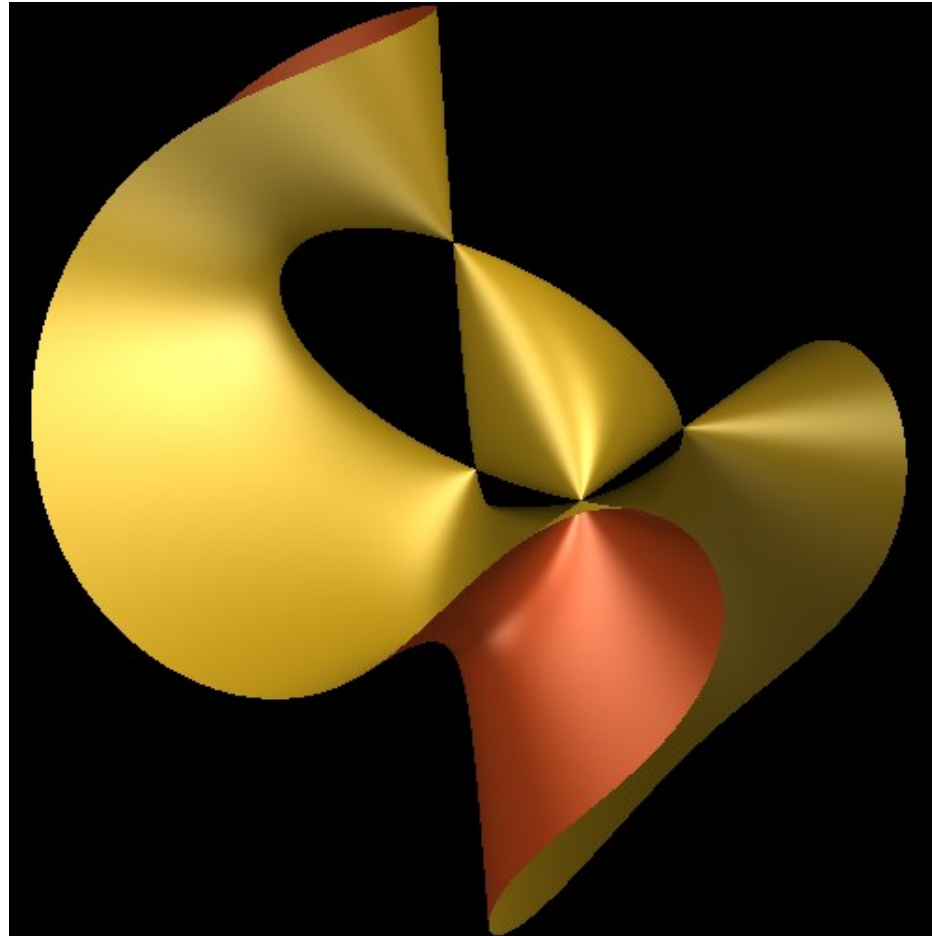
Polynomial programming: objective function and constraints are multivariate polynomials, or equivalently, we minimize a linear function on a (non-convex, non-connected, discrete..) semialgebraic set

Typically, the semialgebraic set is described by **polynomial matrix inequalities** (PMIs)

Semidefinite programming (SDP): linear programming in the (convex but non-polyhedral) cone of positive semidefinite matrices, also called **linear matrix inequality** (LMI) optimisation

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & F(x) = \sum_{\alpha} F_{\alpha} x^{\alpha} \succeq 0 \end{aligned}$$

An LMI set (and something more)



## Outline

1. Quasi-Hankel matrices
2. Generalized companion matrices
3. Systems control applications

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## Convex vs. non-convex optimisation

Build and solve a **hierarchy** of successive **convex LMI relaxations** of increasing size, whose optima are **guaranteed** to converge asymptotically to the global optimum

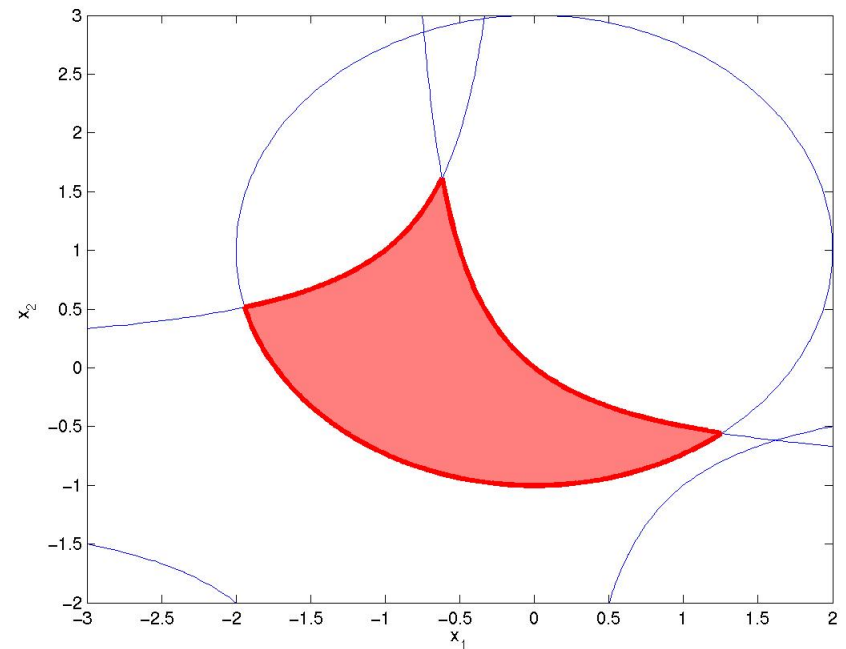


Relaxations are built thanks to results on the primal theory of **moments** and the dual theory of **sum-of-squares** (SOS) decompositions of multivariate polynomials

## LMI relaxations: an example

Quadratic problem

$$\begin{aligned} \max \quad & x_2 \\ \text{s.t.} \quad & 3 - 2x_2 - x_2^2 - x_1^2 \geq 0 \\ & -x_1 - x_2 - x_1x_2 \geq 0 \\ & 1 + x_1x_2 \geq 0 \end{aligned}$$



**Non-convex** feasible set delimited by circular and hyperbolic arcs

## Lifting

LMI relaxation built by replacing each monomial  $x_1^i x_2^j$  with a **lifting variable**  $y_{ij}$

For example, quadratic expression  $3 - 2x_2 - x_2^1 - x_2^2 \geq 0$  replaced with linear expression  $3 - 2y_{01} - y_{20} - y_{02} \geq 0$

Lifting variables  $y_{ij}$  satisfy **non-convex** relations such as  $y_{10}y_{01} = y_{11}$  or  $y_{20} = y_{10}^2$

**Relax** these non-convex relations by enforcing LMI constraint

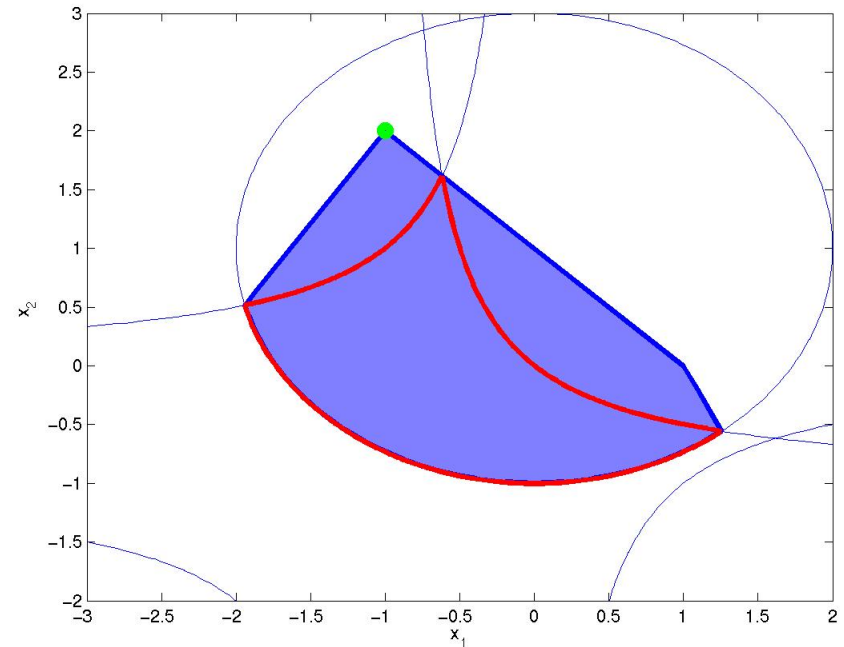
$$M_1^1(y) = \left[ \begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array} \right] \succeq 0$$

**Moment** matrix of first order

## Projecting

First LMI relaxation

$$\begin{aligned} \max \quad & y_{01} \\ \text{s.t.} \quad & \begin{bmatrix} 1 & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0 \\ & 3 - 2y_{01} - y_{20} - y_{02} \geq 0 \\ & -y_{10} - y_{01} - y_{11} \geq 0 \\ & 1 + y_{11} \geq 0 \end{aligned}$$



Projection onto the plane  $y_{10}, y_{01}$  of first-order moments

LMI optimum = 2 = **upper-bound** on global optimum

## Higher order relaxation

To build second LMI relaxation, the **moment matrix** must capture expressions of degrees up to 4

$$M_2^2(y) = \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}$$

Constraints are also lifted and relaxed with the help of **localizing matrices**

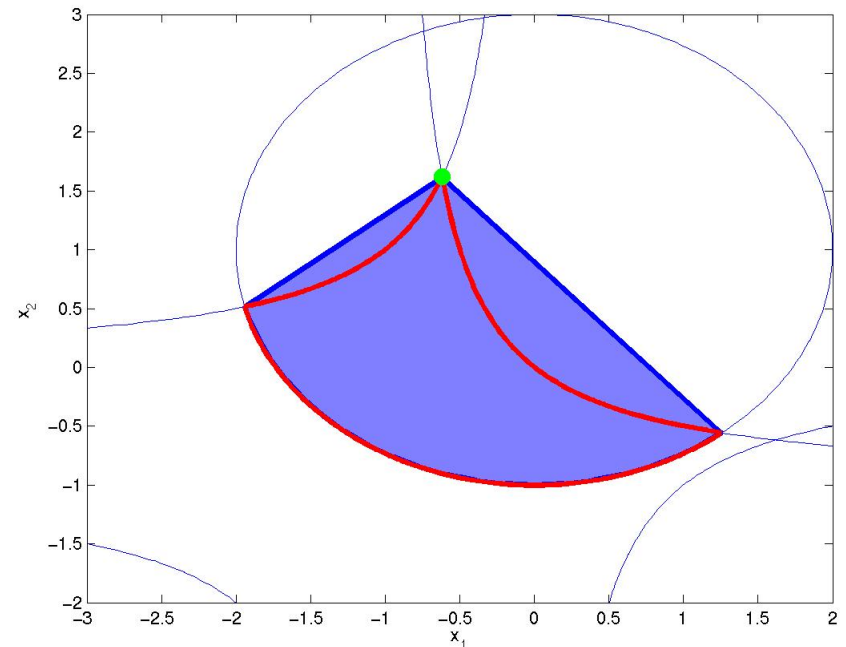
Second LMI provides **tighter** relaxation

$$\begin{array}{l}
\max \quad y_{01} \\
\text{s.t.} \quad \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0 \\
\begin{bmatrix} 3 - 2y_{01} - y_{20} - y_{02} & 3y_{10} - 2y_{11} - y_{30} - y_{12} & 3y_{01} - 2y_{02} - y_{21} - y_{03} \\ 3y_{10} - 2y_{11} - y_{30} - y_{12} & 3y_{20} - 2y_{21} - y_{40} - y_{22} & 3y_{11} - 2y_{12} - y_{31} - y_{13} \\ 3y_{01} - 2y_{02} - y_{21} - y_{03} & 3y_{11} - 2y_{12} - y_{31} - y_{13} & 3y_{02} - 2y_{03} - y_{22} - y_{04} \end{bmatrix} \succeq 0 \\
\begin{bmatrix} -y_{10} - y_{01} - y_{11} & -y_{20} - y_{11} - y_{21} & -y_{11} - y_{02} - y_{12} \\ -y_{20} - y_{11} - y_{21} & -y_{30} - y_{21} - y_{31} & -y_{21} - y_{12} - y_{22} \\ -y_{11} - y_{02} - y_{12} & -y_{21} - y_{12} - y_{22} & -y_{12} - y_{03} - y_{13} \end{bmatrix} \succeq 0 \\
\begin{bmatrix} 1 + y_{11} & y_{10} + y_{21} & y_{01} + y_{12} \\ y_{10} + y_{21} & y_{20} + y_{31} & y_{11} + y_{22} \\ y_{01} + y_{12} & y_{11} + y_{22} & y_{02} + y_{13} \end{bmatrix} \succeq 0
\end{array}$$

## Global optimality

Optimal value of 2nd LMI relaxation = 1.6180  
= **global optimum** within numerical accuracy  
Numerical **certificate** = moment matrix has rank one

First order moments  
 $(y_{10}^*, y_{01}^*) = (-0.6180, 1.6180)$   
provide optimal solution  
of original problem



## Quasi-Hankel structure

Structure of moment matrices

univariate = Hankel

multivariate = sparse quasi-Hankel

$$\begin{bmatrix} 1 & x_{10} & x_{01} & \mathbf{x_{20}} & x_{11} & x_{02} \\ x_{10} & \mathbf{x_{20}} & x_{11} & x_{30} & x_{21} & \mathbf{x_{12}} \\ x_{01} & x_{11} & x_{02} & x_{21} & \mathbf{x_{12}} & x_{03} \\ \mathbf{x_{20}} & x_{30} & x_{21} & x_{40} & x_{31} & x_{22} \\ x_{11} & x_{21} & \mathbf{x_{12}} & x_{31} & x_{22} & x_{13} \\ x_{02} & \mathbf{x_{12}} & x_{03} & x_{22} & x_{13} & x_{04} \end{bmatrix}$$

Known **sparsity pattern**: LMI matrix coefficients  
in pre-factored form (Cholesky)

## Quasi-Hankel structure

Efficient [low-rank algebra](#) to compute gradient and Hessian in interior-point codes ?

Theoretical results by B. Mourrain and V. Pan

So far our implementation in GloptiPoly for Matlab

[www.laas.fr/~henrion/software/gloptipoly](http://www.laas.fr/~henrion/software/gloptipoly)

uses a general purpose SDP solver (SeDuMi by default)

## Outline

1. Quasi-Hankel matrices
2. Generalized companion matrices
3. Systems control applications

## Detecting global optimality

Global optimisation problem

$$p^* = \min_x g_0(x) \\ \text{s.t. } g_i(x) \geq 0, i = 1, 2..$$

Let  $\deg g_i(x) = 2d_i - 1$  or  $2d_i$  and  $d = \max_i d_i$

LMI relaxation of order  $k$

$$p_k^* = \min_y \sum_{\alpha} (g_0)_{\alpha} y_{\alpha} \\ \text{s.t. } M_k(y) \succeq 0 \\ M_{k-d_i}(g_i y) \succeq 0, i = 1, 2..$$

with solution  $y_k^*$

Hierarchy with convergence guarantee

$$p_d^* \leq p_{d+1}^* \leq \dots \leq p_{k^*}^* = p^*.$$

for (generally) small  $k^*$

Sufficient condition for global optimality

rank of moment matrices

$$\text{rank } M_k(y_k^*) = \text{rank } M_{k-d}(y_k^*)$$

## Need for extraction procedures

Because rank condition is only **sufficient**  
any other global optimality certificate is welcome

If there is a finite number of global optimisers,  
they generate a zero-dimensional **variety**

Moment matrices are multiplication matrices in the quotient  
**ideal**, a finite-dimension vector space

Key ideas:

- **Cholesky** decomposition of moment matrix
- column reduced **echelon** form
- computation of **common eigenvalues**

## Cholesky factorization

Extract Cholesky factor  $V$  of moment matrix

$$M_k(y_k^*) = VV'$$

Matrix  $V$  has  $r$  columns, corresponding to  $r$  globally optimal solutions  $x_j^*$ ,  $j = 1, 2, \dots, r$  (provided global optimum was reached)

Choose e.g.  $v = [1 \ x_1 \ x_2 \ \dots \ x_n \ x_1^2 \ x_1x_2 \ \dots \ x_1x_n \ x_2^2 \ x_2x_3 \ \dots \ x_n^2 \ \dots \ x_n^k]'$  as a basis for polynomials of degree at most  $k$

By definition of the moment matrix:

$$M_k(y_k^*) = V^*(V^*)'$$

where

$$V^* = [ v_1^* \ v_2^* \ \dots \ v_r^* ]$$

and  $v_j^*$  is polynomial basis  $v$  evaluated at solution  $x_j^*$

Extracting solutions amounts to finding **linear transformation** between Cholesky factors  $V$  and  $V^*$

## Reduction to column echelon form

Next step is reduction of  $V$  into column echelon form

$$U = \begin{bmatrix} 1 & & & & & \\ * & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ * & * & * & & & \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & \\ * & * & * & \cdots & * & \\ & \vdots & & & \vdots & \\ * & * & * & \cdots & * & \end{bmatrix}$$

by **Gaussian elimination** with column pivoting

Each row in  $U =$  monomial  $x_\alpha$  in basis  $v$

**Pivot** entry in  $U =$  monomial  $x_{\beta_j}$  in **generating basis** of the set of solutions

In other words, denoting  $w = [x_{\beta_1} \quad x_{\beta_2} \quad \dots \quad x_{\beta_r}]'$  it holds

$$v = Uw$$

for all solutions  $x_j^*$ ,  $j = 1, 2, \dots, r$

## Multiplication matrices

For each first degree monomial  $x_i$  extract from  $U$  the  $r$ -by- $r$  **multiplication matrix**  $N_i$  containing coefficients of product monomials  $x_i x_{\beta_j}$  in generating basis  $w$ , i.e. such that

$$N_i w = x_i w \quad i = 1, 2, \dots, n$$

Given matrices  $N_i$  finding scalars  $x_i$  is an eigenvalue problem

Extracting solutions amounts to solving an **eigenvalue problem**

Eigenvectors  $w$  are **shared** by commuting matrices  $N_i$  so it is a particular **common eigenvalue problem**

## Common eigenvalue problem

Build combination of multiplication matrices

$$N = \sum_{i=1}^n \lambda_i N_i$$

where  $\lambda_i$  are **random** positive numbers (summing up to one)

Compute **ordered Schur decomposition**

$$N = QTQ'$$

where  $Q = [q_1 \ q_2 \ \cdots \ q_r]$  is orthogonal and  $T$  upper triangular

Finally, due to orthogonality of vectors  $q_i$ ,  
 $i$ th entry in solution vector  $x_j^*$  is given by

$$(x_j^*)_i = q_j' N_i q_j, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, r$$

## Number of solutions

No easy way to **control** number of extracted solutions in case of multiple global optima

Number of solutions = rank of moment matrix, but enforcing rank in an LMI is a difficult **non-convex** problem

By default GloptiPoly **minimizes the trace** (sum of eigenvalues) of the moment matrix, which may indirectly minimize the rank (number of non-zero eigenvalues)

Practical experiments reveal that low rank moment matrices ensure **faster convergence** of LMI relaxations to global optimum

## Example of extraction

Quadratic problem

$$\begin{aligned} p^* &= \max_x (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - 3)^2 \\ \text{s.t. } &(x_1 - 1)^2 \leq 1 \\ &(x_1 - x_2)^2 \leq 1 \\ &(x_2 - 3)^2 \leq 1 \end{aligned}$$

First LMI relaxation yields  $p_1^* = 3$  and  $\text{rank } M_1(y^*) = 3$ ,  
extraction algorithm fails due to incomplete monomial basis

Second LMI relaxation yields  $p_2^* = 2$  and

$$\text{rank } M_1(y^*) = \text{rank } M_2(y^*) = 3$$

so rank condition ensures **global optimality**

## Cholesky factor

Moment matrix of order  $k = 2$  reads

$$M_2(y^*) = \begin{bmatrix} 1.0000 & 1.5868 & 2.2477 & 2.7603 & 3.6690 & 5.2387 \\ 1.5868 & 2.7603 & 3.6690 & 5.1073 & 6.5115 & 8.8245 \\ 2.2477 & 3.6690 & 5.2387 & 6.5115 & 8.8245 & 12.7072 \\ 2.7603 & 5.1073 & 6.5115 & 9.8013 & 12.1965 & 15.9960 \\ 3.6690 & 6.5115 & 8.8245 & 12.1965 & 15.9960 & 22.1084 \\ 5.2387 & 8.8245 & 12.7072 & 15.9960 & 22.1084 & 32.1036 \end{bmatrix}$$

Positive semidefinite with rank 3

Cholesky factor

$$V = \begin{bmatrix} -0.9384 & -0.0247 & 0.3447 \\ -1.6188 & 0.3036 & 0.2182 \\ -2.2486 & -0.1822 & 0.3864 \\ -2.9796 & 0.9603 & -0.0348 \\ -3.9813 & 0.3417 & -0.1697 \\ -5.6128 & -0.7627 & -0.1365 \end{bmatrix}$$

satisfies

$$M_k(y_k^*) = VV'$$

## Column echelon form

Gaussian elimination on  $V$  yields

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \\ -4 & 2 & 2 \\ -6 & 0 & 5 \end{bmatrix} \begin{matrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{matrix}$$

$1 \quad x_1 \quad x_2$

which means that solutions to be extracted satisfy the following system of polynomial equations

$$\begin{aligned} x_1^2 &= -2 + 3x_1 \\ x_1x_2 &= -4 + 2x_1 + 2x_2 \\ x_2^2 &= -6 + 5x_2 \end{aligned}$$

## Extraction

Multiplication matrices of monomials  $x_1$  and  $x_2$  in polynomial basis  $1, x_1, x_2$  are extracted from  $U$ :

$$N_1 = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -4 & 2 & 2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 1 \\ -4 & 2 & 2 \\ -6 & 0 & 5 \end{bmatrix}$$

Random linear combination

$$N = 0.6909N_1 + 0.3091N_2$$

Schur decomposition of  $N = QTQ'$  yields

$$Q = \begin{bmatrix} 0.4082 & 0.1826 & -0.8944 \\ 0.4082 & -0.9129 & -0.0000 \\ 0.8165 & 0.3651 & 0.4472 \end{bmatrix}$$

Projections of orthogonal columns of  $Q$  onto  $N$  yield the 3 expected globally optimal solutions

$$x_1^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad x_2^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad x_3^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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1. Quasi-Hankel matrices
2. Generalized companion matrices
3. **Systems control applications**

## Robust stability

Robust stability of linear system

$$\dot{x}(t) = A(q)x(t)$$

where uncertain parameter  $q$  belongs to a compact set  $Q$

Characteristic polynomial

$$p(s) = \det(sI - A(q)) = \sum_k s^k p_k(q)$$

has roots in the open left half-plane iff

$$H(p) = \sum_i \sum_j p_i(q)p_j(q)H_{ij} \succ 0$$

with Hermite matrix  $H(p)$  (Bézoutian)

## Robust stability

Assuming nominal system stable ( $q = 0$ ), uncertain system remains stable if and only if the multivariate polynomial

$$\det H(p) = \sum_{\alpha} q^{\alpha} h_{\alpha}$$

remains positive for all  $q \in Q$

General framework of  $\mu$ -analysis (Doyle, Safonov)

Assessing robust stability of linear systems amounts to **polynomial minimisation** over compact sets (typically the unit box)

## Gain scheduling

Linear system

$$z(s) = \frac{b(s, q)}{a(s, q)} u(s)$$

whose transfer function depends on  
an exogeneous parameter  $q \in Q$

Find a linear controller

$$u(s) = \frac{y(s, q)}{x(s, q)} z(s)$$

also depending on  $q$  (scheduled in  $q$ ) and ensuring  
closed-loop **stability** and **performance** ( $H_2$  or  $H_\infty$  norm)

## Gain scheduling

Controller design can be reformulated as  
a parametrized PMI problem

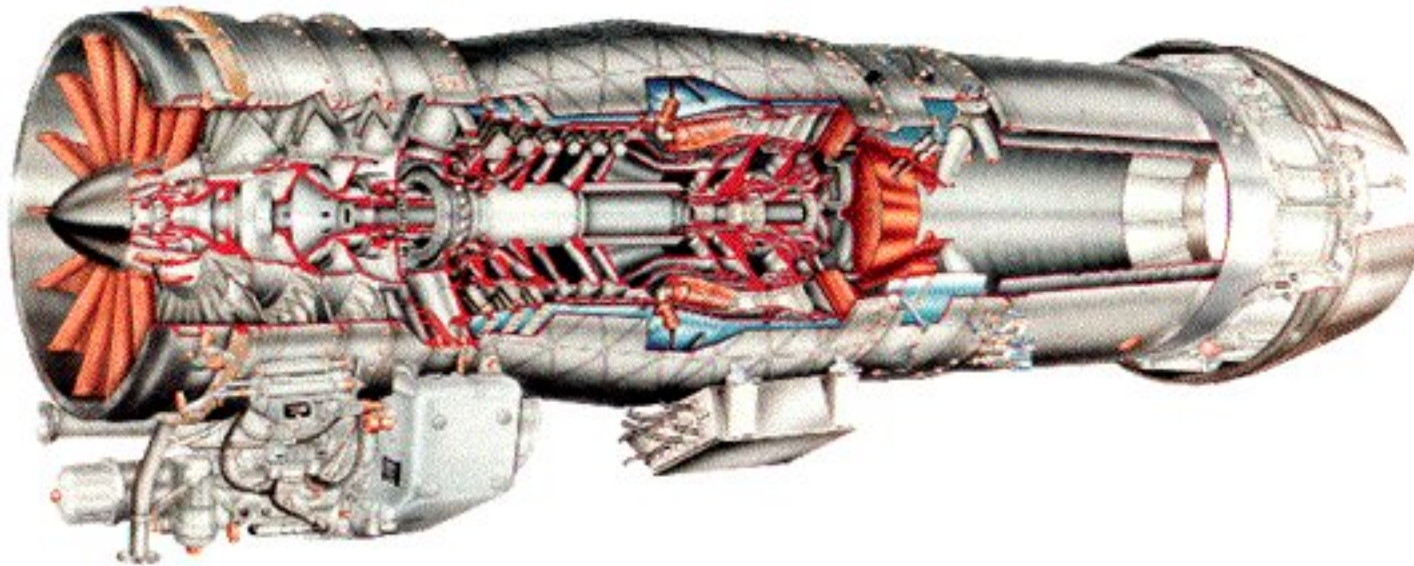
$$F(k, q) \succeq 0$$

where  $k$  is a variable to be found (parametrizing the controller)  
and  $q$  can be anywhere in  $Q$

Difficult semi-infinite non-convex optimisation problem for which  
a hierarchy of finite-dimensional LMI problems can be devised

## Aircraft turbofan engines

Research contract with Snecma since 2000



Inputs: fuel flow WF32, nozzle area A8

Outputs: fan speed XN2, compressor speed XN25

Scheduling parameter  $q$ : pressure in combustion chamber

## Main points

Convex optimisation can be used to solve numerically non-convex optimisation problems, specifically **polynomial matrix inequalities**

**Numerical linear algebra** is a key ingredient

Applications in many branches of engineering and applied maths, in particular **systems control**

## Current limitations

In general, **no bounds** available on the size of the LMI problem to be solved - we can construct worst-case exponential instances

**No working implementation** of SDP solver exploiting the sparse quasi-Hankel structure

**Numerical stability** of solution extraction mechanism remains unclear = unknown sensitivity w.r.t. problem formulation