Sparse Days at CERFACS

Regularization by Mollification

Pierre Maréchal

Mathematical Institute of Toulouse
Outline

- Introduction
- Fourier synthesis
- Asymptotic analysis *(with N. Alibaud and Y. Saesor)*
- Extension *(with X. Bonnefond)*
- A note on proximal inversion *(with A. Rondepierre)*
Outline

- Introduction
- Fourier synthesis
- Asymptotic analysis
- Extension
- A note on proximal inversion
Fourier Synthesis

Recover a function from a partial and approximate knowledge of its Fourier transform.
Example 1: Fourier series

\[ f \in L^2(C_a) \quad \text{where} \quad C_a := [-a/2, a/2]^d \]
Example 1: Fourier series

\[ f \in L^2(C_a) \quad \text{where} \quad C_a := [-a/2, a/2]^d \]
Example 1: Fourier series

\[ f \in L^2(C_a) \quad \text{where} \quad C_a := [-a/2, a/2]^d \]

\[
f(x) = \frac{1}{a^n} \sum_{k \in \mathbb{Z}^n} \hat{f}\left(\frac{k}{a}\right) \exp \left[2i\pi \left\langle \frac{k}{a}, x \right\rangle \right] 1_{C_a}(x)
\]
Example 1: Fourier series

\[ f \in L^2(C_a) \text{ where } C_a := [-a/2, a/2]^d \]

\[ \hat{f}(\xi) = \sum_{k \in \mathbb{Z}^n} \hat{f}\left(\frac{k}{a}\right) \text{sinc } \pi a \left(\xi - \frac{k}{a}\right) \]
Example 2: Aperture synthesis
Example 2: Aperture synthesis
Andromeda Galaxy

WSRT

Courtesy of National Radio Astronomy Observatory / Associated Universities, Inc. / National Science Foundation
Example 3: MRI

Standard acquisitions:
Example 3: MRI

Non-Cartesian and sparse acquisitions:
Fourier extrapolation
Let $V$ and $W$ be subsets of $\mathbb{R}^d$. Assume that $V$ is bounded and that $W$ has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on $W$. 
Fourier extrapolation

Let $V$ and $W$ be subsets of $\mathbb{R}^d$. Assume that $V$ is bounded and that $W$ has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on $W$.

Fourier extrapolation

Let $V$ and $W$ be subsets of $\mathbb{R}^d$. Assume that $V$ is bounded and that $W$ has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on $W$.


*Truncated Fourier operator:*

$$T_W : L^2(V) \rightarrow L^2(W)$$

$$f \mapsto T_W f := 1_W \hat{f} = 1_W U f.$$
Fourier extrapolation

Let \( V \) and \( W \) be subsets of \( \mathbb{R}^d \). Assume that \( V \) is bounded and that \( W \) has a non-empty interior. Recover \( f_0 \in L^2(V) \) from the knowledge of its Fourier transform on \( W \).


**Truncated Fourier operator:**

\[
T_\Omega: \quad L^2(V) \rightarrow L^2(\Omega)
\]

\[
f \quad \mapsto \quad T_\Omega f := 1_\Omega \hat{f} = 1_\Omega U f.
\]
Properties of $T_W$
Properties of $T_W$

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} 1_V(x) 1_W(\xi) f(x) \, dx.$$

$\alpha(x, \xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$
Properties of $T_W$

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} 1_V(x) 1_W(\xi) f(x) \, dx.$$  

$\alpha(x, \xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$

$\implies T_W$ is Hilbert-Schmidt
Properties of $T_W$

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} 1_V(x) 1_W(\xi) f(x) \, dx.$$  

$\alpha(x, \xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$

$\rightarrow$ $T_W$ is Hilbert-Schmidt

Reminder: The Fourier transform of compactly supported functions are entire functions
Properties of $T_W$

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} \mathbf{1}_V(x) \mathbf{1}_W(\xi) f(x) \, dx.$$ 

$\alpha(x, \xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$

$\implies T_W$ is Hilbert-Schmidt

Reminder: The Fourier transform of compactly supported functions are entire functions

$\implies T_W$ is injective
Thus, $T_W^*T_W$ is compact, injective, Hermitian, positive.
Properties of $T_W$

Thus, $T_W^* T_W$ is **compact**, **injective**, Hermitian, positive.

$T_W^{-1} : \text{ran } T_W \to L^2(V)$ is unbounded
Properties of $T_W$

Thus, $T_W^* T_W$ is **compact**, **injective**, Hermitian, positive.

$\implies T_W^{-1} : \text{ran } T_W \rightarrow L^2(V)$ is unbounded

$\implies \text{ran } T_W$ is not closed
Thus, $T_W^* T_W$ is compact, injective, Hermitian, positive.

$\leftarrow T_W^{-1} : \text{ran } T_W \rightarrow L^2(V)$ is unbounded

$\leftarrow \text{ran } T_W$ is not closed

$\leftarrow T_W^+ \text{ is unbounded and } \mathcal{D}(T_W^+) \subsetneq L^2(W)$
Properties of $T_W$

Thus, $T_W^* T_W$ is compact, injective, Hermitian, positive.

\[
\mapsto T_W^{-1} : \text{ran } T_W \to L^2(V) \text{ is unbounded}
\]

\[
\mapsto \text{ran } T_W \text{ is not closed}
\]

\[
\mapsto T_W^+ \text{ is unbounded and } \mathcal{D}(T_W^+) \subsetneq L^2(W)
\]

\[
\mathcal{D}(T_W^+) \text{ is a dense subset of } L^2(W)
\]
Properties of $T_W$

Thus, $T_W^* T_W$ is compact, injective, Hermitian, positive.

$\iff T_W^{-1} : \text{ran } T_W \rightarrow L^2(V)$ is unbounded

$\iff \text{ran } T_W$ is not closed

$\iff T_W^+$ is unbounded and $\mathcal{D}(T_W^+) \not\subseteq L^2(W)$

$\mathcal{D}(T_W^+)$ is a dense subset of $L^2(W)$

The operator equation $T_W f = g$ is ill-posed
General framework

Ill-posed equation: \( T f_0 = g \) with: \( T : F \rightarrow G \)
General framework

Ill-posed equation: \( T f_0 = g \) with: \( T: F \rightarrow G \)

Minimize \( \frac{1}{2} \left\| g - T f \right\|^2 + \alpha \mathcal{H}(f) \)

s.t. \( f \in F \)
General framework

Ill-posed equation: \( T f_0 = g \) with: \( T: F \rightarrow G \)

Minimize \( \frac{1}{2} \| g - T f \|^2 + \alpha \mathcal{H}(f) \)

s.t. \( f \in F \)

Main issues

Well-posedness

Asymptotic behavior (\( \alpha \downarrow 0 \))
Alternative approach

Ill-posed equation: \( T f_0 = g \) with: \( T: F \rightarrow G \)
Ill-posed equation: \( T f_0 = g \) with: \( T: F \rightarrow G \)

\[ f_0 = C_\beta f_0 + (I - C_\beta) f_0 \]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)
Alternative approach

Ill-posed equation: \( T f_0 = g \) with: \( T: F \to G \)

\[ f_0 = C_\beta f_0 + (I - C_\beta) f_0 \]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)

Assume there exists an operator \( \Phi_\beta: G \to G \) such that

\( TC_\beta = \Phi_\beta T \)
Alternative approach

Ill-posed equation: \( T f_0 = g \) with: \( T: F \to G \)

\[
f_0 = C_\beta f_0 + (I - C_\beta) f_0
\]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)

Assume there exists an operator \( \Phi_\beta: G \to G \) such that

\[
T C_\beta = \Phi_\beta T
\]

\( T f_0 \approx g \leadsto T C_\beta f_0 = \Phi_\beta T f_0 \approx \Phi_\beta g \)
Alternative approach

Ill-posed equation: \( Tf_0 = g \) with: \( T: F \to G \)

\[ f_0 = C_\beta f_0 + (I - C_\beta) f_0 \]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)

Assume there exists an operator \( \Phi_\beta: G \to G \) such that

\[ TC_\beta = \Phi_\beta T \]

\[ Tf_0 \approx g \quad \rightarrow \quad TC_\beta f_0 = \Phi_\beta Tf_0 \approx \Phi_\beta g \]

Minimiser

\[ \frac{1}{2} \| \Phi_\beta g - Tf \|_G^2 + \frac{\alpha}{2} \| (I - C_\beta)f \|_F^2 \]

- p. 13/35
Minimize \[ \frac{1}{2} \left\| \Phi \beta g - T f \right\|_G^2 + \frac{\alpha}{2} \left\| (I - C \beta) f \right\|_F^2 \]
Main issues

Minimize \( \frac{1}{2} \| \Phi_{\beta} g - T f \|_G^2 + \frac{\alpha}{2} \| (I - C_{\beta}) f \|_F^2 \)

Well-posedness
Main issues

Minimize \( \frac{1}{2} \| \Phi_{\beta} g - T f \|_G^2 + \frac{\alpha}{2} \| (I - C_{\beta}) f \|_F^2 \)

Well-posedness

Asymptotic behavior \((\alpha \downarrow 0 \text{ and/or } \beta \downarrow 0)\)
Examples

\[ TC_\beta = \Phi_\beta T \quad \text{with} \quad C_\beta := U^{-1}\hat{\phi}_\beta U \]
Examples

\[ TC_\beta = \Phi_\beta T \quad \text{with} \quad C_\beta := U^{-1} \hat{\phi}_\beta U \]

\[ T = T_W \]

\[ TC_\beta = 1_W U U^{-1} \hat{\phi}_\beta U = \hat{\phi}_\beta 1_W U = \hat{\phi}_\beta T \]

\[ \Phi_\beta = (g \mapsto \hat{\phi}_\beta g) \]
Examples

\[ TC_\beta = \Phi_\beta T \text{ with } C_\beta := U^{-1}\hat{\phi}_\beta U \]

\[ T = K = U^{-1}\hat{k}U, \text{ convolution by } k \]

\[ \Phi_\beta = C_\beta \]
Examples

\[ TC_\beta = \Phi_\beta T \quad \text{with} \quad C_\beta := U^{-1} \hat{\phi}_\beta U \]

\[ T = R, \text{ the Radon operator} \]

\[ (Rf)(\theta, s) = \int f(x) \delta(s - \langle \theta, x \rangle) \, dx \]

\[ R(f_1 * f_2) = Rf_1 \ast Rf_2 \]

\[ \ast \text{ convolution w.r.t. } s \]

\[ TC_\beta f = T(\phi_\beta * f) = T\phi_\beta \ast Tf \]

\[ \Phi_\beta = (g \mapsto T\phi_\beta \ast g) \]
Regularization of $T_W f = g$
Regularization of $T_W f = g$

Minimize

$$\frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2$$

s.t. $f \in L^2(V)$
Regularization of $T_W f = g$

Minimize

$$\frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2$$

s.t. $f \in L^2(V)$

Regularized data: $g_\beta := \Phi_\beta g = \hat{\phi}_\beta g$
Regularization of $T_W f = g$

Minimize $\frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2$

s.t. $f \in L^2(V)$

Regularized data: $g_\beta := \Phi_\beta g = \hat{\phi}_\beta g$

$\phi_\beta(x) = \frac{1}{\beta^d} \phi \left( \frac{x}{\beta} \right) \quad \hat{\phi}_\beta(\xi) = \hat{\phi}(\beta \xi)$
Regularization of $T_W f = g$

Minimize

\[
\frac{1}{2} \left\| \hat{\phi} \beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi} \beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2
\]

s.t. \quad f \in L^2(V)

Regularized data: \( g_\beta := \Phi \beta g = \hat{\phi} \beta g \)

\[\phi \beta(x) = \frac{1}{\beta^d} \phi \left( \frac{x}{\beta} \right) \quad \hat{\phi} \beta(\xi) = \hat{\phi}(\beta \xi)\]

Proposition Let \( \alpha, \beta > 0 \) be fixed. Then \((P_{\alpha,\beta})\) has a unique solution \( f_{\alpha,\beta} \), which depends continuously on \( g \in L^2(W) \).
Outline

- Introduction
- Fourier synthesis
- Asymptotic analysis
- Extension
- A note on proximal inversion
Approximation of $T_{W}^{+}$: $\alpha \downarrow 0$
Approximation of $T_W^+$: $\alpha \downarrow 0$

$\mathcal{P}_{0,\beta}$

Minimize $\frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2$

s.t. $f \in L^2(V)$
Approximation of $T^+_W$: $\alpha \downarrow 0$

\[(P_{0,\beta}) \quad \text{Minimize} \quad \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 \]

\text{s.t.} \quad f \in L^2(V)

Unique solution: $T^+_W(\hat{\phi}_\beta g)$
Approximation of $T_W^+$: $\alpha \downarrow 0$

\[(P_{0,\beta}) \quad \text{Minimize} \quad \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 \]

\[\text{s.t.} \quad f \in L^2(V)\]

Unique solution: $T_W^+(\hat{\phi}_\beta g)$

**Theorem** Let $\beta > 0$ be fixed and let $g \in \mathcal{D}(T_W^+)$. 

(i) If $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$, then $f_{\alpha,\beta} \to T_W^+(\hat{\phi}_\beta g)$ as $\alpha \downarrow 0$.

(ii) If $\hat{\phi}_\beta g \notin \mathcal{D}(T_W^+)$, then $\|f_{\alpha,\beta}\|_{L^2(V)} \to \infty$ as $\alpha \downarrow 0$. 
Approximation of $T_W^+$: $\alpha \downarrow 0$

$$(P_{0,\beta}) \quad \begin{aligned} \text{Minimize} & \quad \frac{1}{2}\|\hat{\phi}_\beta g - T_W f\|^2 \\ \text{s.t.} & \quad f \in L^2(V) \end{aligned}$$

Unique solution: $T_W^+ (\hat{\phi}_\beta g)$

**Proposition** Assume that $\phi \in L^1(\mathbb{R}^d)$ is such that $\hat{\phi}$ is analytical, and let $\beta > 0$ be fixed and $g \in \mathcal{D}(T_W^+)$. Then, the following are equivalent:

(i) $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$;

(ii) $\text{supp} (\phi_\beta * T_W^+ g) \subseteq V$. 
Approximation of $T^+_W$: $\beta \downarrow 0$
Approximation of $T_W^+: \beta \downarrow 0$

**Theorem**  Assume that

- $\alpha > 0$ (fixed)
Approximation of $T^+_{W}$: $\beta \downarrow 0$

**Theorem**  
Assume that

- $\alpha > 0$ (fixed)
- $\phi \in L^1(\mathbb{R}^d)$ with $\int_\mathbb{R}^d \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
Approximation of $T_W^+$: $\beta \downarrow 0$

**Theorem**  Assume that

- $\alpha > 0$ (fixed)
- $\phi \in L^1(\mathbb{R}^d)$ with $\int \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
- $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K \|\xi\|^s$ for some $K, s > 0$
Approximation of $T_W^+$: $\beta \downarrow 0$

**Theorem**  Assume that

- $\alpha > 0$ (fixed)
- $\phi \in L^1(\mathbb{R}^d)$ with $\int \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
- $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K \|\xi\|^s$ for some $K, s > 0$
- $\forall \xi \in \mathbb{R}^d \setminus \{0\}, \hat{\phi}(\xi) \neq 1$
Approximation of $T_W^+$: $\beta \downarrow 0$

Theorem  Assume that

- $\alpha > 0$ (fixed)
- $\phi \in L^1(\mathbb{R}^d)$ with $\int \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
- $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K \|\xi\|^s$ for some $K, s > 0$
- $\forall \xi \in \mathbb{R}^d \setminus \{0\}, \hat{\phi}(\xi) \neq 1$

If $g \in T_W(L^2(V) \cap H^s(\mathbb{R}^d))$, then $f_{\alpha, \beta} \to T_W^+ g$ strongly as $\beta \downarrow 0$. 
Overview of the proof
Overview of the proof

Step 1: \((f_{\alpha, \beta})_{\beta \in (0,1]}\) is bounded
Overview of the proof

Step 1: \( (f_{\alpha,\beta})_{\beta \in (0,1]} \) is bounded

Step 2: \( (f_{\alpha,\beta})_{\beta \in (0,1]} \) converges weakly to \( T_W^+ g \)
Overview of the proof

Step 1: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) converges weakly to \(T_W^+ g\)

\[ \beta_n \downarrow 0, \ f_n := f_{\alpha,\beta_n} \]
Overview of the proof

Step 1: \( (f_{\alpha, \beta})_{\beta \in (0,1]} \) is bounded

Step 2: \( (f_{\alpha, \beta})_{\beta \in (0,1]} \) converges weakly to \( T_W^+ g \)

\[ \beta_n \downarrow 0, \ f_n := f_{\alpha, \beta_n} \]

\[ \exists (f_{n_k}) \rightharpoonup T_W^+ g \]
Overview of the proof

Step 1: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) converges weakly to \(T_W^+ g\)

\[\beta_n \downarrow 0, \ f_n := f_{\alpha,\beta_n}\]

\[\exists (f_{n_k}) \to T_W^+ g\]

Step 3: the convergence is in fact strong
Overview of the proof

Step 1: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) converges weakly to \(T^+_W g\)

\[ \beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n} \]

\[ \exists (f_{n_k}) \rightharpoonup T^+_W g \]

Step 3: the convergence is in fact strong

\[ (f_n) \text{ bounded} \]

\[ \lim_{R \to \infty} \sup_n \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0 \]

\[ \sup_n \|T_h f_n - f_n\| \to 0 \text{ as } \|h\| \to 0 \]

\Rightarrow (f_n) \text{ precompact}
Overview of the proof

Step 1: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) converges weakly to \(T_{W}^{+} g\)

\[\beta_n \downarrow 0, \ f_n := f_{\alpha,\beta_n}\]

\[\exists (f_{n_k}) \rightarrow T_{W}^{+} g\]

Step 3: the convergence is in fact strong

\[(f_n)\] is bounded (Step 1)
Overview of the proof

Step 1: $(f_{\alpha,\beta})_{\beta \in (0,1]}$ is bounded

Step 2: $(f_{\alpha,\beta})_{\beta \in (0,1]}$ converges weakly to $T_W^+ g$

- $\exists (f_{n_k}) \rightharpoonup T_W^+ g$

Step 3: the convergence is in fact strong

- $(f_n)$ is bounded (Step 1)

- $V$ bounded $\iff \lim_{R \to \infty} \sup_n \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0$
Overview of the proof

Step 1: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\alpha,\beta})_{\beta \in (0,1]}\) converges weakly to \(T_W^+ g\)

\[ \beta_n \downarrow 0, \ f_n := f_{\alpha,\beta_n} \]

\(\exists (f_{n_k}) \rightarrow T_W^+ g\)

Step 3: the convergence is in fact strong

\((f_n)\) is bounded (Step 1)

\(V\) bounded \(\iff\) \(\lim_{R \to \infty} \sup_n \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0\)

\(\sup_n \|T_h f_n - f_n\| \rightarrow 0\) as \(\|h\| \rightarrow 0\)
Examples: Lévy kernels
Examples: Lévy kernels

\[ 1 - \hat{\phi}(\xi) \sim_{\xi \to 0} \| \xi \|^s \]
Examples: Lévy kernels

\[ |1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} \| \xi \|^s \]

\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \]
Examples: Lévy kernels

\[ 1 - \hat{\phi}(\xi) \sim_{\xi \to 0} \| \xi \|^s \]

\( \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \)

\( \hat{\phi}: \xi \mapsto \exp(-\|\xi\|^s), \quad s \in [0, 2] \)
Examples: Lévy kernels

\[ |1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} \| \xi \|^s \]

\( \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \)

\( \hat{\phi}: \xi \mapsto \exp\left(-\|\xi\|^s\right), \quad s \in [0, 2] \)

\( \phi: x \mapsto U^{-1} \exp\left(-\| \cdot \|^s\right)(x) \)
Examples: Lévy kernels

\[ |1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} \| \xi \|^s \]

\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \]

\[ \hat{\phi}: \xi \mapsto \exp(-\|\xi\|^s), \quad s \in [0, 2] \]

\[ \phi: x \mapsto U^{-1} \exp(-\| \cdot \|^s)(x) \]

\[ \phi \text{ is positive, isotropic, radially decreasing, } C^\infty \]
Examples: Lévy kernels

Point spread functions

Filters

\( \phi_\beta(x) \)

\( U_\beta(\xi) \)
Examples: Lévy kernels

**Cauchy filter (s=1)**

**Cauchy kernel (s=1)**

**Filter for s=0.6**

**Kernel for s=0.6**
Outline

- Introduction
- Fourier synthesis
- Asymptotic analysis
- Extension
- A note on proximal inversion
Ill-posed equation: $T f_0 = g$ with: $T: F \rightarrow G$
Ill-posed equation: \( T f_0 = g \) with: \( T: F \to G \)

\[
f_0 = C_\beta f_0 + (I - C_\beta) f_0
\]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)
Extension

Ill-posed equation: \( Tf_0 = g \) with: \( T: F \to G \)

\[
f_0 = C_\beta f_0 + (I - C_\beta) f_0
\]

where \( C_\beta \text{ approaches } I \text{ as } \beta \downarrow 0 \)

Assume that there is no operator \( \Phi_\beta : G \to G \)

such that \( TC_\beta = \Phi_\beta T \)
Ill-posed equation: \( Tf_0 = g \) with: \( T: F \rightarrow G \)

\[
f_0 = C_\beta f_0 + (I - C_\beta) f_0
\]

where \( C_\beta \) approaches \( I \) as \( \beta \downarrow 0 \)

Assume that there is no operator \( \Phi_\beta: G \rightarrow G \)

such that \( TC_\beta = \Phi_\beta T \)

\[\]

Minimize \[ \frac{1}{2} \left\| TC_\beta - XT \right\|^2 \]

s.t. \( X \in L(G) \), \( X = 0 \) on \( (\text{ran } T)^\perp \)
$T : L^2(V) \rightarrow G$, $G$ Hilbert space
$C_\beta$ convolution by $\phi_\beta$
Assume that $T$ is well-defined on $\text{ran } C_\beta$ (for $\beta \in (0, \nu)$)
Extension

\[ T: L^2(V) \to G, \quad G \text{ Hilbert space} \]

\[ C_\beta \text{ convolution by } \phi_\beta \]

Assume that \( T \) is well-defined on \( \text{ran } C_\beta \) (for \( \beta \in (0, \nu) \))

\[
(P_\beta) \quad \text{Minimize } \quad \frac{1}{2} \left\| \Phi_\beta g - Tf \right\|_G^2 + \frac{\alpha}{2} \left\| (I - C_\beta)f \right\|_{L^2(\mathbb{R}^d)}^2
\]
Extension

\[ T: L^2(V) \rightarrow G, \quad G \text{ Hilbert space} \]

\[ C_\beta \text{ convolution by } \phi_\beta \]

Assume that \( T \) is well-defined on \( \text{ran} \ C_\beta \) (for \( \beta \in (0, \nu) \))

\[(\mathcal{P}_\beta) \quad \text{Minimize} \quad \frac{1}{2} \left\| \Phi_\beta g - T f \right\|^2_G + \frac{\alpha}{2} \left\| (I - C_\beta) f \right\|^2_{L^2(\mathbb{R}^d)} \]

\[(\mathcal{Q}_\beta) \quad \begin{aligned} \text{Minimize} & \quad X \mapsto \| TC_\beta - XT \| \\ \text{s.t.} & \quad X \in L(G), \quad X = 0 \quad \text{on} \quad (\text{ran} \ T)^\perp \end{aligned} \]
Proposition If $TC_\beta T^+$ is bounded, then $TC_\beta T^+$ admits a continuous extension on $G$ which is a solution to $(Q_\beta)$. 
Solving $(Q_\beta)$

**Proposition** If $TC_\beta T^+$ is bounded, then $TC_\beta T^+$ admits a continuous extension on $G$ which is a solution to $(Q_\beta)$.

**Remark** $TCT^+$ is bounded if and only if there exists a positive number $K$ such that

\[ \forall f \in (\ker T)^\perp, \quad \|TCf\|_F \leq K\|Tf\|_G. \]
**Proposition** Let $T$ be the integral operator of kernel $\alpha$, that is: $T f(x) = \int \alpha(x, y) f(y) \, dy$, with $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$. Assume that

(i) $\int_{\mathbb{R}^d \times \mathbb{R}^d} |\alpha(x, y)|^2 \, dx \, dy < \infty$

(ii) for all $x, y, z \in \mathbb{R}^d$, $\alpha(x, y + z) = \alpha(x, y)g(x, z)$;

(iii) there exists a positive constant $M_\phi$, depending on $\phi$ only, such that

$$\forall x \in \mathbb{R}^d, \quad \left| \int_{\mathbb{R}^d} \phi(z)g(x, z) \, dz \right| < M_\phi.$$ 

Then $T$ is well-defined on $\text{ran} \, C$ and $TCT^+$ is bounded on its domain.
Remarks

1) Problem $(Q_\beta)$ is ill-posed. However, the target $TC_\beta$ does not undergo any perturbation. A proximal strategy may be suitable.
Remarks

1) Problem \((Q_\beta)\) is ill-posed. However, the target \(TC_\beta\) does not undergo any perturbation. A proximal strategy may be suitable.

2) The convergence theorem (as \(\beta \downarrow 0\)) remains valid in this extended context.
Remarks

1) Problem \((Q_\beta)\) is ill-posed. However, the target \(TC_\beta\) does not undergo any perturbation. A proximal strategy may be suitable.

2) The convergence theorem (as \(\beta \downarrow 0\)) remains valid in this extended context.

3) A numerical study is under consideration.
Remarks

1) Problem \( (Q_\beta) \) is ill-posed. However, the target \( TC_\beta \) does not undergo any perturbation. A proximal strategy may be suitable.

2) The convergence theorem (as \( \beta \downarrow 0 \)) remains valid in this extended context.

3) A numerical study is under consideration.

4) Other operators than \( C_\beta \) may be considered.
Minimize $X \mapsto \|TC_\beta - XT\|$ 

s.t. $X \in M_{m \times n}(\mathbb{R})$, $X = 0$ on $(\text{ran } T)^\perp$
For simplicity, we consider here the problem of computing the pseudo-inverse of an ill-posed matrix $M$. 
For simplicity, we consider here the problem of computing the pseudo-inverse of an ill-posed matrix $M$.

**Theorem** The pseudo-inverse of a matrix $M \in M_{m \times n}(\mathbb{R})$ is the solution of minimum Frobenius norm of the optimization problem

\[
\text{(P)} \quad \begin{align*}
\text{Minimize} & \quad f(\Phi) := \frac{1}{2} \| M\Phi - I \|_F^2 \\
\text{s.t.} & \quad \Phi \in M_{n \times m}(\mathbb{R})
\end{align*}
\]
The proximal algorithm
The proximal algorithm

The proximal point algorithm is a general algorithm for computing zeros of maximal monotone operators.
The proximal algorithm

A well-known application is the minimization of a convex function $f$ by finding a zero in its subdifferential.
The proximal algorithm

In our setting, it consists in the following steps:

1. Choose an initial matrix $\Phi_0 \in M_{m \times n}(\mathbb{R})$;

2. Generate a sequence $(\Phi_k)_{k \geq 0}$ according to the formula

$$
\Phi_{k+1} = \arg\min_{\Phi \in M_{n \times m}(\mathbb{R})} \left\{ f(\Phi) + \frac{1}{2\mu_k} \| \Phi - \Phi_k \|^2 \right\},
$$

in which $(\mu_k)_{k \geq 0}$ is a sequence of positive numbers, until some stopping criterion is satisfied.
Optimality conditions

\[ \Phi_{k+1} = \arg\min_{\Phi \in M_{n \times m}(\mathbb{R})} \left\{ \frac{1}{2} \| M\Phi - I \|_F^2 + \frac{1}{2\mu_k} \| \Phi - \Phi_k \|_2^2 \right\} \]
Optimality conditions

\[ \Phi_{k+1} = \arg\min_{\Phi \in M_{n \times m}(\mathbb{R})} \left\{ \frac{1}{2} \| M\Phi - I \|_F^2 + \frac{1}{2\mu_k} \| \Phi - \Phi_k \|_2^2 \right\} \]

\[ M^{\top}(M\Phi_{k+1} - I) + \frac{1}{\mu_{k+1}}(\Phi_{k+1} - \Phi_k) = 0 \]
Optimality conditions

\[ \Phi_{k+1} = \underset{\Phi \in M_{n \times m}(\mathbb{R})}{\text{argmin}} \left\{ \frac{1}{2} \| M\Phi - I \|_F^2 + \frac{1}{2\mu_k} \| \Phi - \Phi_k \|_2^2 \right\} \]

\[ M^\top (M\Phi_{k+1} - I) + \frac{1}{\mu_k} (\Phi_{k+1} - \Phi_k) = 0 \]

\[ (I + \mu_k M^\top M)\Phi_{k+1} = \Phi_k + \mu_k M^\top \]
Optimality conditions

\[ \Phi_{k+1} = \arg\min_{\Phi \in \mathcal{M}_{n \times m}(\mathbb{R})} \left\{ \frac{1}{2} \| M\Phi - I \|_F^2 + \frac{1}{2\mu_k} \| \Phi - \Phi_k \|_2^2 \right\} \]

\[ M^\top (M\Phi_{k+1} - I) + \frac{1}{\mu_k} (\Phi_{k+1} - \Phi_k) = 0 \]

\[ (I + \mu_k M^\top M)\Phi_{k+1} = \Phi_k + \mu_k M^\top \]

Since \( M^\top M \) is positive semi-definite and \( \mu_k \) is positive for all \( k \), the matrix \((I + \mu_k M^\top M)\) is nonsingular and the proximal iteration also reads

\[ \Phi_{k+1} = (I + \mu_k M^\top M)^{-1} \left( \Phi_k + \mu_k M^\top \right) . \]
Remarks
1) In the case where $\mu_k = \mu$ for all $k$, each proximal iteration involves the multiplication by the same inverse matrix $(I + \mu M^\top M)^{-1}$, and that the latter inverse may be quite easy to compute numerically, if the matrix $I + \mu M^\top M$ is well-conditioned.
Remarks

1) In the case where $\mu_k = \mu$ for all $k$, each proximal iteration involves the multiplication by the same inverse matrix $(I + \mu M^\top M)^{-1}$, and that the latter inverse may be quite easy to compute numerically, if the matrix $I + \mu M^\top M$ is well-conditioned.

2) The proximal iteration may be performed by means of any efficient minimization algorithm.
Convergence

**Proposition** Let $\alpha_1$ denote the smallest nonzero eigenvalue of $M^\top M$ and let $E_1$ be the corresponding eigenspace. Assume that $\mu_k = \mu$ for all $k$ and that $\Phi_0$ is not orthogonal to the eigenspace $E_1$. Then,

$$\frac{\|M(\Phi_{k+1} - \Phi_k)\|}{\|\Phi_{k+1} - \Phi_k\|} \rightarrow \frac{1}{1 + \alpha_1 \mu}$$

and

$$\frac{\Phi_{k+1} - \Phi_k}{\|\Phi_{k+1} - \Phi_k\|} \rightarrow \Psi_1$$

in which $\Psi_1$ is a unit eigenvector in $E_1$. Moreover the sequence $(\Phi_k)$ generated by the proximal algorithm converges linearly to the orthogonal projection of $\Phi_0$ onto the solution set $\arg\min f = M^+ + \ker M$. 

– p. 34/35
**Tikhonov approximation.** A standard approximation of the pseudo-inverse of an ill-conditioned matrix $M$ is $(M^\top M + \varepsilon I)^{-1} M^\top$, where $\varepsilon$ is a small positive number. This approximation is nothing but the Tikhonov regularization of $M^+$, with regularization parameter $\varepsilon$.

The choice $\Phi_0 = 0$ yields the latter approximation for $\varepsilon = 1/\mu$ after one proximal iteration.
Link with existing iterative methods. In the case where \( \mu_k = \mu \) for all \( k \), the proximal algorithm belongs to the class of fixed point methods, along with the algorithms of Jacobi, Gauss-Seidel, SOR and SSOR.

It is easy to check that \( M^+ \) satisfies the fixed point equation

\[
\Phi = \varphi(\Phi) := B\Phi + C,
\]

with

\[
B := (I + \mu M^\top M)^{-1} \quad \text{and} \quad C := (M^\top M + \mu^{-1} I)^{-1} M^\top.
\]

Clearly, \( \varphi \) is a contraction and, if \( M \) is invertible, then \( M^\top M \) is positive definite and \( \varphi \) is a strict contraction.