A note on the normwise perturbation theory for the regular generalized eigenproblem $Ax = \lambda Bx$

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Abstract

In this paper, we present a normwise perturbation theory for the regular generalized eigenproblem $Ax = \lambda Bx$, when λ is a simple and finite eigenvalue, which departs from the classical analysis with the chordal norm [9]. A backward error and a condition number are derived for a choice of flexible measure to represent independent perturbations in the matrices A and B. The concept of optimal backward error, associated with an eigenvalue only is also discussed.

Keywords : regular generalized eigenproblem, condition number, backward error.

1 Introduction

The generalized eigenproblem (P) : $Ax = \lambda Bx$, where A and B are two square matrices, often appears at the heart of physical applications. The problem (P), or equivalently the matrix pair (A, B) is said to be regular if there exists at least one complex number z for which $\det(A - zB)$ is nonzero. Otherwise it is said singular. Computing eigenvalues of singular matrix pairs belongs to the class of ill-posed problems [4, 5], i.e. their eigenvalues can change discontinuously as functions of A and B. In this paper, we restrict our study to eigenproblems for regular matrix pairs which are well-posed problems. Regular matrix pairs (A, B) for which B is nonsingular have the same spectrum as the matrix $B^{-1}A$. When B is singular, the matrix pair admits at least one infinite eigenvalue.

The perturbation theory aims at deriving condition numbers and backward errors which are extremely useful for assessing the computation or the numerical simulation. In [9], Stewart and Sun detail the condition number of the regular generalized eigenproblem for the choice of a specific norm, namely the *chordal norm*. The main motivation for using such a norm is to treat finite and infinite eigenvalues within the same formalism. In such a case, it is useful to represent the eigenvalue as a pair of numbers and the chordal norm arises naturally to cope with this situation. The price to pay to handle infinite eigenvalues is to deal with very particular norms, such as the chordal norm, which are more difficult

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to interpret in terms of physical perturbations.

However many situations arise in practice where one seeks a finite eigenvalue only, or where at least one of the two matrices A or B is nonsingular. It is then possible to use more *natural* norms to represent perturbations on the data A and B and on the solution λ, that is norms appropriate to represent easily physical perturbations. Our formulation relies upon a measure which has the property to be flexible, i.e. which can be absolute or relative with respect to the matrix A and / or B according to the user's prescription. The distance between the two matrix pairs (A, B) and (A', B') is defined by

$$\delta = \min \{\omega > 0; \|A - A'\| \leq \omega\alpha \text{ and } \|B - B'\| \leq \omega\beta\}.$$

In the context of a perturbational analysis, keeping A (resp., B) unchanged amounts to set α (resp., β) to zero. Setting $\alpha = \|A\|$ and $\beta = \|B\|$ gives the classical normwise measure. Usually, the physical context will guide the choice of these parameters. In this note, we base the perturbation theory for regular matrix pairs on this measure. In Section 2, we establish the backward analysis for a finite eigenvalue and its corresponding eigenvector, and we introduce the concept of optimal backward error associated with the eigenvalue only. In Section 3, we give the condition number for a simple finite eigenvalue. Section 4 is devoted to numerical illustrations through various examples: sensitivity analysis, backward error analysis and stopping criterion for iterative solvers.

From now on, let α and β be two positive scalars, $\| \,.\, \|$ be any vector and subordinate matrix norm, and $\| \,.\, \|_*$ be the dual vector norm. We assume that (A, B) forms a regular pair of matrices and that λ is a *semi-simple* and *finite* eigenvalue of (A, B). The vector x (resp., x_*) is the right (resp., left) eigenvector associated with λ.

2 Backward error for a finite eigenvalue

Definition 2.1 *Let $(\tilde{\lambda}, \tilde{x}) \in \mathbb{C} \times \mathbb{C}^n$ be an approximate eigenpair of the generalized eigenproblem $Ax = \lambda Bx$. The normwise backward error η associated with $(\tilde{\lambda}, \tilde{x})$ is defined by*

$$\eta(\tilde{\lambda}, \tilde{x}) = \min \{ \quad \varepsilon > 0; \quad (A + \Delta A)\tilde{x} = \tilde{\lambda}(B + \Delta B)\tilde{x} \quad \textit{such that} \quad \|\Delta A\| \leq \varepsilon\alpha \textit{ and } \|\Delta B\| \leq \varepsilon\beta \ \}.$$

Lemma 2.1 *The backward error at $(\tilde{\lambda}, \tilde{x})$ can be computed as*

$$\eta(\tilde{\lambda}, \tilde{x}) = \frac{\left\|A\tilde{x} - \tilde{\lambda}B\tilde{x}\right\|}{(\alpha + \left|\tilde{\lambda}\right| \beta) \|\tilde{x}\|}. \tag{1}$$

Proof

Let $r = A\tilde{x} - \tilde{\lambda}B\tilde{x}$ and $\theta = \|r\| / ((\alpha + \left|\tilde{\lambda}\right| \beta) \|\tilde{x}\|)$. It is straightforward to show that θ is a lower bound for η. The perturbations that realize the equality $\theta = \eta$ are

$$\Delta A = \frac{-\alpha}{(\alpha + \left|\tilde{\lambda}\right| \beta) \|\tilde{x}\|} r y^* \text{ and } \Delta B = \frac{\bar{\tilde{\lambda}}}{\left|\tilde{\lambda}\right|} \frac{\beta}{(\alpha + \left|\tilde{\lambda}\right| \beta) \|\tilde{x}\|} r y^*,$$

where $\bar{\tilde{\lambda}}/\left|\tilde{\lambda}\right|$ is replaced by 1 if $\tilde{\lambda} = 0$ and where $y^* x = \|x\|$ and $\|y\|_* = 1$.

$\square$

Remark: Let us examine the possibilities for the term $\alpha + \left|\tilde{\lambda}\right|\beta$ in the denominator to become zero. If the residual is zero too, then the approximate eigenpair is the exact one and the backward error must be taken as zero. We assume now that this residual is not zero. The case $\alpha = \beta = 0$ forbids any perturbations on A and B: the backward error is infinite because there is no admissible perturbations. The remaining case $\alpha = \left|\tilde{\lambda}\right| = 0$ forbids perturbations on A. Since B plays no role (because $\tilde{\lambda} = 0$), there is no way to find an admissible perturbation and the backward error is infinite.

In many situations, one seeks eigenvalues only, and the corresponding eigenvectors are not computed: the previous definition of the backward error no longer helps to check the numerical quality of the computed eigenvalue. The notion of *optimal backward error* [10] permits to judge the numerical quality of an eigenvalue without knowing any associate eigenvector.

Definition 2.2 *Let $\tilde{\lambda} \in \mathbb{C}$ be an approximate eigenvalue of the generalized eigenproblem $Ax = \lambda Bx$. The normwise backward error η associated with $\tilde{\lambda}$ is defined by*

$$\eta_{\text{opt}}(\tilde{\lambda}) = \min \{ \quad \varepsilon > 0; \quad \exists u \neq 0,\ (A + \Delta A)u = \tilde{\lambda}(B + \Delta B)u \quad \textit{such that} \quad \|\Delta A\| \leq \varepsilon\alpha \textit{ and } \|\Delta B\| \leq \varepsilon\beta \ \}.$$

Lemma 2.2 *The optimal backward error at $\tilde{\lambda}$ can be computed as*

$$\eta_{\text{opt}}(\tilde{\lambda}) = \frac{1}{(\alpha + \left|\tilde{\lambda}\right|\beta)\left\|(A - \tilde{\lambda}B)^{-1}\right\|}. \tag{2}$$

Proof
It is straightforward to see that $\eta_{\text{opt}}(\tilde{\lambda}) = \min_{u\neq 0} \eta(\tilde{\lambda}, u)$. Therefore, $\eta_{\text{opt}}(\tilde{\lambda}) = (\alpha + \left|\tilde{\lambda}\right|\beta)^{-1} \min_{u\neq 0} \left\|(A - \tilde{\lambda}B)u\right\| / \|u\| = (\alpha + \left|\tilde{\lambda}\right|\beta)^{-1} \left\|(A - \tilde{\lambda}B)^{-1}\right\|^{-1}$. This justifies also the terminology of optimal backward error.

$\square$

Clearly the optimal backward error is expensive to compute and is interesting mainly from a conceptual point of view. However, it permits to shed an interesting light on the convergence behaviour of iterative solvers for computing eigenvalues, as will be shown in the last numerical example in Section 4.

3 Condition number for a simple finite eigenvalue

Lemma 3.1 *The relative normwise condition number for the problem (P) associated with the* simple *eigenvalue λ is*

$$K = \frac{(\alpha + |\lambda|\beta)\|x\|\|x_*\|_*}{|\lambda||x_*^* Bx|}. \tag{3}$$

Proof
Let A and B be submitted to a perturbation of amplitude ε with respect to the measure defined in the introduction. Let the pair $(\lambda + \Delta\lambda, x + \Delta x)$ be the solution of the problem $(A + \Delta A)(x + \Delta x) = (\lambda + \Delta\lambda)(B + \Delta B)(x + \Delta x)$. We then have to the first order $\Delta\lambda Bx = A\Delta x + \Delta A x - \lambda B \Delta x - \lambda \Delta B x$, and therefore

$$\frac{|\Delta\lambda|}{|\lambda|} \leq \varepsilon \frac{(\alpha + |\lambda|\,\beta)\,\|x\|\,\|x_*\|_*}{|\lambda|\,|x_*^* B x|}.$$

Equality is achieved for the specific perturbations $\Delta A = \varepsilon\alpha w v^*$ and $\Delta B = -s\varepsilon\beta w v^*$ where v (resp., w) is such that $v^* x = \|x\|$ and $\|v\|_* = 1$ (resp., $x_*^* w = \|x_*\|_*$ and $\|w\| = 1$), and where $s = \bar{\lambda} / |\lambda|$ if λ is nonzero and 1 otherwise.

□

Setting $\alpha = \beta = 1$ gives the absolute condition number $K_{\text{abs}} = (1 + |\lambda|)\,\|x\|\,\|x_*\|_* \,/\, |x_*^* B x|$. In [9], Stewart and Sun establish an upper bound for the first order perturbation of an eigenvalue. The eigenvalue λ is represented as a pair of numbers $< a, b >$ such that $bAx = aBx$ and $\lambda = a/b$ is $b \neq 0$. If $< \tilde{a}, \tilde{b} >$ represents the approximate eigenvalue, then Stewart and Sun show that

$$\chi(< a, b >, < \tilde{a}, \tilde{b} >) \leq \frac{\|x\|_2\,\|x_*\|_2}{\sqrt{|a|^2 + |b|^2}}\,\|(\Delta A, \Delta B)\|_2\,, \tag{4}$$

where $\chi(.,.)$ represents the chordal distance. The quantity $C_{\text{chordal}} = \|x\|_2\,\|x_*\|_2 \,/ \sqrt{|a|^2 + |b|^2}$ is called the chordal condition number[1]. Setting inequality (4) in our notations leads to

$$|\Delta\lambda| \leq \sqrt{1 + |\lambda|^2}\,\frac{\|x\|_2\,\|x_*\|_2}{|x^* B x|}\,\|(\Delta A, \Delta B)\|_2\,, \tag{5}$$

up to the first order. The factor $C_{\text{abs}} = \sqrt{1 + |\lambda|^2}\,\|x\|_2\,\|x_*\|_2 \,/\, |x^* B x|$ is very similar to our absolute condition number. We will denote by C the relative condition number[1], $C = C_{\text{abs}}\,\|(A, B)\|_2 \,/\, |\lambda|$. The main advantage of Stewart and Sun's approach is that it permits to handle the case of an infinite eigenvalue ($b = 0$) which occurs when B is singular. However, the chordal distance behaves counter-intuitively for large numbers (see [9, p. 284]); this can be a drawback for evaluating the error in the approximation of a large eigenvalue.
In our formulation, we cannot treat the case of an infinite eigenvalue but our approach offers several other advantages:

- it permits to handle independent perturbations on A and B, in either a relative or an absolute sense,

- it is valid for any subordinate norm and not restricted to the 2-norm,

[1]The quantities C and C_{chordal} might be only upper bounds of the condition number. The present authors are unaware of any proof showing the existence of perturbations permitting to achieve C or C_{chordal}.

- K is the condition number and not only an upper bound for it: there exist perturbations for which equality in (3) is achieved,
- both the condition number and the backward error are available: this permits to predict the forward error associated with an approximate eigenvalue,
- setting $\beta = 0$ and $B = I$ permits to retrieve the condition number and the backward error for the standard eigenvalue problem.

4 Applications

In this section we present three numerical examples that illustrate the interest of the condition number and the backward errors described above. All the experiments have been performed with MATLAB.

4.1 Sensitivity analysis

Given the regular pair (A, B), we generate a perturbed pair $(A+\Delta A, B+\Delta B)$ by imposing random perturbations on the entries of A and B. Let λ be an eigenvalue of (A, B) and $\tilde{\lambda} = \lambda + \Delta\lambda$ the corresponding eigenvalue of $(A + \Delta A, B + \Delta B)$. We compare the relative error $E = |\Delta\lambda| \,/\, |\lambda|$ with its prediction by

- our formulation for the condition number: $E_1 = K \times \max\{\|\Delta A\|_2 \,/\, \|A\|_2 \,, \|\Delta B\|_2 \,/\, \|B\|_2\}$,
- Stewart and Sun's condition number: $E_2 = C \times \|(\Delta A, \Delta B)\|_2 \,/\, \|(A, B)\|_2$.

We will give also the chordal distance E_{chordal} between λ and $\tilde{\lambda}$ together with its prediction E_3 via inequality (4). In the experiments reported here, the matrix A of order n is a bidiagonal matrix with 20 on the main diagonal and n on the subdiagonal. The matrix B is a diagonal matrix such that $b_{ii} = \sqrt{i}/n$. The reference eigenvalue is then known exactly : $\lambda = 20n$ and its eigenvector is e_1, the first vector of the canonical basis. The corresponding eigenvalue of $(A+\Delta A, B+\Delta B)$ is computed by QZ. We have systematically checked that the backward error associated with the computed eigenvalue $\tilde{\lambda}$ with respect to (A, B) is smaller than the perturbation imposed on A and B, so that our estimations reflect the mathematical sensitivity of the eigenpair and not a numerical artefact due to rounding errors.

Results reported in Tables 1 and 2 correspond to perturbations on the data with relative size close to 10^{-12}. Due to the difference in the choice of the norm for measuring the data perturbations, Stewart and Sun's condition number C can be significantly larger than K (by a factor of 10^2 to 10^3). This fact entails an overestimation by E_2 of the relative error E in the eigenvalue, whereas E_1 estimates E more faithfully (overestimation by a factor of 10 only, in general). This experiment suggests that our approach permits to perform efficiently a sensitivity analysis.

Table 2 shows the chordal error and its estimation E_4 (see equation 4). As expected, we see that the chordal error is significantly smaller than the prediction E_4, especially for large eigenvalues (that is for large values of n in our example). The chordal condition

n	C	E_2/E	E	E_1/E	K
30	4.6e+04	2.5e+04	1.1e-12	1.4e+02	2.5e+02
35	1.4e+05	1.5e+04	6.1e-12	6.9e+01	6.5e+02
40	4.7e+05	2.5e+03	1.3e-10	1.0e+01	1.8e+03
45	1.6e+06	8.0e+03	1.5e-10	2.7e+01	5.4e+03
50	5.6e+06	5.0e+03	8.6e-10	1.5e+01	1.7e+04
55	2.1e+07	1.2e+04	1.4e-09	3.3e+01	5.7e+04
60	8.2e+07	4.5e+03	1.5e-08	1.1e+01	2.0e+05
65	3.4e+08	8.1e+03	3.6e-08	2.9e+01	7.5e+05
70	1.5e+09	1.4e+04	9.3e-08	2.9e+01	3.0e+06
75	6.8e+09	7.1e+03	9.3e-07	1.3e+01	1.3e+07
80	3.3e+10	1.2e+04	2.7e-06	2.1e+01	5.7e+07

Table 1: Sensitivity analysis: condition numbers and error estimations

number is, by far, much smaller than K or C: the eigenproblem is less sensitive to perturbations in the chordal sense than in the classical sense. However, the estimation E_4 can overestimate significantly the chordal error.

4.2 Backward error analysis

Generally a backward error analysis is associated with the study of a computational process. It helps answering the essential question in Numerical Software : does an algorithm produces the exact solution of a nearby problem ? The following experiment will enlight the interest of having extended the notion of backward error to the generalized eigenproblem. Let $(\tilde{\lambda}, \tilde{x})$ be the solution of the eigenproblem $Ax = \lambda Bx$ computed by the QZ algorithm from Matlab. We define

$$\eta_1 = \frac{\left\| A\tilde{x} - \tilde{\lambda} B\tilde{x} \right\|}{\left(\|A\| + \left|\tilde{\lambda}\right| \|B\|\right) \|\tilde{x}\|} \quad \text{and} \quad \eta_2 = \frac{\left\| B^{-1}A\tilde{x} - \tilde{\lambda}\tilde{x} \right\|}{\|B^{-1}A\| \, \|\tilde{x}\|}.$$

The backward error η_1 is associated with the generalized eigenproblem and η_2 with the corresponding standard problem $B^{-1}Ax = \lambda x$. Of course η_2 is defined only for regular matrices B. To be meaningful, η_2 must be computed with a backward stable algorithm to avoid any arithmetic artefact due to the computation of B^{-1}; the following numerical experiments have even been checked with Mathematica. Figure 1 shows η_1 (oo) and η_2 (++) as functions of the order n of the matrices. Here A is the `dingdong` matrix and B the `lehmer` matrix from Higham's collection [8]. We see on this example that η_2 increases from 10^{-13} to 10^{-11} approximately; therefore QZ cannot be considered as an algorithm which is normwise backward stable (to machine precision) with respect to the standard

n	C_{chordal}	E_4	E_{chordal}
30	1.5e+00	4.5e-11	1.8e-15
35	3.8e+00	1.3e-10	8.8e-15
40	9.8e+00	4.1e-10	1.6e-13
45	2.7e+01	1.3e-09	1.7e-13
50	8.0e+01	4.3e-09	8.6e-13
55	2.5e+02	1.6e-08	1.3e-12
60	8.5e+02	5.6e-08	1.2e-11
65	3.1e+03	2.2e-07	2.7e-11
70	1.2e+04	9.6e-07	6.6e-11
75	4.7e+04	4.4e-06	6.2e-10
80	2.0e+05	2.0e-05	1.7e-09

Table 2: Chordal condition number and error estimation

eigenproblem $B^{-1}Ax = \lambda x$. However it is backward stable with respect to the generalized eigenproblem $Ax = \lambda Bx$ because η_1 remains of the order of machine precision. This result seems reasonable. Indeed η_2 follows from an implicit model of perturbations on $B^{-1}A$; but of course this matrix is not explicitly built by the QZ algorithm. On this example, the model of independent perturbations on A and B which yields η_1 seems more appropriate to analyze the computation.

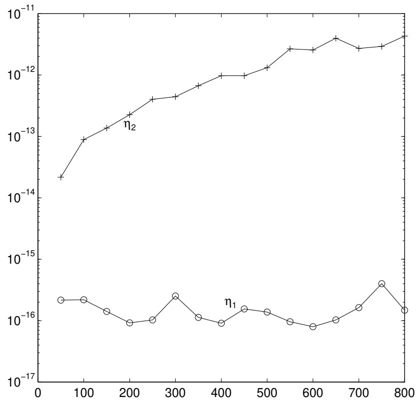

Figure 1: Backward errors η_1 and η_2 for the QZ algorithm

4.3 Stopping criteria for iterative solvers

This last numerical experiment aims at showing the interest of the optimal backward error as a theoretical concept. We consider the generalized eigenproblem $Ax = \lambda Bx$ where B is symmetric positive definite with the Cholesky factorization $B = L^*L$. Solving $Ax = \lambda Bx$ amounts to solve

$$L^{-*}AL^{-1}y = \lambda y \tag{6}$$

and to set $x = L^{-1}y$. Equation (6) can be solved by the standard Lanczos algorithm. Let (λ_i, y_i, x_i) be the approximations of (λ, x, y) at the i^{th} step of the Lanczos algorithm. Note that usually x_i is not computed at each step but only when the solution of (6) has converged.

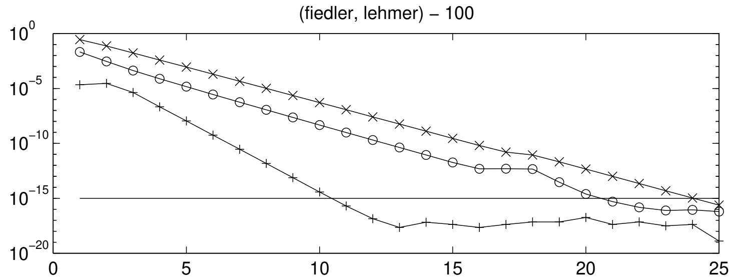

Figure 2: Lanczos – Stopping criterion and backward errors versus the iteration number

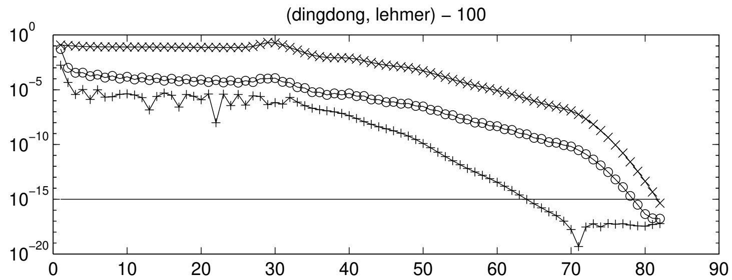

Figure 3: Lanczos – Stopping criterion and backward errors versus the iteration number

The Lanczos process for solving (6) is usually stopped when the value of a "normalized" residual is smaller than a prescribed tolerance. The residual is not computed explicitly but obtained directly from the computation of the tridiagonal matrix in the Lanczos process. For this experiment, we will choose $st_i = \|r_i\| \,/\, \|L^{-*}AL^{-1}\|$ (where r_i is the Lanczos residual associated with Equation (6)) as the stopping criterion: it can be interpreted as the backward error associated with λ_i with respect to the standard eigenvalue problem (6). We will now compare the stopping criterion st_i, the backward error for the generalized eigenproblem $\eta_(\lambda_i, x_i)$ and the optimal backward error $\eta_{\text{opt}}(\lambda_i)$.

Figure 2 shows $\eta(\lambda_i, x_i)$ ('o'), $\eta_{\text{opt}}(\lambda_i)$ ('+') and st_i ('$\times$') for the matrix pair (`fiedler`, `lehmer`). Figure 3 shows the same quantities for the matrix pair (`dingdong`, `lehmer`). In both examples the matrices, taken from [8], are of order 100. The tolerance is set to 10^{-15} as indicated on the figures by a continuous horizontal line.

According to the stopping criterion st_i, convergence is achieved after 25 (resp., 82) iterations for the first (resp., second) example. However, the backward error $\eta(\lambda_i, x_i)$ was already smaller than 10^{-15} after 21 (resp., 79) iterations. This is the same phenomenon as for the previous example with QZ: it seems more appropriate to consider λ_i as an eigenvalue of a perturbed generalized eigenproblem $(A + \Delta A, B + \Delta B)$ than of a perturbed standard eigenproblem $(L^{-*}AL^{-1} + E)$.
Even more interesting is the comparison between the $\eta(\lambda_i, x_i)$ and $\eta_{\text{opt}}(\lambda_i)$. The optimal backward error matches the required tolerance after 11 (resp., 64) iterations in the first (resp., second) example: this is due to the quadratic convergence of the eigenvalue (in the symmetric case[2]). Therefore, iterations 12 to 25 (resp., 82 to 94) help only refining the computed eigenvector. If the user is interested by the eigenvalue only and not by the eigenvector, then these iterations are superfluous. The optimal backward error permits to detect this early convergence and avoid additional iterations. However, the optimal backward error suffers from a high cost of computation which makes it more interesting from a conceptual than from a practical point of view. More work should be done in order to investigate about the availability of a cheap estimate for the optimal backward error.

5 Conclusion

In this note, we have completed the normwise perturbation theory for a finite eigenvalue of a regular matrix pair. The availability of both the condition number (for a simple eigenvalue) and of the backward error permits to efficiently carry the assessment of the numerical quality of computed eigenpairs. The optimal backward error, which reflects the quality of the eigenvalue only with no reference to the corresponding eigenvector, offers at this stage a more theoretical than practical interest because of its high computational cost. However, it has proved to be helpful in some physical applications where the eigenvector is not available [7]. Finally, the normwise model chosen here yields a wide flexibility for choosing or constraining the perturbations on the data and allows the user to treat independently each matrix of the pair.

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[2]If A is symmetric, then $L^{-*}AL^{-1}$ is also symmetric.

References

[1] F. Chaitin-Chatelin and V. Frayssé. *Lectures on Finite Precision Computations.* SIAM, Philadelphia, 1996.

[2] F. Chatelin. *Valeurs propres de matrices.* Masson, Paris, 1988.

[3] F. Chatelin. *Eigenvalues of matrices.* Wiley, Chichester, 1993. Enlarged Translation of [2].

[4] J. W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: robust software with error bounds and applications. Part I: Theory and Algorithms. *ACM Trans. Math. Software*, 19:160–174, 1993.

[5] J. W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: robust software with error bounds and applications. Part II: Software and Applications. *ACM Trans. Math. Software*, 19:175–201, 1993.

[6] L. Elsner and J. Sun. Perturbation theorems for the generalized eigenvalue problem. *Linear Algebra Appl.*, 48:341–357, 1982.

[7] V. Frayssé, M. Gueury, F. Nicoud, and V. Toumazou. Generalized spectral portraits. Technical Report TR/PA/96/19, CERFACS, 1996.

[8] N. J. Higham. A collection of test matrices in MATLAB. *ACM Trans. Math. Software*, 17:289–305, September 1991.

[9] G. W. Stewart and J. Sun. *Matrix Perturbation Theory.* Academic Press, New York, 1991.

[10] V. Toumazou. *Portraits spectraux de matrices: un outil d'analyse de la stabilité.* Ph.D. dissertation, Université H. Poincaré, Nancy, 1996.