Spectral portraits for matrix pencils

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Abstract

Spectral portraits and pseudospectra have recently attracted the attention of many scientists as a tool of choice for investigating spectral properties of nonnormal matrices. Until now, most of the papers devoted to this topic were generally dealing with the standard eigenproblem $Ax = \lambda x$. However in practice, physical applications often give rise to generalized eigenproblems $Ax = \lambda Bx$. It is therefore natural to extend the notion of ε-pseudospectrum and spectral portrait to matrix pencils (A, B). In this paper, we explain how to define and compute spectral portraits for matrix pencils and we illustrate their importance on and example in Computational Fluid Dynamics.

Keywords : generalized eigenproblem, pseudospectrum, spectral portrait, nonnormality.

1 Introduction

Eigenvalues of nonsymmetric matrices play an essential role in physical problems. These eigenvalues may be used either for their physical meaning (vibration modes, stability region,...) or for their numerical meaning (convergence of numerical schemes,...). However, as a matrix gets far from normal, its spectrum gets more and more unstable. On such matrices, iterative methods can become backward unstable (if not implemented carefully) [2], [3]; even stable algorithms such as QR give wrong approximations for eigenvalues due to the huge spectral ill-conditioning [3], [6]. Moreover, we still have no access to a condition number for clustered eigenvalues; only upper bounds exist in the literature [5], [6]. In this context, a computed eigenvalue may loose its meaning if no information is known about its stability. The trend today is to consider not an eigenvalue by itself but its topological neighbourhood. Such an approach permits to understand the influence of perturbations on the spectrum of nonnormal matrices [18]. The spectral portrait is a tool of choice for understanding the influence of an eigenvalue. It is a way to visualize the ε-pseudospectrum

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[22] that is $\Lambda_\varepsilon(A) = \{z \in \mathbb{C};\ z$ is an eigenvalue of $(A + \Delta A)$ with $\|\Delta A\|_2 \leq \varepsilon \|A\|_2\}$. The spectral portrait has been recently the subject of many publications (see [3], [9], [19], [20], [11], [1]). It consists in the representation of the map $z \longrightarrow \mathrm{spp}(z) = log_{10} \|(A - zI)^{-1}\|_2 \|A\|_2$ in a prescribed region of the complex plane. The contour lines of level ε of the spectral portrait are the frontiers of the ε-pseudospectrum of the matrix A : they enclose all the eigenvalues of the matrices $A + \Delta A$ with $\|\Delta A\|_2 \leq \varepsilon \|A\|_2$.

However, it is often the case in physics that the eigenproblem to be solved is a generalized eigenproblem and not the standard one. An immediate possibility is to study the spectral portrait of $B^{-1}Ax = \lambda x$. This implicitly refers to study global perturbations on the matrix product $B^{-1}A$. However, physicists may ask for the set of eigenvalues for the pairs $(A + \Delta A, B + \Delta B)$, where $\|\Delta A\|_2$ and $\|\Delta B\|_2$ are within given tolerances α and β respectively. These tolerances are usually depending on physical and numerical parameters. In order to answer this question, we have extended the notion of spectral portraits to the generalized eigenproblem. This extension is easily performed from the perturbation theory developed in [8]. This paper will serve as an illustration for this perturbation theory. This work relies upon an appropriately defined measure which has the property to be flexible i.e. which can be absolute or relative with respect to the matrix A and / or B, according to the user's prescription. The distance between the two pairs (A, B) and (A', B') is defined by

$$\delta = \min \{\omega > 0; \|A - A'\| \leq \omega\alpha \text{ and } \|B - B'\| \leq \omega\beta\}.$$

The properties of this measure and its comparison with the CHOrdal norm [16], more classically used for matrix pencils, is studied in [8]. In the context of a perturbational analysis, keeping A (resp., B) unchanged amounts to set α (resp., β) to zero. Setting $\alpha = \|A\|$ and $\beta = \|B\|$ gives the classical normwise measure. Usually, the physical context will guide the choice of these parameters.

In Section 2, we define and characterize the generalized spectral portrait. In Section 3, we describe an application of the generalized spectral portrait on a problem arising in a CFD. Finally, In Section 4, we briefly present a parallel code based on the Lanczos method.

2 Definition and characterizations

Let α and β be two positive scalars, and $\| \,.\, \|$ be any vector subordinate norm. Let A, B, ΔA and ΔB be square matrices from $\mathbb{C}^{n\times n}$ such that (A, B) and $(A + \Delta A, B + \Delta B)$ form regular pairs of matrices[1]. We denote by $\{\sigma_i(A)\}_{i=1,\ldots,n}$ the singular values of the matrix A sorted by decreasing order.

Definition 2.1 *The normwise ε-pseudospectrum for the regular pencil (A, B) is defined by*

$$\Lambda_\varepsilon(A, B) = \{ \quad z \in \mathbb{C}; \quad \exists u \neq 0 \in \mathbb{C}^n \text{ such that } (A + \Delta A)u = z(B + \Delta B)u$$

[1]The matrix pair (A, B) is said to be regular if there exists at least one complex number z for which $\det(A - zB)$ is nonzero. Otherwise it is said singular.

$$\textit{where} \quad \|\Delta A\| \leq \varepsilon\alpha \textit{ and } \|\Delta B\| \leq \varepsilon\beta \,\} . \tag{1}$$

An equivalent definition for $\Lambda_\varepsilon(A, B)$ can be based on the optimal backward error. In [8], we have defined and characterized the optimal normwise backward error for the generalized eigenproblem. We recall now its definition

Definition 2.2 *Let $\tilde{\lambda} \in \mathbb{C}$ be an approximate eigenvalue of the generalized eigenproblem $Ax = \lambda Bx$. The optimal normwise backward error η associated with $\tilde{\lambda}$ is defined by*

$$\begin{aligned} \eta(\tilde{\lambda}) &= \min \{\, \varepsilon > 0; \exists u \neq 0,\ (A + \Delta A)u = \tilde{\lambda}(B + \Delta B)u \\ &\quad \textit{with } \|\Delta A\| \leq \varepsilon\alpha \textit{ and } \|\Delta B\| \leq \varepsilon\beta \,\} \\ &= \frac{1}{(\alpha + \left|\tilde{\lambda}\right| \beta) \left\|(A - \tilde{\lambda}B)^{-1}\right\|}. \end{aligned}$$

The concept of *optimal* backward error has been introduced in [8] to assess the numerical quality of a computed eigenvalue, independently of any approximate eigenvector. This definition permits to interpret the normwise generalized ε-pseudospectrum $\Lambda_\varepsilon(A, B)$ as the set of the complex points z for which the normwise backward error $\eta(z)$ is less or equal than ε. This point of view is very informative and useful to interpret and use the pseudospectrum on practical applications. The following theorem gives now three equivalent ways to characterize the normwise ε-pseudospectrum with any subordinate norm.

Theorem 2.1 *The three following assertions are equivalent :*

i) *z is an eigenvalue of the matrix pair $(A + \Delta A, B + \Delta B)$ where $\|\Delta A\| \leq \varepsilon\alpha$ and $\|\Delta B\| \leq \varepsilon\beta$.*

ii) *There exists a unit vector $u \neq 0$ in $\mathbb{C}^n$ such that $\|(A - zB)u\| \leq \varepsilon(\alpha + |z|\,\beta)$.*

iii) $\|(A - zB)^{-1}\| \geq \dfrac{1}{\varepsilon(\alpha + |z|\,\beta)}$.

Proof
The proof is straightforward from the characterization of the optimal backward error [8].

□

Corollary 2.2 *With the Euclidean norm, the property*

iv) $\sigma_n(A - zB) \leq \varepsilon(\alpha + |z|\,\beta)$

is also equivalent to the properties i) *to* iii) *of Theorem 2.1.*

The above results offer us a simple way to compute the generalized spectral portrait.

Definition 2.3 *The generalized spectral portrait is the plot of the map*

$$z \longmapsto \mathrm{gspp}(z) = \log_{10}\left[(\alpha + |z|\,\beta)\left\|(A - zB)^{-1}\right\|_2\right],$$

for z in a prescribed region of the complex field.

We see that the generalized spectral portrait is a simple extension of the classical spectral portrait that involves the generalized resolvent $(A - zB)^{-1}$. Results on the classical spectral portrait can be retrieved immediately by setting $B = I$ and $\beta = 0$.

Remark 1: The definitions given above remain valid when B is singular. If B is regular, the pair (A, B) and the matrix $B^{-1}A$ have the same spectrum and the ε-pseudospectrum of (A, B) is bounded. In [21], van Dorsselaer studies in detail the pseudospectrum of (A, B) where B is singular and remains unperturbed, and shows that for a large enough ε, the ε-pseudospectrum of (A, B) is the entire complex plane. Our work differ from [21] in that we allow B to vary and in the choice of *relative* perturbations instead of absolute ones.

Remark 2: Earlier works [14], [15] considered also pseudospectra for matrix pencils but again allowing perturbations on A only.

3 Applications

The use of spectral portrait is not restricted to the assessment of the numerical quality in finite precision computations. The main purpose for which we were lead to work on the generalized eigenproblem was to study the sensitivity of such problems to external (physical) perturbations. Generally, the matrices A and B are obtained at the final stage of the treatment of a physical problem in numerical simulation. They are many reasons why the entries of A and B can be subject to perturbations. They can simulate physical perturbations (such as deformations), to take into account data uncertainty (limited accuracy of physical measurements), and also deal with numerical perturbations such as the truncation error of an approximation scheme, discretization of derivatives, iterative methods, rounding errors,.... In such cases, the perturbations are known independently on A and B and it would not be appropriate to work on $B^{-1}A$ (this might even be impossible if B were singular). After explaining the use of a spectral portrait on an academic example, we will detail an application from Computational Fluid Dynamics for which a sensitivity analysis with respect to perturbations of A and B is required.

3.1 Test case

Let A be the companion matrix of order 10 associated with the monic polynomial $p(x) = (x-1)^3(x-2)^3(x-3)^3(x-4) = x^{10} + \sum_{i=0}^{9} a_i x^i$. The ten coefficients are $a_0 = 864$, $a_1 = -4968$, $a_2 = 12492$, $a_3 = -18086$, $a_4 = 16703$, $a_5 = -10290$, $a_6 = 4287$, $a_7 = -1194$, $a_8 = 213$ and $a_9 = -22$. Let B be the diagonal matrix whose i^{th} entry equals $1/i$.

Figure 1 gives the generalized spectral portrait for the matrix pair (A, B), and for $\alpha = \|A\|$ and $\beta = \|B\|$. The mesh size is 128×128. The choice of a scale in shades of grey permits to see easily the contours of the ε-pseudospectra for values of ε from 10^{-5} to 10^{-11}. Consider the region enclosed by the border associated with $\varepsilon = 10^{-9}$: this region is the set of all eigenvalues of all the matrix pairs $(A + \Delta A, B + \Delta B)$ for $\|\Delta A\| \le 10^{-9}\,\|A\|$, $\|\Delta B\| \le 10^{-9}\,\|B\|$.

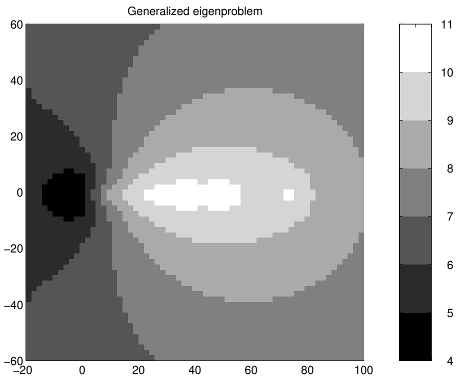

Figure 1: *Generalized spectral portrait for* (A, B)
$\alpha = \|A\|$ and $\beta = \|B\|$

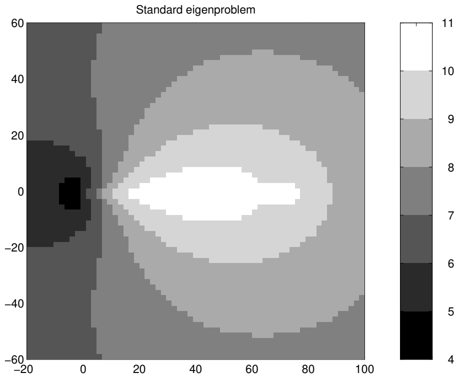

Figure 2: *Spectral portrait for* $B^{-1}A$
$\alpha = \|B^{-1}A\|$

Another interpretation is the following. Suppose that the matrices A and B are known to the normwise relative precision 10^{-7} for instance. Then, the region enclosed by the border associated with $\varepsilon = 10^{-7}$ is the set of all possible eigenvalues for the matrix pair. In our case, this region is very large because the matrix A is a nonnormal matrix with defective eigenvalues.

Figure 2 gives the classical spectral portrait of the matrix $B^{-1}A$. We can see that the ε-pseudospectra are rather different than in the generalized spectral portrait : they tend to be larger. This portrait can be useful for analyzing the quality of a computed eigenvalue $\tilde{\lambda}$ of the matrix pair (A, B) obtained with an algorithm using (implicitly or explicitly) the matrix $B^{-1}A$. If we compute the backward error $\eta(\tilde{\lambda})$ associated with $\tilde{\lambda}$, the spread of the pseudospectrum of level $-\log_{10}\eta(\tilde{\lambda})$ gives information about the accuracy of the computed eigenvalue $\tilde{\lambda}$ for $B^{-1}A$.

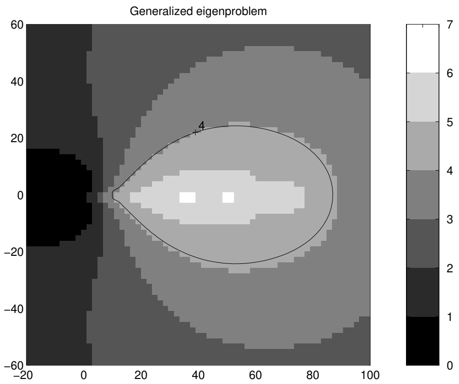

Figure 3: *Spectral portrait for* (A, B)
$\alpha = 1$ and $\beta = 10^{-3}$

Suppose now that the matrix A is known within an absolute tolerance of 10^{-4} and B within an absolute tolerance of 10^{-7}. We want to visualize all the possible eigenvalues for all the pairs representing (A, B) within the given tolerance:

$$\begin{cases} \|\Delta A\| \le 10^{-4} = \alpha\varepsilon, \\ \|\Delta B\| \le 10^{-7} = \beta\varepsilon. \end{cases}$$

We can choose $\alpha = 1$, $\beta = 10^{-3}$ and $\varepsilon = 10^{-4}$: this corresponds to the spectral portrait given in Figure 3. The contour line of level $-\log_{10}\varepsilon = 4$ encloses the desired set. It is displayed on Figure 3. Of course, there is an infinite number of possible choices for the triple $(\alpha, \beta, \varepsilon)$. All the triples such that $\alpha\varepsilon = 10^{-4}$ and $\beta\varepsilon = 10^{-7}$ are admissible and would give the same information. It is remarkable that *one* curve (the border of the ε-pseudospectrum) encloses all the eigenvalues for pairs of matrices resulting from *independent* perturbations on A and B. This makes the spectral portrait particularly easy to interpret and a tool of choice for a sensitivity analysis.

3.2 An example from CFD

The use of spectral portraits can be very interesting in the framework of problems coming from physics.
Let us examine, on a scholar case, the information that can be given by the pseudospectra. For this case, we make the following assumptions :

- the physicist has to solve (P), an exact eigenproblem $Ax = \lambda x$ where A is a $n \times n$ matrix.

- From the localization of an eigenvalue denoted by λ_*, the physicist can conclude on the stability or the instability of the model for the physical problem. λ_* is, for example, the eigenvalue of largest real part. The region where λ_* must be located to ensure the stability of the physical problem (Π) is called the *stability domain* of (Π).

- The problem (P) is approximated by the problem $(\tilde{P})$, $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{x}$ where $\tilde{A}$ is an $n \times n$ matrix. We denote by $\tilde{\lambda}_*$ the eigenvalue of largest real part of $(\tilde{P})$.

The question for the physicist could be the following one : if $(\tilde{P})$ is solved and if $\tilde{\lambda}_*$ is located in the stability domain of (Π), it is possible to conclude that the problem (Π) is stable? That is, although λ_* is not explicitly computed, is it possible to conclude from the resolution of $(\tilde{P})$ that λ_* is located in the stability domain of (Π)?
One of the possible answers based on pseudospectra can be expressed as follows : we compute $\varepsilon = \left\|\tilde{A} - A\right\|_2 / \left\|\tilde{A}\right\|_2$ and we represent the ε-pseudospectrum of $\tilde{A}$ around $\tilde{\lambda}_*$. The eigenvalue λ_* belongs to this ε-pseudospectrum.

If the pseudospectrum is totally included in the stability (resp., instability) domain of (Π), one can conclude that λ_* is itself in the stability domain and therefore that the physical problem is stable (resp., unstable).

On the other hand, if the ε-pseudospectrum lies partly in the instability domain, one cannot conclude about the stability of the problem (II) from the computed solution of the problem $(\tilde{P})$.

One other possible answer would be to compute the spectral portrait of A, and not $\tilde{A}$, and to determine the value of ε from which all the ε-pseudospectra around λ_* lie entirely in the stability (or instability) domain. Let ε' be this value. The physicist can use all the matrices $\tilde{A}$ such that $\left\|\tilde{A} - A\right\|_2 \leq \varepsilon' \|A\|_2$ to approximate (P) by the $(\tilde{P})$ problems.

In practice, such academic cases are dream cases. Actual cases should in fact have the following characteristics :

- the reference problem is continuous, $Tx = \lambda x$ where T is a differential or integral operator,
- when A is a matrix, $\left\|\tilde{A} - A\right\|_2$ can be difficult to estimate or can even be unbounded (in the case of differential operators [4]).

The following example deals with a problem submitted by J. R. Angilella and F. Nicoud from the CFD Project at CERFACS: it concerns the linear stability of a generalized Couette flow with injection and a pressure gradient.
Briefly presented, the problem has the following features :

- the flow is defined by its stream function $\psi(x,y)$ separated into a mean $\bar{\psi}$ and a fluctuating part $\hat{\psi}$ such that $\psi(x,y) = \bar{\psi}(x,y) + \hat{\psi}(x,y,t)$,
- the perturbation $\hat{\psi}(x,y,t)$ takes the form

  $$\hat{\psi}(x,y,t) = \phi(y) \exp^{i\alpha(x-ct)}$$

  where $\phi(y)$ is the complex amplitude of the fluctuating stream function,
- $\phi(y)$ verifies the Orr-Sommerfeld equation and roughly speaking, it satisfies the generalized eigenvalue problem

  $$(P): \qquad A\phi(y) = \lambda B\phi(y)$$

  where the operators A and B are linear combinations of differential operators involving $\phi^{(i)}$ for $i = 0, ..., 4$.

The stability of the flow depends on the eigenvalue of (P) with largest imaginary part. If this imaginary part is positive, the flow is unstable; otherwise, it is stable.
In order to study this flow, Angilella and Nicoud use a discretization with n grid points along the y axis. The problem (P) is approximated by the problem $(P_{n,p})$

$$(P_{n,p}) \qquad A_{n,p}\phi = \lambda B_{n,p}\phi$$

where the components of the matrices $A_{n,p}$ and $B_{n,p}$ are computed by Taylor expansions of order p.
Their question was how to affirm that, when the eigenvalue of largest imaginary part of $(P_{n,p})$ is in the stability or instability domain, the eigenvalue of largest imaginary part of the problem (P) is also in the same domain.
The difficulty in answering their question derives from the fact that the problem (P) is continuous and also from the fact that we do not have any information about the problem except for the physical parameters.
One possible solution consists in building the problem $(P_{n,p})$ for p tending to infinity and considering the problem $(P_{n,\infty})$ as the quasi-exact problem. By computing, $\|A_{n,\infty} - A_{n,p}\|_2$ and $\|B_{n,\infty} - B_{n,p}\|_2$, one could estimate ε and represent the ε-pseudospectrum of a pencil $(A_{n,p}, B_{n,p})$. Unfortunately, the matrices involved in the problems $(P_{n,p})$ are approximations of differential operators and their differences are unbounded : such an approach therefore makes no sense (see [4]).
The only information we have, a priori, on the solution of (P) comes from an article of Orszag (see [13]) in which, for a particular flow of this type, the author gives the eigenvalue of largest imaginary part, denoted below by λ_{ref},

$$\lambda_{\text{ref}} = 0,23752649 + 0,00383967i\,.$$

Owing to the fact that Orszag uses spectral methods, considered as very accurate to solve the problem (P), we have assumed that λ_{ref} is exact and we have considered it as the only known reference to the exact problem.
For the example detailed in [13], Angilella and Nicoud have built the matrix pencils of $(P_{n,p})$ for different values of n and p. From this set of matrix pencils $(A_{n,p}, B_{n,p})$, we have tried, by a qualitative approach, to determine the matrix pencil that realizes the best compromise between the cost for the physicist (size of discretization, order of the scheme for finite differences...) and the quality of the approximation.

A problem $(P_{n,p})$ will be considered as a good approximation of (P) if the error committed on the eigenvalue of largest imaginary part is small and if the pencil $(A_{n,p}, B_{n,p})$ does not exhibit, in the neighbourhood of this eigenvalue, a high instability with respect to the pseudospectra.

The set of matrix pencils $(A_{n,p}, B_{n,p})$ has been built for $p = 2, 4$ and 6 and $n = 50, 100, 200, \ldots, 1000$.
Figure 4 shows, in logarithmic scale, for the considered values of p and n, the evolution of the optimal backward error associated with λ_{ref} when λ_{ref} is considered as a solution of $(A_{n,p}, B_{n,p})$. The case $p = 2$ is represented by '*', the case $p = 4$ by '+' and the case $p = 6$ by 'o'.
As we expected, the higher the order of the scheme, the better the optimal backward error associated to λ_{ref}. λ_{ref} is a solution at machine precision of $(P_{n,p})$ for $n = 1000$ and $p = 6$. Therefore, from this point of view, the approximation of (P) by $(P_{1000,6})$ is optimal.

However, since Angilella and Nicoud were working with $p = 6$ and $n = 300$, we have

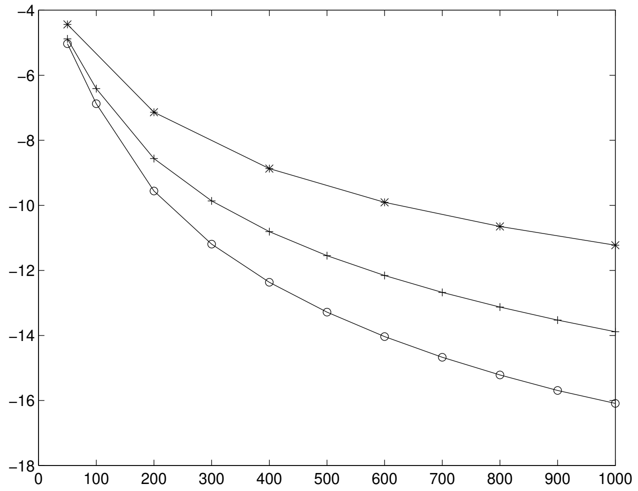

Figure 4: Optimal backward error associated to λ_{ref} as a solution of $(P_{n,p})$ vs. n for $p = 2$ (*), $p = 4$ (+) and $p = 6$ (o)

compared the quality of the approximation of (P) by $(A_{300,6}, B_{300,6})$, $(A_{400,4}, B_{400,4})$ and $(A_{1000,2}, B_{1000,2})$ by looking at their respective spectral portraits. Indeed, for each of these pencils, the optimal backward error associated to λ_{ref} is of the same order, 10^{-11}, and we have tried to determine which one was the best to use.
On the spectral portraits given by figures 5, 6 and 7, we have plotted the border of the

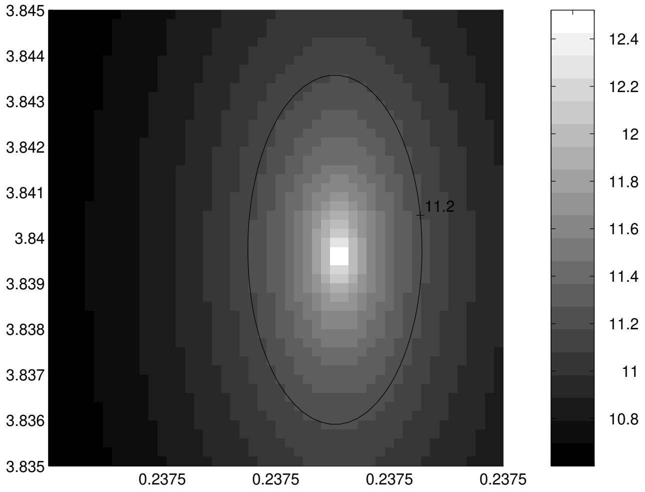

Figure 5: Spectral portrait of $(A_{300,6}, B_{300,6})$ and level curve $\varepsilon = 10^{-11,2}$

ε-pseudospectrum where ε is equal to the optimal backward error $\eta(\lambda_{\mathrm{ref}})$ associated to λ_{ref} for each of the pencils.
The spread of the $\eta(\lambda_{\mathrm{ref}})$-pseudospectra of these three matrix pencils gives us two types of information :

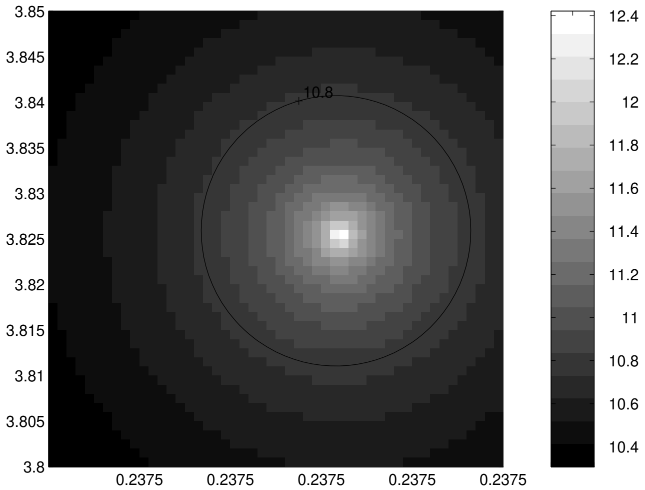

Figure 6: Spectral portrait of $(A_{400,4}, B_{400,4})$ and level curve $\varepsilon = 10^{-10,8}$

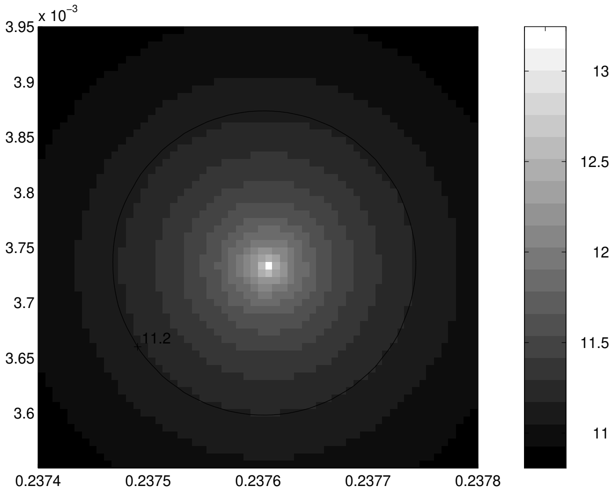

Figure 7: Spectral portrait of $(A_{1000,2}, B_{1000,2})$ and level curve $\varepsilon = 10^{-11,2}$

- it measures the maximal direct error that may be committed during the approximation of $\lambda_{\rm ref}$ since $\lambda_{\rm ref}$ is located on the border of the ε-pseudospectrum,

- it allows the comparison of the spectral instabilities of the matrix pencils under consideration since the corresponding values of ε approximations of the same order (resp., $10^{-11.2}$, $10^{-10.8}$, $10^{-11.2}$).

The maximal diameter between two points of the ε-pseudospectrum is of order 10^{-5} for $(A_{300,6}, B_{300,6})$, 3×10^{-5} for $(A_{400,4}, B_{400,4})$ and 3×10^{-4} for $(A_{1000,2}, B_{1000,2})$. Therefore, $(P_{300,6})$ can be considered as a better approximation than $(P_{400,4})$ and $(P_{1000,2})$ for (P) in the sense previously defined : the maximal direct error that can be committed should be smaller than with the other matrix pencils and since the ε-pseudospectrum of $(A_{300,6}, B_{300,6})$ is the smallest of the three plotted, this matrix pencil can be considered as the most stable one in the neighbourhood of $\lambda_{\rm ref}$.

Figures 8 and 9 show the spectral portraits of $(A_{50,2}, B_{50,2})$ and $(A_{50,6}, B_{50,6})$. The border

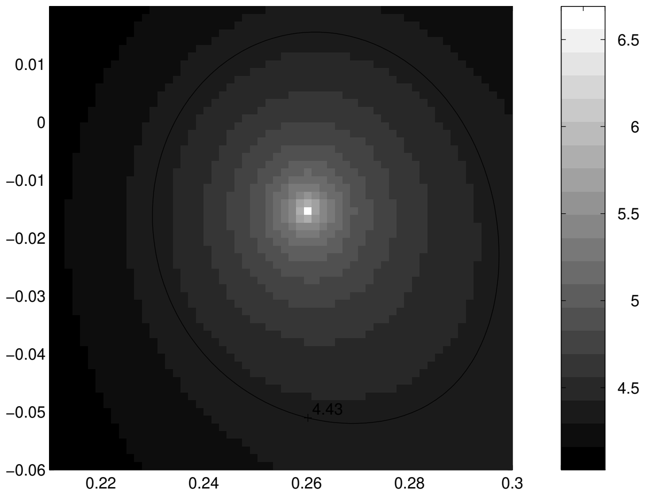

Figure 8: Spectral portrait of $(A_{50,2}, B_{50,2})$ and level curve $\varepsilon = 10^{-4,4}$

of the ε-pseudospectrum has also been plotted where ε still corresponds to the optimal backward error associated with $\lambda_{\rm ref}$ considered as an eigenvalue of the respective matrix pencils.
As can be observed, both ε-pseudospectra lie in both the stability and the instability domains implying that the approximations $(P_{50,2})$ and $(P_{50,6})$ do not accurately represent the reality of the physical problem. Moreover, on the spectral portrait of $(A_{50,6}, B_{50,6})$, one can distinguish two eigenvalues (points of maximal amplitude) in the same region of the ε-pseudospectrum. Due to this collective behaviour, $(P_{50,6})$ can be said to be more unstable than $(P_{50,2})$ in the neighbourhood of $\lambda_{\rm ref}$.

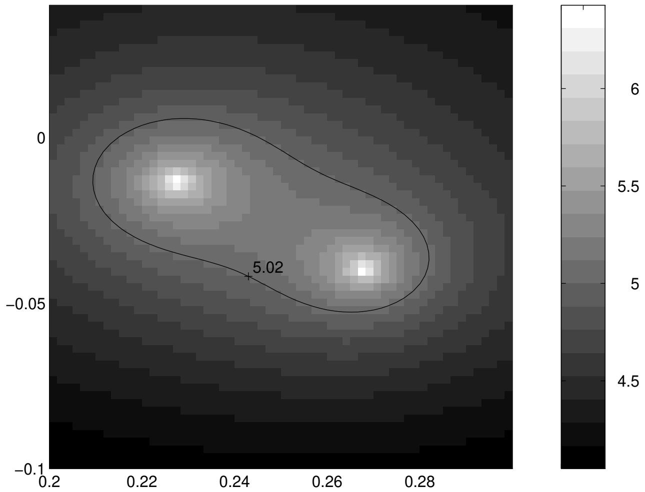

Figure 9: Spectral portrait of $(A_{50,6}, B_{50,6})$ and level curve $\varepsilon = 10^{-5,02}$

4 Implementation

In [11], [12], [17], several methods to compute the spectral portrait are described. They implement various methods for computing the norm of the resolvent $(A - zI)^{-1}$. Let $C = A - zB$. The 2-norm of C^{-1} is the reciprocal of the smallest singular value of C or equivalently the square root of the reciprocal of the smallest eigenvalue of C^*C. This eigenvalue can be easily computed with a restarted Lanczos method combined with a shift and invert technique around 0. Our code is tuned for sparse matrices (Compressed Sparse Row format) and uses routines from BLAS, LAPACK, SPARSKIT and the Harwell subroutine library.

The parallelism in the computation of the spectral portraits is easy to exploit. Suppose we have meshed the interesting region of $\mathbb{C}$. It is clear that the evaluation of gspp(z) can be done independently on any point of the grid; this gives rise to embarrassing parallelism. But an important aspect is that the time for evaluating gspp(z) may vary a lot from one point to another. Therefore, we have designed a parallel code targeted for distributed memory machines that implements a load balancing strategy. The grid is divided into subdomains which are dynamically allocated to the first processor available. The only required communications are the initial broadcast of the matrix to all slaves, each allocation of a subdomain (master to slave) and the set of values gspp(z) on the subdomain (slave to master). There is neither synchronization nor communication between the slaves.

A complete description of the parallel code implemented with PVM is available in [7] where performance obtained on the Meiko CS2 machine are reported.

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5 Conclusion

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