About Definitions of Pseudospectra of Closed Operators in Banach Spaces

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Abstract

Our goal, in this paper, is to study some equivalent definitions of the pseudospectra of a closed linear operator. We start from two definitions, one with strict inequality and the other with non-strict inequality. The first definition makes the pseudospectrum open and the second makes it closed. We will show the subtle difference between these seemingly almost identical definitions and the relations between them.

key words: pseudospectra, closed linear operator, Banach and Hilbert space.

Introduction

In this paper, we consider two definitions for the pseudospectra of closed linear operators in a Banach space. The first one is the conventional definition of pseudospectra with non-strict inequality [9]. It makes the pseudospectrum a closed set and represents the natural extension of the definition of pseudospectra for matrices (finite dimension) to operators (infinite dimension). The second definition is obtained from the first one by taking a strict inequality instead of the non-strict inequality [4, 5]. This definition makes

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the pseudospectrum an open set. We call the corresponding set the *spectral value set* [5].

We will give some equivalent definitions for both the spectral value set (open) and the pseudospectrum (closed). The question arises naturally whether the pseudospectrum is the closure of the spectral value set. The answer is difficult, in the case of a Banach space with infinite dimension, because we do not know if the norm of the resolvent of a closed operator can or cannot be constant in an open set of the resolvent set: this is an open theoretical question in functional analysis (see Harrabi [6, 7]).

Let us first give the notations which will be used in the paper.

Notations

- X is real or complex Banach space (i.e., a complete normed vector space). $\| . \|$ is the norm on X.
- $\mathcal{H}$ is a Hilbert space (i.e., Banach space with a norm generated by an inner product: $\|x\| = (x, x)^{1/2}$).
- T is a closed linear operator on X (resp., on $\mathcal{H}$).
- $D(T) = \{x \in X : \; Tx \in X\}$ is the domain of definition of T.
- X^*, the *adjoint* space of X, is the space of all bounded semilinear forms on X.
- $\mathcal{C}(X)$ (resp., $\mathcal{C}(\mathcal{H})$) is the space of all closed linear operators in X (resp., in $\mathcal{H}$).
- $\mathcal{L}(X)$ (resp., $\mathcal{L}(\mathcal{H})$) is the space of all bounded linear operators on X (resp., on $\mathcal{H}$) with domain equal to X (resp., $\mathcal{H}$).
- the resolvent set of T is $\rho(T) = \{z \in \mathbb{C} : \; (zI - T)^{-1} \in \mathcal{L}(X)\}$ where I is the identity operator in X.
- the spectrum $\sigma(T)$ of T is the complementary set in $\mathbb{C}$ of $\rho(T)$.
- $\sigma_p(T)$ denotes the point spectrum of T (see Chatelin [3, pp. 96-97]).
- $cl(S)$ is the closure of the set S.

The reader can find in Chatelin [3] more details about linear operators in Banach spaces.

1 Definitions of the pseudospectrum and of the spectral value set of a closed linear operators

Let T be a closed linear operator on a Banach space X, and let $\epsilon > 0$. We give the following definitions.

Definition 1 *The ϵ-pseudospectrum of T is (see [2, 9])*

$$\sigma_\epsilon(T) \quad = \quad \left\{ z \in \rho(T) : \ \|(zI - T)^{-1}\| \geq \frac{1}{\epsilon} \right\} \cup \sigma(T). \tag{1}$$

The ϵ-spectral value set of T is (see [5])[1]

$$\omega_\epsilon(T) \quad = \quad \left\{ z \in \mathbb{C} : \ \|(zI - T)^{-1}\| > \frac{1}{\epsilon} \right\} \cup \sigma(T). \tag{2}$$

The ϵ-pseudospectrum is a closed set for all $\epsilon \geq 0$, and $(\sigma_\epsilon(T))_{\epsilon>0}$ is a nested closed family with $\lim_{\epsilon\to 0^+} \sigma_\epsilon(T) = \sigma(T)$. In contrast, the ϵ-spectral value set is an open set for all $\epsilon > 0$, and $(\omega_\epsilon(T))_{\epsilon>0}$ is a nested open family with $\lim_{\epsilon\to 0^+} \omega_\epsilon(T) = \sigma(T)$.

Next, we define the ϵ-approximate point spectrum of a closed linear operator T, which is closely related to the spectral value set and the pseudospectrum.

Definition 2 *The ϵ-approximate point spectrum of T is (see [8])*

$$\Sigma_\epsilon(T) = \{ z \in \mathbb{C} : \ \|(zI - T)x\| < \epsilon, \ \textit{for some } x \in D(T), \|x\| = 1 \}. \tag{3}$$

The ϵ-approximate point spectrum of T is an open set for all $\epsilon > 0$, and $(\Sigma_\epsilon(T))_{\epsilon>0}$ is a nested open family with $\lim_{\epsilon\to 0^+} \Sigma_\epsilon(T) = \sigma_a(T)$ where $\sigma_a(T)$ is the approximate point spectrum of T (see Chatelin [3, pp. 96-97]).
We note that the three sets $\sigma_\epsilon(T)$, $\omega_\epsilon(T)$ and $\Sigma_\epsilon(T)$ grow to fill the whole complex plane as $\epsilon \to \infty$.

[1] The definition given in [5] is slightly more general than (2) given above.

2 Equivalent definitions for the spectral value set

The following theorem is devoted to give some equivalent definitions of $\omega_\epsilon(T)$.

Theorem 1 *Let T be a closed linear operator in Banach space X, and $\epsilon > 0$. The set $\omega_\epsilon(T)$ defined by* (2) *has the three following equivalent definitions:*

$$\{z \in \mathbb{C} : \exists\, E \in \mathcal{L}(X),\ \|E\| < \epsilon \textit{ such that } z \in \sigma(T+E)\}, \tag{4}$$

$$\{z \in \mathbb{C} : \exists\, E \in \mathcal{L}(X),\ \|E\| < \epsilon \textit{ such that } z \in \sigma_p(T+E)\} \cup \sigma(T), \tag{5}$$

$$\Sigma_\epsilon(T) \cup \sigma(T). \tag{6}$$

Remark

As already mentioned, the spectral value set defined in [5] is slightly more general than the set defined by (4): the structured perturbations considered in [5] include unbounded perturbations of T. The reader is referred to Gallestey, Hinrichsen and Pritchard [5] for more information about the spectral value set. Note that the equivalence (5) $\Leftrightarrow$ (2) is already given in [5].

Proof

To prove this theorem, we proceed as follows:

$$\begin{aligned} (5) &\Rightarrow (4) \\ (4) &\Rightarrow (2) \\ (2) &\Rightarrow (6) \\ (6) &\Rightarrow (5) \end{aligned}$$

- It is easily seen that (5) $\Rightarrow$ (4), since $\sigma_p(T) \subseteq \sigma(T)$ for any closed linear operator T.
- To prove that (4) $\Rightarrow$ (2), we note first that $\sigma(T)$ is included in the set defined by (4).
  Let $z \in \rho(T)$ be such that $z \in \sigma(T+E)$ for some $E \in \mathcal{L}(X)$ and

$\|E\| < \epsilon$. As z is in the resolvent set, then $(zI - T)^{-1} \in \mathcal{L}(X)$. We can write

$$zI - T - E \;=\; (zI - T)\left(I - (zI - T)^{-1}E\right).$$

We suppose that $\|(zI - T)^{-1}\| \leq \frac{1}{\epsilon}$. This implies that $\|(zI - T)^{-1}E\| \leq \|(zI - T)^{-1}\|\,\|E\| < 1$, which implies in turn that $I - (zI - T)^{-1}E$ is invertible, and its inverse is in $\mathcal{L}(X)$ [3]. We get finally that $zI - T - E$ is invertible as the product of two invertible operators in $\mathcal{L}(X)$ [3]. This contradicts the fact that $z \in \sigma(T + E)$. Therefore $\|(zI - T)^{-1}\| > \frac{1}{\epsilon}$. This proves that (4) $\Rightarrow$ (2).

- To prove that (2) $\Rightarrow$ (6), we need to show that if $z \in \rho(T)$ such that $\|(zI - T)^{-1}\| > \frac{1}{\epsilon}$ then $z \in \Sigma_\epsilon(T)$. To show this, we use some equivalent definitions of the norm of the resolvent. In fact, for $z \in \rho(T)$, we have

$$\begin{aligned}
\|(zI - T)^{-1}\| &= \sup_{y \in X,\ y \neq 0} \frac{\|(zI - T)^{-1}y\|}{\|y\|}, \\
&= \sup_{x \in D(T),\ x \neq 0} \frac{\|x\|}{\|(zI - T)x\|}, \\
&= \sup_{x \in D(T),\ \|x\| = 1} \frac{1}{\|(zI - T)x\|}, \\
&= \frac{1}{\inf\limits_{x \in D(T),\ \|x\| = 1} \|(zI - T)x\|}.
\end{aligned}$$

Let $z \in \rho(T)$ be such that $\|(zI - T)^{-1}\| > \frac{1}{\epsilon}$. Then

$$\inf_{x \in D(T),\ \|x\| = 1} \|(zI - T)x\| \;<\; \epsilon.$$

The definition of the *infimum* implies that there exists some normalized $x \in D(T)$ such that $\|(zI - T)x\| < \epsilon$.

- To prove (6) $\Rightarrow$ (5), we just need to prove that all $z \in \Sigma_\epsilon(T)$ are in the set defined by (5), since this set contains $\sigma(T)$. Let $z \in \mathbb{C}$ be such that there exists some $x_0 \in D(T)$ satisfying $\|x_0\| = 1$, and $\|(zI - T)x_0\| < \epsilon$. The fact that $\|x_0\| = 1$ implies the existence of some $y^* \in X^*$ with $\|y^*\|_* = 1$ ($\|.\|_*$ is the dual norm) such that $< y^*, x_0 >= \|x_0\| = 1$. Define the operator E by $Ex = \overline{< y^*, x >}(zI - T)x_0$. It is easy to

see that E is a linear bounded operator, and $\|E\| < \epsilon$. The point z is in the point spectrum of $T+E$. In fact z is an eigenvalue of $T+E$ associated with x_0 as an eigenvector: $(T+E)x_0 = Tx_0 + Ex_0 = Tx_0 + < y^*, x_0 > (zI - T)x_0 = Tx_0 + (zI - T)x_0 = zx_0$, since $Ex_0 = < y^*, x_0 > (zI - T)x_0$ and $< y^*, x_0 > = \|x_0\| = 1$.
This proves that (6) $\Rightarrow$ (5), and finishes the proof of the theorem.

□

3 The closure of the spectral value set

The set $\omega_\epsilon(T)$ is an open set for all $\epsilon > 0$, in contrast with the set $\sigma_\epsilon(T)$ which is closed. The two sets $\omega_\epsilon(T)$ and $\sigma_\epsilon(T)$ are related as follows:

$$\sigma_\epsilon(T) \;\; = \;\; \omega_\epsilon(T) \cup \left\{ z \in \rho(T) : \;\; \|(zI - T)^{-1}\| = \frac{1}{\epsilon} \right\}.$$

We therefore ask the question: is the closure of $\omega_\epsilon(T)$ equal to $\sigma_\epsilon(T)$, or equivalently, is the border of $\omega_\epsilon(T)$ equal to $\{z \in \rho(T) : \;\; \|(zI - T)^{-1}\| = \frac{1}{\epsilon}\}$?
This brings us to the following question:
if a closed operator T is given in a Banach space X, can the norm of the resolvent of T be constant on an open set ?

We note that the answer is no in a Hilbert space, or in a Banach space of finite dimension (see [1, 6]).

The resolvent $R(z,T) = (zI - T)^{-1}$ is an holomorphic function in the resolvent set $\rho(T)$. In particular, the resolvent satisfies that, for all $z_0 \in \rho(T)$ and all $r > 0$ such that $\{z \in \mathbb{C} : \;\; |z - z_0| \leq r\} \subseteq \rho(T)$, then

$$\|(z_0 I - T)^{-1}\| \;\; \leq \;\; \sup_{0 \leq \theta \leq 2\pi} \|(r\, e^{i\theta} I - T)^{-1}\|.$$

This means that, if the norm of the resolvent is not constant in the neighborhood of z_0, then there exists a sequence $(z_n)_{n \in \mathbb{N} - \{0\}}$ such that $z_n \to z_0$ as $n \to \infty$ and $\|(z_n I - T)^{-1}\| > \|(z_0 I - T)^{-1}\|$ for all $n \in \mathbb{N} - \{0\}$.

We define the following hypothesis for T:

$$(H) \qquad \left\{ \begin{array}{l} \textit{there is no open set in } \rho(T) \textit{ on which} \\ \textit{the norm of the resolvent of } T \textit{ is constant.} \end{array} \right.$$

As shown above, the hypothesis (H) implies that, for all $z_0 \in \rho(T)$, there exists some sequence $(z_n)_{n\in\mathbb{N}-\{0\}} \in \rho(T)$ such that $z_n \to z_0$, and $\|(z_nI - T)^{-1}\| > \|(z_0I - T)^{-1}\|$.
On the other hand, if T is a closed operator such that for all $z \in \rho(T)$ there exists some sequence $(z_n)_{n\in\mathbb{N}} \in \rho(T)$ such that $z_n \to z$ sa $n \to \infty$ and $\|(z_nI - T)^{-1}\| > \|(zI - T)^{-1}\|$ for all $n \in \mathbb{N}$, then T satisfies the hypothesis (H) i.e., that the norm of the resolvent of T is not constant in an open set.

Under the hypothesis (H) for T, the following lemma gives the relationship between the ϵ-pseudospectrum $\sigma_\epsilon(T)$ and the ϵ-spectral value set $\omega_\epsilon(T)$.

Lemma 2 *Let T be a closed operator satisfying the hypothesis (H), and let $\epsilon > 0$. Then*

$$cl(\omega_\epsilon(T)) \quad = \quad \sigma_\epsilon(T) \tag{7}$$

where $cl(\omega_\epsilon(T))$ denotes the closure of $\omega_\epsilon(T)$.

Proof

- In fact, we have $cl(\omega_\epsilon(T)) \subseteq \sigma_\epsilon(T)$ since $\omega_\epsilon(T) \subseteq \sigma_\epsilon(T)$ and $\sigma_\epsilon(T)$ is closed.

- Reciprocally, if $z \in \sigma(T)$ or $z \in \rho(T)$ with $\|(zI - T)^{-1}\| > \frac{1}{\epsilon}$ then $z \in \omega_\epsilon(T)$ by definition of $\omega_\epsilon(T)$ set, then $z \in cl(\omega_\epsilon(T))$. If, $z \in \rho(T)$ and $\|(zI - T)^{-1}\| = \frac{1}{\epsilon}$ (i.e., $z \in \sigma_\epsilon(T) - \omega_\epsilon(T)$), then, by the hypothesis (H), there exists some sequence $(z_n)_{n\in\mathbb{N}} \in \rho(T)$ such that $\|(z_nI - T)^{-1}\| > \|(zI - T)^{-1}\| = \frac{1}{\epsilon}$ and $z_n \to z$. This implies that $z_n \in \omega_\epsilon(T)$ and $z_n \to z$. Consequently, this implies that $z \in cl(\omega_\epsilon(T))$, and shows that $cl(\omega_\epsilon(T)) = \sigma_\epsilon(T)$.

□

4 Equivalent definitions of the pseudospectrum

In the following theorem, we give some equivalent definitions of the pseudospectrum of an operator T satisfying the hypothesis (H).

Theorem 3 *Let T be a closed operator in a Banach space X satisfying the hypothesis (H). Let $\epsilon > 0$, then the ϵ-pseudospectrum $\sigma_\epsilon(T)$ defined by* (1) *has the five following equivalent definitions:*

$$\begin{aligned}
&\{z \in \mathbb{C} : \lim_{n\to\infty} \|(zI - T)x_n\| \leq \epsilon \textit{ where } (x_n)_{n\in\mathbb{N}} \in D(T), \\
&\qquad \textit{and } \|x_n\| = 1\} \cup \sigma(T), && (8)\\
&cl(\{z \in \mathbb{C} :\ \exists\ E \in \mathcal{L}(X),\ \|E\| \leq \epsilon \textit{ such that } z \in \sigma(T+E)\}), && (9)\\
&cl(\{z \in \mathbb{C} :\ \exists\ E \in \mathcal{L}(X),\ \|E\| \leq \epsilon \textit{ such that } z \in \sigma_p(T+E)\}) \cup \sigma(T), && (10)\\
&cl(\{z \in \mathbb{C} :\ \exists\ E \in \mathcal{L}(X),\ \|E\| < \epsilon \textit{ such that } z \in \sigma(T+E)\ \}), && (11)\\
&cl(\{z \in \mathbb{C} :\ \exists\ E \in \mathcal{L}(X),\ \|E\| < \epsilon \textit{ such that } z \in \sigma_p(T+E)\}) \cup \sigma(T), && (12)\\
&cl(\Sigma_\epsilon(T)) \cup \sigma(T). && (13)
\end{aligned}$$

The definitions (8), (9) and (13) were given in [6]. Definitions (10), (11) and (12) are new, but their proofs are similar to that of the other definitions and are easily deduced from the following two facts:

- The sets defined by (11), (12), and (13) are just the closure of $\omega_\epsilon(T)$ for all $\epsilon > 0$.

- It is easy to show that $\omega_\epsilon(T) \subset (10) \subseteq (9) \subseteq (8) = \sigma_\epsilon(T)$ (here, (8) denotes the set defined by (8), etc...).

We end this section with the two remarks:

- The closure appears in the definition (9) and (10), because we do not know if these sets are open or closed, or neither open nor closed.

- As shown above, the hypothesis that the norm of the resolvent is not constant in an open domain is used to show the equality between the closure of $\omega_\epsilon(T)$ and $\sigma_\epsilon(T)$.

Conclusion

In this paper, we have compared two possible definitions of pseudospectra of a closed linear operator in Banach space with non-strict inequality (1) and strict inequality (2). Although the two definitions (1) and (2) seem quite similar, it has been shown that, in a Banach space, only the definition (2) lends itself to useful equivalent formulations for any closed operator T. With definition (1) on the contrary, similar equivalent formulations hold, to the best of our knowledge, only for these closed operators T which satisfy (H). Whether any closed operator T satisfies (H) remains an open question. In view of the above study, it seems that definition (2) should be preferred for pseudospectra in Banach spaces because of the elegant treatment it allows.

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