About Singularities in Inexact Computing

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Abstract

This report details the technical content of the new theory of **Homotopic Deviation** presented in [1, 2]. It elaborates on new insight presented in [9, 11]. It contains all the mathematical background necessary about the singularities which may occur in the homotopic unfolding of a matrix A by means of a deviation matrix E. The definitions of Exact and Inexact Computing are given in [1, 2]. The theory of Homotopic Deviation studies the spectral field of the matrix family $A(t) = A + tE$ for $t = re^{i\theta}, 0 \leq \theta \leq 2\pi$ and $r \in \mathbb{R}^+$. Special attention is given to the case $r \longrightarrow \infty$. The classical theory of Homotopic Pertubation corresponds to $t \in [0,1] \subset \mathbb{R}$.

Keywords: homotopy, deviation, perturbation, singularity, orbit, ray, spectral radius, resolvent, Neumann series, Inexact Computing, backward error, critical point, kernel point, quadratic eigenproblem, rank deficiency, Jordan structure.

1 The problem

A and E are given matrices in $\mathbb{C}^{n\times n}$, t is a complex parameter. We set $A(t) = A + tE$, such that $A(0) = A$. The family $A(t)$ realises an *homotopic deviation* of A by the deviation matrix E. We use the word *deviation* for E when no metric assumption such as $||E||$ small enough is made. Otherwise, we say that E is a *perturbation* of A.

We relate the properties of $A(t)$ to those of A by means of:

$$A + tE - zI = (I + tE(A - zI)^{-1})(A - zI)$$

where $z \in \mathbb{C}$ is not a eigenvalue of A, that is $z \notin \sigma(A)$, the spectrum of A, the set of eigenvalues.

Let μ_z denote an eigenvalue of $F_z = -E(A - zI)^{-1}$ for $z \notin \sigma(A)$.

Lemma 1.1 *The point $z \notin \sigma(A)$ is an eigenvalue of $A+tE$ iff there exists an eigenvalue $\mu_z \neq 0$ of F_z such that $t\mu_z = +1$.*

Proof Let $y \neq 0$ be an eigenvector of F_z associated with $\mu_z \neq 0$.

$$-E(A - zI)^{-1}y = \mu_z y, \quad y \neq 0 \qquad \Leftrightarrow \qquad \mu_z(A - zI)u = -Eu$$

for $u = (A - zI)^{-1}y \neq 0 \quad \Leftrightarrow \quad (A + \frac{1}{\mu_z}E)u = zu$.

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This lemma shows that when $\rho(F_z) = 0$ for a certain z, then this z cannot be an eigenvalue of $A + tE$, because t is not defined. So any z in $\mathbb{C} - \sigma(A)$ is an eigenvalue of at least one matrix $A(t)$, as long as $0 < \rho(F_z) < \infty$.

In parallel to this interpretation, any z such that $0 < \rho(F_z) < \infty$ can receive the interpretation of being a regular point of $A(t) - zI$: $A + tE - zI$ is invertible for any t such that $t \neq \frac{1}{\mu_z}$ for any eigenvalue $\mu_z \neq 0$ of F_z.

The two interpretations (z eigenvalue of $A + tE$, versus $A + tE - zI$ invertible) hold in parallel (for different values of t !) for any z in $\mathbb{C} - \sigma(A)$ such that $\rho(F_z) > 0$.

Remark:
1) $z \in \sigma(A) \quad \Rightarrow \quad \rho(E(A - zI)^{-1}) = \infty$ for an arbitrary $E : \lim_{z \to \lambda} \rho(E(A - zI)^{-1}) = \infty$ for $\lambda \in \sigma(A)$, see theorem 2.2.10, p.59 in [4].
Of course, when E is a spectral projection for A associated with λ, then the limit is finite.
2) The fact that the eigenvalues μ_z are in $\mathbb{C}$ forces the parameter t to be complex to satisfy $t\mu_z = 1$.

1.1 The set of critical points $\kappa(A, E)$

We introduce the

Definition 1.1 *A point $z \in \mathbb{C} - \sigma(A)$ such that $\rho(E(A - zI)^{-1}) = 0$ is a critical point associated with (A, E). The set of critical points is denoted $\kappa(A, E)$.*

The question whether $\kappa(A, E)$ is empty or not will be studied later (Sections 3 and 6). We set $\Sigma(A, E) = \sigma(A) \cup \kappa(A, E)$. We remark that $\kappa(A, E)$ and $\sigma(A)$ are disjoint sets by definition.

1.2 Homotopic backward error

Let $\mu_{iz}, i = 1, \cdots, n$ denote the eigenvalues of F_z. The same z is an eigenvalue of the n (not necessarily distinct) matrices $A + t_i E$, with $t_i = \frac{1}{\mu_{iz}}, i = 1, \cdots, n$, if all $\mu_{iz} \neq 0$. Amongst these n matrices, one at least is closest to A which is associated with t_* of minimum modulus:

$$|t_*| = min_i |t_i| = \frac{1}{max_i |\mu_{iz}|}$$

$A(t_*)$ is the matrix of the family $A(t)$ closest to A, if the distance is measured by the modulus of t.

Definition 1.2 *The homotopic distance to spectral singularity of z is*

$$|t_*| = min(|t|, \quad z \, is \, an \, eigenvalue \, of \, A + tE)$$

It is also called **homotopic backward error** $BE(z)$. One has:

$$|t_*| = BE(z) = \frac{1}{\rho(E(A-zI)^{-1})}$$

where $|t_*|$ measures by how much z, which is an eigenvalue of $A(t_*) = A + t_*E$, fails to be an *eigenvalue* of A [10].

2 Properties of the resolvent matrix $R(t, z)$

By definition [4, 5], the matrix

$$R(t, z) = (A + tE - zI)^{-1}$$

is called the *resolvent matrix*, because it allows to resolve the associated linear system. Formally, one can write, for $z \notin \sigma(A)$:

$$R(t, z) = (A + tE - zI)^{-1} = (A - zI)^{-1}(I - tF_z)^{-1}$$

From the point of view of solving the system

$$(A + tE - zI)x(t, z) = y, \tag{2.1}$$

three questions about $R(t, z)$ are in order:

i) existence ($\Rightarrow$ (2.1) has a unique solution $x(t, z)$)

ii) non existence ($\Rightarrow$ z is an eigenvalue of $A(t)$)

iii) analyticity of the map $t \to R(t, z)$, for z fixed, whenever it is defined ($\Rightarrow$ $x(t, z)$ can be computed from its series expansion in t whenever convergent).

i) We have seen that i) is satisfied for $z \in \mathbb{C} - \sigma(A)$ and $t \neq \frac{1}{\mu_z}$, $\mu_z \neq 0$. It is satisfied, in particular, for any $t \in \mathbb{C}$ when $z \in \kappa(A, E) \neq \emptyset$ because, then, $|t| < \infty$ ensures $t \neq \frac{1}{\mu_z}$ with $\mu_z = 0$.

ii) Secondly ii) is satisfied for $t = 0$ iff $z \in \sigma(A)$, and for $t \neq 0$ such that $t = \frac{1}{\mu_z}$ with $\mu_z \neq 0$, for $z \in \mathbb{C} - \Sigma(A, E)$. Note again that if $z \in \kappa(A, E) \neq \emptyset$, then $A(t) - zI$ is regular: z cannot be an eigenvalue of $A(t)$ for any $t \in \mathbb{C}$.

iii) Thirdly, for $z \in \mathbb{C} - \Sigma(A, E)$, $R(t, z)$ is analytic for $|t| < \frac{1}{\rho(F_z)}$. This follows from the Neumann series expansion

$$(I - tF_z)^{-1} = I + \Sigma_1^{\infty} t^k F_z^k, \tag{2.2}$$

which converges for $|t|\rho(F_z) < 1$.

However, for $z \in \kappa(A, E) \neq \emptyset$, there is a vast simplification: the above asymptotic expansion in t, (2.2), becomes a *polynomial* in t.
Indeed, $\rho(F_z) = 0 \Longleftrightarrow F_z$ is nilpotent. There exists an integer $a \in \mathbb{N}$ such that $F_z^a = 0$ and $F_z^{a-1} \neq 0$. F_z is a *defective* matrix, rather than being semi-simple

(i.e. diagonalizable). We shall prove later that a is determined by the structure of (A, E), and may depend on z.
Therefore (2.2) becomes

$$(I - tF_z)^{-1} = I + tF_z + \cdots + t^{a-1}F_z^{a-1} \tag{2.3}$$

This shows that $R(t, z)$ is a polynomial in t of degree $a - 1$: there is no condition on t for its existence. In other words, at any critical point $z \in \kappa(A, E)$, the resolvent matrix $R(t, z)$ has an algebraic representation in t, for any $t \in \mathbb{C}$ (as opposed to analytic for $|t| < \frac{1}{\rho(F_z)}$).

3 The critical set $\kappa(A, E)$

It is known that $\lim_{|z| \longrightarrow \infty} ||(A - zI)^{-1}|| = 0$ (Corollary 2.2.5, p.56, [4]). Therefore $\lim_{|z| \longrightarrow \infty} \rho(E(A - zI)^{-1}) \quad \leq \quad ||E|| \lim_{|z| \longrightarrow \infty} ||(A - zI)^{-1}|| = 0.$

Do there exist points z at finite distance such that $\rho(F_z) = 0$?

The answer is yes when **E is of rank** 1 such that $E^2 \neq 0$, and under the assumption of no algebraic simplification for $(A - zI)^{-1}$, $z \notin \sigma(A)$. By this, we mean that the rational form $(A - zI)^{-1}$ is the quotient of a polynomial of degree $n - 1$ by that of degree n, for $z \in \mathbb{C} - \sigma(A)$ with no common root.

Lemma 3.1 *Let $E = \alpha\beta^H$, with $\alpha, \beta \in \mathbb{C}^n$ such that $\beta^H\alpha \neq 0$. Under the assumption of no algebraic simplification for $(A - zI)^{-1}$, the equation $\beta^H(A - zI)^{-1}\alpha = 0$ leads to a polynomial equation of degree $n - 1$ in $z \in \mathbb{C} - \sigma(A)$.*

Proof We write $(A - zI)^{-1}$ under rational form. Its ij^{th} element is of the form $\frac{\alpha_{ij}(z)}{det(A - zI)}$, where $\psi(z) = det(A - zI)$ is the characteristic polynomial of A of degree n, and where $\alpha_{ij}(z)$ for $i \neq j$ (resp. $\alpha_{ii}(z)$) are polynomials of degree $\leq n - 2$ (resp. $n - 1$). Therefore the numerator of $\beta^H(A - zI)^{-1}\alpha$ is a polynomial π of degree $n - 1$ with leading coefficient $\beta^H\alpha \neq 0$ (modulo the sign). The condition $\psi(z) = det(A - zI) \neq 0$ for z in $\mathbb{C} - \sigma(A)$ implies that the equation $\beta^H(A - zI)^{-1}\alpha = 0$ is satisfied whenever the numerator satisfies $\pi(z) = 0$ for $z \in \mathbb{C} - \sigma(A)$.

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Remark 3.1:

1) The condition $\beta^H\alpha \neq 0$ implies that E is semi-simple: it is diagonalizable with zero as a semi-simple eigenvalue of multiplicity $n - 1$, and $\beta^H\alpha$ as a simple nonzero eigenvalue. As a corollary, $E^2 \neq 0$.

2) When $\beta^H\alpha = 0$, E is defective: $E^2 = 0$, and the numerator $\pi(z)$ is a polynomial of degree $< n - 1$. An important example is provided by the **incomplete Arnoldi decomposition** of size $k < n$: $E = v_{k+1}v_k^H$ is defective ($v_k^H v_{k+1} = 0$) [15, 16].

Proposition 3.2 *The conditions E of rank 1, $E^2 \neq 0$ and no simplification for $(A - zI)^{-1}$, imply that $\kappa(A, E)$ consists of $n - 1$ points in $\mathbb{C} - \sigma(A)$. At such critical points z, F_z is nilpotent such that $F_z^2 = 0$.*

Proof The critical points are the roots of π which are not eigenvalues of A. See the Remark 3.2 below. Now, $F_z = \alpha\beta^H(A-zI)^{-1}$ satisfies $F_z^2 = \alpha\beta^H(A-zI)^{-1}\alpha\beta^H(A-zI)^{-1}$, which shows that $F_z^2 = 0$ and $F_z \neq 0 \Longleftrightarrow \beta^H(A-zI)^{-1}\alpha = 0$.

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As an example we consider the companion matrix A associated with the polynomial x^n and the deviation matrix $E = -ae_n^T$, with $a = (a_0, ..., a_{n-1})^T$. The matrix $A+E$ is the companion matrix associated with the polynomial $p(x) = x^n + \Sigma_{i=0}^{n-1} a_i x^i = x^n + q(x)$.

Lemma 3.3 *The critical points of* (A, E) *are the zeros of* $q(z)$ *in* $\mathbb{C} - \sigma(A)$, *for the example above.*

Proof We have

$$A = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

of order n.

$$(A-zI)^{-1} = \tfrac{-1}{z^n} \begin{pmatrix} z^{n-1} & & & 0 \\ z^{n-2} & \ddots & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & z^{n-2} & z^{n-1} \end{pmatrix},$$

and $e_n^T(A-zI)^{-1}a = \dfrac{-1}{z^n}(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}) = \dfrac{q(z)}{-z^n}$.

If $a_0 \neq 0, q(z)$ does not admit the root $z = 0$. Therefore the critical points are the $n-1$ roots of $q(z)$. Note that $q(z)$ is of degree $n-1$ iff $a_{n-1} = e_n^T a \neq 0$, that is $E^2 \neq 0$.

If $a_0 = 0$, as well as $a_1 = \cdots = a_{l-1} = 0$, then one should simplify:

$$-\frac{q(z)}{z^n} = -\frac{a_l + a_{l+1}z + \cdots + a_{n-1}z^{n-l-1}}{z^{n-l}}$$

There are exactly $n-l-1$ roots of $q(z)$ which are not eigenvalues of A, that is which are nonzero.

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Remark 3.2: Under the assumption of Proposition 3.2, there are exactly $n-1$ critical points because the polynomials $\pi(z)$ and $\psi(z) = det(A-zI)$ have no common root. If they have l common roots (counting multiplicity) then there are $n-l-1$ critical points (after algebraic simplification) which are not eigenvalues of A.

What can be said if $rank(E) = 2$?

We saw that when E is of rank 1, $E = \alpha\beta^H$, then the eigenvalue of $F_z = -E(A-zI)^{-1}$ which is not necessarily 0 is the scalar $\mu_z = -\beta^H(A-zI)^{-1}\alpha$.

Therefore $F_z^2 = \alpha\beta^H(A-zI)^{-1}\alpha\beta^H(A-zI)^{-1} = \mu_z F_z = 0$ at the critical points. The degree of nilpotency of F_z depends on its rank, which is 1 here, and not on z.

Now let us set $E = UV^H$ where $U, V \in \mathbb{C}^{n\times 2}$ and V^HU is a 2×2 regular matrix. The nonzero eigenvalues of F_z are those of 2×2 matrix: $M_z = -V^H(A-zI)^{-1}U$. And
$F_z^3 = -UV^H(A-zI)^{-1}UV^H(A-zI)^{-1}UV^H(A-zI)^{-1} = -UM_z^2V^H(A-zI)^{-1}$.

It is clear that $F_z^3 = 0 \iff M_z^2 = 0 \iff M_z$ is a nilpotent matrix $\iff trM_z = 0$ and $detM_z = 0$.
We set:

$$M_z = \frac{1}{det(A-zI)}\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z)\end{pmatrix}, \quad trM_z = \frac{[a(z)+d(z)]}{det(A-zI)}$$

$$detM_z = \frac{[a(z)d(z)-b(z)c(z)]}{det^2(A-zI)}$$

where a, b, c, d are polynomials in z of degree $\leq n-1$.

Imposing on M_z to be nilpotent yields a set of two rational equations which in general will have no common root in $\mathbb{C}$. So, in general, $\kappa(A, E)$ is empty for E of rank 2.

We can go further in the analysis by exploiting the structure of F_z and M_z with $E = UV^H$, where $U, V \in \mathbb{C}^{n\times\widehat{r}}$. We assume that rank U = rank $V = \widehat{r} < n$. Therefore 0 is a semi-simple eigenvalue for E (resp. F_z) of multiplicity $\widehat{g} = n - \widehat{r}$ iff the $\widehat{r}\times\widehat{r}$ matrix $C = V^HU$ (resp. $M_z = -V^H(A-zI)^{-1}U$) is regular (rank $\widehat{r}$). The associated eigenspace is Ker$E$ (resp. KerF_z) of dimension $\widehat{g}$. When z varies in $\mathbb{C}-\sigma(A)$, the geometric multiplicity of $0 \in \sigma(F_z)$ remains constant at the value $\widehat{g} = n - \widehat{r}$.
For almost all z in $\mathbb{C}-\sigma(A)$, $0 \in \sigma(F_z)$ is semi-simple. However it is possible that its algebraic multiplicity increases for a finite number of points z in $\mathbb{C}-\sigma(A)$. Such points will be characterized in Section 6 (Propositions 6.2 and 6.5). For the time being, we analyse, under the hypothesis $C = V^HU$ regular, what can happen at a point z such that $\mu_z = 0$, where μ_z is an eigenvalue of M_z. Note that the eigenvalues of F_z which are not necessarily zero when $\widehat{r} < n$ are exactly the eigenvalues of $M_z = C^{-1}V^HF_zU$.

Lemma 3.4 *We assume $C = V^HU$ regular. For any z in $\mathbb{C}-\sigma(A)$ to be such that $0 \in \sigma(M_z)$ has geometric multiplicity $\gamma \leq \widehat{r}$, it is necessary that $\gamma \leq \widehat{g}$.*

Proof We use the fact that $KerF_z = (A-zI)V^\perp = (A-zI)KerE$ has constant dimension $\widehat{g}$ as z varies in $\mathbb{C}-\sigma(A)$. Therefore $0 \in \sigma(M_z)$ implies that $0 \in \sigma(F_z)$ becomes defective with exactly $\widehat{g}$ Jordan blocks, one at least of size ≥ 2. In the process, the γ eigenvectors for $0 \in \sigma(M_z)$ should be matched with at most $\widehat{g}$ eigenvectors for $0 \in \sigma(F_z)$. That is $\gamma \leq \widehat{g}$, hence $\gamma \leq min(\widehat{r}, n-\widehat{r})$.

△

As a direct consequence, we get the

Proposition 3.5 *Under the assumption $C = V^HU$ regular, a critical point z implies the nilpotency of F_z with a degree of nilpotency δ between 2 and $\widehat{r}+1$.*

Proof By definition, $\rho(M_z) = \rho(F_z) = 0$ charaterises a critical point z. When $\sigma(F_z) = \{0\}$, the Jordan structure for F_z which yields the largest ascent l corresponds to the maximum $l_{max} = n - \widehat{g} - 1 = \widehat{r} + 1$. Therefore $F_z^{\widehat{r}+1} = 0$ and $F_z^{\widehat{r}} \neq 0$. This unique (nontrivial) Jordan block for F_z induces a unique Jordan block for M_z of size $\widehat{r}$, and $M_z^{\widehat{r}} = 0$. The value $\widehat{r} + 1$ (resp. $\widehat{r}$) corresponds to the *maximum* degree of nilpotency achievable for F_z (resp. M_z).
Let us look now at the smallest ascent for F_z.
When $\widehat{r} \leq \widehat{g}$, that is $2\widehat{r} \leq n$, M_z can be semi-simple if $\gamma = \widehat{r}$ (which implies $M_z = 0$). The Jordan structure of F_z consists of $\widehat{r}$ Jordan blocks of size 2, plus $n - 2\widehat{r}$ trivial blocks of size 1. Therefore $F_z^2 = 0$.
When $\widehat{r} > \widehat{g}$, $\gamma = \widehat{g} < \widehat{r}$, therefore M_z cannot be semi-simple at a critical point. The lower bound $\delta = 2$ is achievable only if $\widehat{r} \leq \widehat{g}$. For $n \geq 3$ and $\widehat{r} > \widehat{g}$, the minimal value for δ is at least 3.

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Proposition 3.5 covers the case $\widehat{r} = 1$ already treated in Proposition 3.2. When $\widehat{r} = 1$, the maximal and minimal degree of nilpotency coalesce. This need not be the case $\widehat{r} > 1$. We shall come back to this question in Section 6.

We complete the analysis by a look at the case $\widehat{r} = n$. Neither F_z nor M_z accept the value 0 as an eigenvalue. This remark will be useful for Section 6.

The last remark concerns the modification, which occurs when E is rank deficient ($\widehat{r} < n$), to the question of the analycity of $t \longrightarrow R(t,z)$. By the Sherman-Morrison formula, we get formally: $R(t,z) = (A - zI + tUV^H)^{-1} = (A - zI)^{-1}[I - tU(I_{\widehat{r}} - tM_z)^{-1}V^H(A - zI)^{-1}]$. Therefore $(I_n - tF_z)^{-1}$ of order n is replaced by $(I_{\widehat{r}} - tM_z)^{-1}$ of order $\widehat{r}$. However $\rho(F_z) = \rho(M_z)$ is unchanged.

4 The map $\varphi : z \longrightarrow \rho(F_z)$

The map $\varphi : z \longrightarrow \rho(F_z)$ provides a useful graphical tool to analyse the spectral field of $A(t)$ under homotopic deviation, when rank $E = 1$, as will be shown in Section 5.

4.1 Properties of φ

We list the properties which explain the fundamental role of φ [2, 3]:

a) $z \longrightarrow \rho(F_z)$ is upper semi-continuous in $z \in \mathbb{C} - \sigma(A)$, by Proposition 2.2.7, p.56, [4], and by the continuity of $z \longrightarrow E(A - zI)^{-1}$ in $\mathbb{C} - \sigma(A)$.

b) φ is subharmonic in $\mathbb{C} - \sigma(A)$, because of the Cauchy integral formula [2]. It satisfies the maximum principle, it cannot have a local maximum (different from $+\infty$) unless it is locally constant. Because $\lim_{|z| \longrightarrow \infty} \rho(F_z) = 0$, if ρ is locally constant, then it is everywhere zero on $\mathbb{C} - \sigma(A)$, [2].

c) φ can have local minima.

This set of properties explain why φ can be used to get a **global picture** of $A(t)$ as well as a local one. In particular, this establishes that the level curves $z \longrightarrow \rho(F_z) =$ constant are closed curves. They enclose either eigenvalues of A

(value $\rho = +\infty$) or local minima.

The singularities of φ consists of the points z where $\rho(F_z)$ takes the special value 0 or $+\infty$:

a) value $+\infty$: ρ is not defined for $z \in \sigma(A)$, unless E is a spectral projection for A associated with certain eigenvalues of A.

b) value 0: it can be shown that the critical points where $\rho = 0$ are singular points for ρ [3].

Outside the singular set $\Sigma(A, E) = \sigma(A) \cup \kappa(A, E)$, the behaviour of φ is regular: it takes values in $]0, \infty[$.

4.2 An analogous map $\psi : z \longrightarrow ||(A - zI)^{-1}||$

During the decade of the 90s, special attention has be given to the map $\psi : z \longrightarrow ||(A - zI)^{-1}||$, defined for $z \in \mathbb{C} - \sigma(A)$. The reason is that the *normwise* backward error, for $z \notin \sigma(A)$,

$$BE(z) = \frac{1}{||(A - zI)^{-1}||}$$

measures by how much z fails to be an eigenvalue of A [10].

The above backward error is called normwise because z is an eigenvalue of $A + \Delta A$, where the perturbation ΔA satisfies the norm condition $||\Delta A|| \leq 1$. The bound

$$\rho(E(A - zI)^{-1}) \leq ||E|| \, ||(A - zI)^{-1}|| \leq ||(A - zI)^{-1}||$$

valid for all E such that $||E|| \leq 1$ is useful to compare homotopic and normwise backward errors.

Like φ, the map ψ is a subharmonic function in $\mathbb{C} - \sigma(A)$ [2]. Its level curve $z \longrightarrow \psi(z) = \frac{1}{r}$ is the border of the normwise r-pseudospectrum [10, 14].

There is a significant difference between the two maps, however. Because of the property of $||.||$, ψ cannot be zero at finite distance. It can only exhibit local minima with positive value [1].

5 Unfolding the spectral field of $A(t)$, for $t \in \mathbb{C}$

We want to represent the spectral field of the family $A(t)$:

$$t \in \mathbb{C} \longrightarrow z \in \mathbb{C} \text{ is an eigenvalue of } A + tE = A(t)$$

We know that the range of this map is $\mathbb{C} - \kappa(A, E)$.
To easily represent the map $\mathbb{C} \longrightarrow \mathbb{C}$, we decompose it into two families of curves, defined by a *real* parameter:

i) the *singular rays* $\Lambda(\theta)$, corresponding to $t = re^{i\theta}$ for a fixed θ in $[0, 2\pi[$, the parameter r varying in $[0, +\infty[$,

ii) the *singular orbits* $\Sigma(r)$, corresponding to $t = re^{i\theta}$ for a fixed $r > 0$ in $]0, +\infty[$, the parameter θ varying in $[0, 2\pi]$.

Because $t\mu_z = 1$ when z is an eigenvalue of $A + tE$ such that $\mu_z \neq 0$, and because $|z| \longrightarrow \infty \Longrightarrow \rho \longrightarrow 0 \Longrightarrow |t| = r \longrightarrow \infty$, we conclude that the singular rays $\Lambda(\theta)$ can extend to infinity, starting from the eigenvalues of A for $r = 0$. The question of the behaviour of the singular rays as $|t| = r \longrightarrow \infty$ will be specifically studied in Section 6.
On the other hand, the singular orbits $\Sigma(r)$ consist of one or several curves connected to the original eigenvalues by the spectral rays.
As $r \longrightarrow 0$, each distinct eigenvalue is enclosed by one closed curve which belongs to $\Sigma(r)$.

We remark that for any z which is not a critical point, there exists at least one t such that z is an eigenvalue of $A + tE$. If E is full rank, so is $F_z = -E(A - zI)^{-1}$ for $z \in \mathbb{C} - \sigma(A)$. Therefore such a z is at the intersection of n pairs of curves $\{\Lambda(\theta_j), \Sigma(r_j)\}$ associated with any $t_j = r_j e^{i\theta_j}$, such that $t_j \mu_{jz} = 1$, $0 \neq \mu_{jz} \in \sigma(F_z)$ for $j = 1, \cdots, n$. Such a z is therefore at the intersection of the two curves $\Lambda(\theta)$ and $\Sigma(r)$ associated with any such $t = re^{i\theta}$.

To proceed with the description of $\Lambda(\theta)$ and $\Sigma(r)$, we distinguish whether rank $E = 1$ or not.

5.1 rank $E = 1$

We already know that rank $E = 1$ implies in general the existence of a critical set which is non empty: $\kappa(A, E) \neq \emptyset$. In addition, rank $E = 1$ implies rank $F_z = 1$: there is **at most one nonzero eigenvalue** μ_z **to** F_z such that $|\mu_z| = \rho(F_z)$. This will have very important consequences, which are given below as propositions.

Proposition 5.1 *rank $E = 1$ implies that $\Sigma(1) = \Gamma$, where Γ is defined as $\{z; \rho(E(A - zI)^{-1}) = 1\}$.*

Proof $\Sigma(1)$ is the orbit $\{z;\ z$ is an eigenvalue of $A + tE$, with $|t| = 1\}$. By assumption, F_z is rank 1, it has one eigenvalue μ_z which is not necessarily 0.
$z \in \Sigma(1) \Longleftrightarrow |\mu_z| = 1 \Longleftrightarrow \rho(E(A - zI)^{-1}) = 1 \Longleftrightarrow z \in \Gamma$.

△

When E is rank 1, the orbits $\Sigma(r)$ and the level curves $\{z,\ \rho(F_z) = \frac{1}{r}\}$ are identical. Therefore the singular rays and orbits realise a network of curves on the surface $z \longrightarrow \rho(F_z)$.

Proposition 5.2 *It takes a multiple $2m\pi$, $m \geq 1$ of 2π to describe an orbit $\Sigma(r)$ which encloses eigenvalues. The integer m is equal to the total algebraic multiplicity of the enclosed eigenvalues.*

Proof Consequence of the Cauchy integral formula.

△

When rank $E = 1$ with $E^2 \neq 0$, the critical set $\kappa(A, E)$ is in general non empty (when $n > 1$). The critical points are at finite distance in $\mathbb{C}$, however they are seen by homotopic deviation as being at *infinite* distance (measured as the homotopic distance $|t|$, or homotopic backward error). As $r \longrightarrow \infty$, certain closed curves of $\Sigma(r)$ enclose the critical points more and more closely.

5.2 rank $E > 1$

Neither Proposition 5.1 nor Proposition 5.2 remain valid for rank $E > 1$. They are modified according to the proposition below.

Proposition 5.3 *rank $E > 1$ implies that there may exist points z inside Γ which lie on $\Sigma(1)$.*

Proof Now F_z has at least two non necessarily zero eigenvalues. Therefore a z such that $\rho(F_z) > 1$ (z inside Γ) may be an eigenvalue of $A + tE$, with $|t| = 1 < \rho(F_z)$.

△

This proposition has two important consequences for the orbits $\Sigma(r)$ and the level curves of ρ:

1) The orbits $\Sigma(r)$ differ from the level curves $z \longrightarrow \rho(F_z) = \frac{1}{r}$. They do not enjoy anymore the topological simplicity of being the level curves of a subharmonic function in $\mathbb{C} - \sigma(A)$.

2) Symmetrically, the level curves for ρ lose the periodicity based on 2π which was induced by the orbits $\Sigma(r)$.

Definition 5.1 *We call synchronicity the coincidence at z of two eigenvalues on the same orbit $\Sigma(r)$ corresponding to two different arguments $\theta_1 \neq \theta_2 \,(mod 2\pi)$.*

Consider $t = re^{i\theta}$, $A(t) = A + re^{i\theta}E$, and suppose, for the sake of simplicity, that rank $E = 2$. Let $\mu_1(\theta)$ and $\mu_2(\theta)$ the two eigenvalues (not necessarily zero) of $tF_z = re^{i\theta}F_z = -re^{i\theta}E(A - zI)^{-1}$. We suppose that they are labeled such that $|\mu_1(\theta)| \geq |\mu_2(\theta)|$. The equality $|\mu_1(\theta)| = |\mu_2(\theta)|$ for $\theta_1 \neq \theta_2 \,(mod 2\pi)$ is possible because E is of rank 2. This implies that $A(t_1)$ and $A(t_2)$ with $t_1 = re^{i\theta_1}$ and $t_2 = re^{i\theta_2}$ share the same eigenvalue z.

Striking examples of these phenomena are given in [2]: see the example **Venice** for $n = 8$, and the example **One** for $n = 8$ and $n = 29$. See also [1].

6 Behaviour of the singular rays $\Lambda(\theta)$ as $r \longrightarrow \infty$

We look more closely, in this section, at the question whether some eigenvalues of $A(t)$ can remain at finite distance when $|t| = r \longrightarrow \infty$, or whether they all go to infinity [9].

From the identity

$$A + tE = t(\frac{1}{t}A + E) = \frac{1}{s}(E + sA),\; s = \frac{1}{t}$$

valid for any $t \neq 0$ (or $|s|$ bounded), we may study the behaviour of the eigenvalues $\lambda(t)$ of $A(t) = A + tE$ as $r = |t| \longrightarrow \infty$ by means of that of $\dfrac{\nu(s)}{s}$, where

$\nu(s)$ is any eigenvalue of $E(s) = E + sA$, as $|s| \longrightarrow 0$. Clearly $E(s) \longrightarrow E$ as $s \longrightarrow 0$. Let ν (resp. $\nu(s)$) denote an eigenvalue of E (resp. $E(s)$).

In full generality, the asymptotic expansion of $\nu(s)$ around $\nu(0) = \nu$, for s small enough, is by means of a *Puiseux series* in $s^{\frac{1}{q}}$ [6, 7, 8]. The exponent $\dfrac{1}{q}$ depends on the size $q \geq 1$ of each of the various Jordan blocks which constitute the Jordan box associated with the eigenvalue ν such that $\nu(s) \longrightarrow \nu$ under consideration [4].

It is therefore clear that a necessary condition for $\lim_{|s|\longrightarrow 0} |\frac{\nu(s)}{s}|$ to be *bounded* is that $\nu = 0$. However, this may not be sufficient if the exponent $\dfrac{1}{q}$ is fractional, i.e. $q > 1$. A sufficient condition is given by the

Lemma 6.1 *If $\nu = 0$ is a semi-simple eigenvalue of E, such that $\lambda(t) = \frac{\nu(s)}{s}$ and $\nu(s) \longrightarrow 0$ as $|s| \longrightarrow 0$, then $\lim_{|t|\longrightarrow\infty} |\lambda(t)| < \infty$.*

Proof Clear from above because $\nu = 0$ semi-simple implies $q = 1$.

△

Remark 6.1: When 0 is a defective eigenvalue of E, certain eigenvalues $\lambda(t)$ may still remain bounded as $|t| \longrightarrow \infty$. The general case can be treated with the help of [8]. See [16] for the case where rank$E = 1$ and E defective, $E^2 = 0$ (that is $q = 2$).

6.1 The kernel points

To ease the presentation, we assume below that E satisfies the hypothesis of lemma 6.1. E is not full rank: its rank $\widehat{r}$ is such that $1 \leq \widehat{r} < n$, and the semi-simple eigenvalue $\nu = 0$ of multiplicity $\widehat{g} = n - \widehat{r}$ has for eigenspace $K = KerE$ of dimension $\widehat{g} = n - \widehat{r}$. Let P denote the eigenprojection for E associated with $0 : ImP = K$. The following proposition was first stated by M. Van Gijzen [9].

Proposition 6.2 *If E has the semi-simple eigenvalue 0 with multiplicity $\widehat{g} = n - \widehat{r}$, there exist in general $\widehat{g}$ eigenvalues $\lambda(t)$ such that $\lim_{|t|\longrightarrow\infty} \lambda(t) = \xi$, where ξ is any one of the $\widehat{g}$ eigenvalues of the $\widehat{g} \times \widehat{g}$ matrix Π representing PAP as a map from K to K.*

Proof The series expansion $\nu(s) = \xi s + O(s^2)$ is valid around $\nu = 0$ for s small enough, therefore $\frac{\nu(s)}{s} = \xi + O(s)$ and $\frac{\nu(s)}{s} \longrightarrow \xi$ as $s \longrightarrow 0$. The characterization of ξ is a simple adaptation of Proposition 4.2.2 and Propriété 4.2.3 of [4], p. 106.

△

Definition 6.1 *The kernel points of (A, E) are the eigenvalues of $\Pi \sim PAP$ which are not eigenvalues of A.*

This definition assumes that $\sigma(\Pi) \cap \sigma(A) = \emptyset$.

Corollary 6.3 *Under the assumption of Proposition* 6.2 *and* $\sigma(\Pi) \cap \sigma(A) = \emptyset$, *there are* $\widehat{g} = n - \widehat{r}$ *eigenvalues* $\lambda(t)$ *which remain finite as* $|t| \longrightarrow \infty$ *and converge to the kernel points in* $\sigma(\Pi)$. *There are* $\widehat{r}$ *eigenvalues* $\lambda(t)$ *whose modulus tends to* ∞.

Proof Clear.

$\triangle$

Lemma 6.4 *Let* $U, V \in \mathbb{C}^{n\times\widehat{r}}$ *be such that* $E = UV^H$ *satisfies the assumption of Proposition 6.2. Then the eigenprojection* P *on* $KerE$ *is defined by the formula* $P = I - U(V^HU)^{-1}V^H$.

Proof Let $P = I - Q$ where Q is the projection on ImE along $KerE$. Then, by the assumption that 0 is semi-simple, the $\widehat{r}\times\widehat{r}$ matrix V^HU is regular (rank $\widehat{r}$), and the formula $Q = U(V^HU)^{-1}V^H$ is straightforward.

$\triangle$

As an illustration, we revisit the example of Lemma 3.3 under the assumption $a_{n-1} = e_n^Ta \neq 0$. The eigenprojection P is, by Lemma 6.4, equal to $I - \frac{1}{a_{n-1}}ae_n^T$. Therefore

$$PA = \begin{pmatrix} 0 & & & \frac{-a_0}{a_{n-1}} & 0 \\ 1 & & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & \frac{-a_{n-2}}{a_{n-1}} & 0 \\ 0 & \cdots & & 0 & 0 \end{pmatrix}$$

and the matrix Π is the companion matrix of order $n-1$ associated with the normalized (monic) polynomial $\frac{q(z)}{a_{n-1}}$ of degree $n-1$, as expected. Here $\widehat{r} = 1$, and when $a_0 \neq 0$, all the $n-1$ kernel points are critical.

The role of the hypothesis that $0 \in \sigma(E)$ is semi-simple can be better appreciated by comparison with, as an example, the analysis for the Arnoldi method, where E is defective, $E^2 = 0$ [16].

6.2 The correspondence $t\mu_z = 1$

The characterisation of z as an eigenvalue of $A + tE$ given by Lemma 1.1 has two exceptions:

i) $t = 0$, which implies $z \in \sigma(A)$ and μ_z not defined,

ii) $\mu_z = 0$, which implies $|t|$ unbounded.

M. Van Gijzen has suggested to look for some connection between the points z such that M_z is singular and the kernel points $\xi \in \sigma(\Pi)$. We still assume that $C = V^HU$ is of rank $\widehat{r}$.

The vector $v = (A - \xi I)u$ represents the residual for A at the pair (ξ, u). Clearly $v = 0 \iff \xi \in \sigma(A)$. By definition, the loss of rank for M_z can occur only for kernel points which do not belong to $\sigma(A)$, therefore the *residual* v *is nonzero* at the kernel points.

Proposition 6.5 *Let (ξ, u) be an eigenpair for $PA_{\restriction K}$ such that $\xi \notin \sigma(A)$. The vector $v = (A - \xi I)u \in ImE$ defines a unique $a \in \mathbb{C}^{\widehat{r}}$, $a \neq 0$, which is an eigendirection for M_ξ associated with 0.*

Proof By assumption $PAu = \xi u$ with $Pu = u \neq 0$. We set $v = (A - \xi I)u$ which satisfies $v \neq 0$ and $Pv = 0$: v belongs to ImE. Therefore, there exists a unique $a \in \mathbb{C}^{\widehat{r}}$, $a = C^{-1}V^H v \neq 0$ such that $v = Ua$. In addition, $-M_\xi a = V^H(A - \xi I)^{-1}Ua = V^H(A - \xi I)^{-1}v = V^H u = 0$ means that $a \neq 0$ defines an eigendirection for M_ξ associated with 0. △

Can we say more about the situation at a kernel point $\xi \notin \sigma(A)$? Indeed, at such a point, 0 is an eigenvalue of the three matrices $\Pi_\xi = \Pi - \xi I$, $M_\xi = V^H(A - \xi I)^{-1}U$ and $F_\xi = UV^H(A - \xi I)^{-1}$ of respective order $\widehat{g} = n - \widehat{r}$, $\widehat{r}$ and n. The geometric multiplicity γ of $0 \in \sigma(M_\xi)$ which satisfies $\gamma \leq min(\widehat{r}, \widehat{g})$ by Lemma 3.4, is easily related to the geometric multiplicity of ξ in $\sigma(\Pi) - \sigma(A)$.

Lemma 6.6 *Let ξ be a kernel point. The geometric multiplicity of ξ in $\sigma(\Pi)$ equals the geometric multiplicity of 0 in $\sigma(M_\xi)$.*

Proof Easy consequence of Proposition 6.5, with $v \neq 0$. △

What consequences can be drawn about the existence of critical points ξ where $\rho(F_\xi) = \rho(M_\xi) = 0$? If a critical point ξ exist, $0 \in \sigma(M_\xi)$ has algebraic multiplicity $\widehat{r}$, and geometric multiplicity $\gamma \leq min(\widehat{r}, \widehat{g})$. This leads to the distinction whether *i)* $\widehat{r} \leq \widehat{g}$ or *ii)* $\widehat{r} > \widehat{g}$.

i) $\widehat{r} \leq \widehat{g} \Longleftrightarrow \widehat{r} \leq [\frac{n}{2}]$.
Because $min(\widehat{r}, \widehat{g}) = \widehat{r}$, it is possible that criticality occurs at ξ with $\gamma = \widehat{r}$ and M_ξ semi-simple $= 0$. Therefore $2 \leq \delta \leq \widehat{r} + 1$, where δ is the degree of nilpotency for F_ξ.

ii) $\widehat{r} > \widehat{g} \Longleftrightarrow \widehat{r} > [\frac{n}{2}]$.
Now $min(\widehat{r}, \widehat{g}) = \widehat{g}$ and the value $\widehat{r}$ cannot be achieved by γ: if criticality occurs at ξ, then necessarily $M_\xi \neq 0$ is defective.
The lowest value achievable by δ is not 2 anymore, it is ≥ 3, and it increases as $\frac{\widehat{r}}{\widehat{g}}$ increases.

Proposition 6.7 *No critical point can exist with $\delta = 2$ if $\widehat{r} > [\frac{n}{2}]$.*

Proof Clear from the above discussion. △

From a computational point of view, this discussion indicates that critical points, when they exist, can be more stable when $\widehat{r} \leq \widehat{g}$.
There are 2 opposite situations in that respect:
a) $\widehat{r} = 1$, corresponding to $\delta = 2$: at most $n - 1$ kernel (critical) points.
b) $\widehat{g} = 1$, corresponding to $\delta = \widehat{r} + 1$: a unique kernel point.

What happens in the particular case where A and E are hermitian ?
We suppose that $A = A^H$, and $E = UU^H$. Therefore the eigenprojection P is orthogonal and PAP is hermitian. Thus the kernels points in $\sigma(\Pi)$ and the eigenvalues of A are *real*.
Now $M_z = -U^H(A - zI)^{-1}U$ is such that $(M_z)^H = -U^H(A - \bar{z}I)^{-1}U = M_{\bar{z}}$: at any z real $\notin \sigma(A)$ $(z = \bar{z})$, M_z is hermitian also.

Proposition 6.8 *When A and E are hermitian, criticality can occur only when $\widehat{r} < [\frac{n}{2}]$ and yields $\delta = 2$.*

Proof The above discussion shows that at a kernel point, which is real, M_z is necessarily hermitian. Therefore $M_z = 0 \Longleftrightarrow \widehat{r} \leq [\frac{n}{2}] \Longleftrightarrow \delta = 2$.

$\triangle$

7 The role of rank deficiency in the deviation matrix E

The presentation that we gave favoured the family $A(t)$ over the family $E(s)$. It is easy to see that they actually play a more balanced role. We write

$$(A + \frac{1}{\mu_z}E)u = zu, \; u \neq 0, \tag{7.1}$$

to say that (z, u) is an eigenpair for $A + \frac{1}{\mu_z}E$. Then $y = (A - zI)u$ is an eigenvector for F_z associated with μ_z, iff $y \neq 0$.
Equivalently, we write

$$(\mu_z A + E)u = \mu_z z u, \; u \neq 0, \tag{7.2}$$

The family $A(t) = A + tE$ (resp. $E(s) = E + sA$) corresponds to $t\mu_z = 1$ (resp. $s = \mu_z = \frac{1}{t}$) for $|t| < \infty$ (resp. $s \neq 0$).
Looking at the singular rays as $|t| \longrightarrow \infty$ has raised the important question about the deviation matrix E:

Is E rank deficient or not ?

7.1 Complete communication: E is full rank ($\widehat{r} = n$)

Then the n singular rays extend to infinity. For $r = |t|$ large enough, the n eigenvalues $\lambda(t)$ orbit on the same large singular orbit, when θ describes $[0, 2\pi]$. There are no kernel points. The solution $x(t, z)$ has an analytic representation for $|t| < \frac{1}{\rho(M_z)}$.

7.2 Reduced communication: $E = UV^H$ is rank-deficient ($\widehat{r} < n$) with $V^H U$ regular

As previously, we assume that 0 is a semi-simple eigenvalue of E of multiplicity $\widehat{g} = n - \widehat{r}$. There are $\widehat{g} = n - \widehat{r}$ kernel points for (A, E) which are the eigenvalues of $PA_{\upharpoonright KerE}$. The situation can be looked at in two equivalent ways:

i) from the point of view of $\lambda(t)$, eigenvalue of $A(t) = A + tE$ as $|t| \longrightarrow \infty$,

ii) from the point of view of $\frac{\nu(s)}{s}$, where $\nu(s)$ is an eigenvalue of $E(s) = E + sA$, as $s \longrightarrow 0$.

The identity $A + tE = t(sA + E)$ for $s = \frac{1}{t}$ proves that $A(t)$ and $E(s)$ share the same spectral structure for s and $t \neq 0$.

Let $P(s)$ denote the spectral projection for $E(s)$ associated with the group of eigenvalues $\nu(s)$ of total multiplicity $\widehat{g}$, which tend to 0 as $s \longrightarrow 0$. Clearly $P(s) \longrightarrow P$, and any vector in Ker$E$ is an eigenvector for E associated with 0. The relationship with $A(t)$ is through $\lambda(t) = \frac{\nu(s)}{s}$: $\lim_{|t| \longrightarrow \infty} \lambda(t) = \xi = \lim_{s \longrightarrow 0} \frac{\nu(s)}{s}$

The $\widehat{g}$ values ξ represent the *slopes* of the $\widehat{g}$ branches $\nu(s)$ generated for $s \neq 0$, small enough, by the semi-simple eigenvalue 0 for E of multiplicity $\widehat{g}$.

These numbers are the eigenvalues of $PA_{\restriction KerE}$ by Proposition 6.2, that is the spectrum $\sigma(\Pi)$. When $\xi \notin \sigma(A)$, Proposition 6.5 relates the eigenpair (ξ, u) for $PA_{\restriction K}$ to the eigenpair $(0, a)$ for M_ξ.

Remark 7.1 If A has an eigenvector $x \in KerE : Ax = \lambda x$, $x \neq 0$, then $\lambda = \xi$ is also an eigenvalue of Π: left multiplication by Π implies $\Pi x = \lambda x$, $x \in KerE$. Then $\sigma(A) \cap \sigma(\Pi) \neq \emptyset$.

We remark that ξ can be an eigenvalue of $A(t)$ for a certain t, if $1 < \widehat{r} < n$. Indeed F_z has $\widehat{r}$ eigenvalues μ_z which are not necessarily zero. If $z = \xi$, there exists at least one such eigenvalue μ_* which is non zero (unless ξ is a critical point), then ξ is an eigenvalue for $A(t_*)$ with $t_* \mu_* = 1$.

Therefore ξ is at a finite homotopic distance $|t_*|$. However, as already seen in Section 5, this is not true any more if $\widehat{r} = 1$.

Criticality with $\delta = 2$ can happen only if the rank of the deviation matrix is low enough: $\widehat{r} \leq [\frac{n}{2}]$.

7.3 Rank 1 deviation ($\widehat{r} = 1$)

This particular case is of special importance in Numerical Analysis, and, more specifically, in Numerical Linear Algebra. Because rank one perturbations are ubiquitous in Backward Error Analysis [10].

When E is rank 1 and $E^2 \neq 0$, Homotopic Deviation theory gets simpler in two fundamental ways:

1) The singular orbits are the level curves of the subharmonic map $\varphi : z \longrightarrow \rho(F_z)$. They exhibit a periodicity based on 2π.

2) The kernel points which are not eigenvalues of A are identical to the critical points of (A, E). There are at most $n - 1$ such points z for which $F_z^2 = 0$ and $R(t, z) = (A - zI)^{-1}(I + tF_z)$. At the critical points, a major *computational change* takes place for the resolution of the equation (2.1). The matrix F_z becomes defective and the resolvent matrix $R(t, z)$ is a polynomial in t of degree 1.

This change is unconditional in t, in particular, it requires no metric constraint of the type $|t|$ small enough, to hold. For any t, the algorithmic process to compute $x(t, z) = R(t, z)y$ consists of **two** steps only, instead of being the limit of an **infinite** number of steps, as is the rule when z is not a critical point. We recall that the incomplete Arnoldi decomposition provides an interesting example where $E^2 = 0$ [15, 16].

7.4 An example of rank deficiency for the deviation matrix E

It is known [4], p. 94, that the study of the dynamics of rotating structures yields an equation of the form

$$M\frac{d^2u}{dt^2} + B\frac{du}{dt} + Ku = f \tag{7.3}$$

where M is the *mass* matrix, (semi) definite positive and symmetric, K is the *stiffness* matrix, and $B = G + C$ is the matrix which takes into account the gyroscopic effect G and the damping effect C. The matrices K and B may not be symmetric.

We can look for $u(t)$ under the form $u(t) = e^{\lambda t}u$. The vibration problem associated with (7.3) yields the *quadratic* eigenproblem:

$$(\lambda^2 M + \lambda B + K)u = 0 \tag{7.4}$$

There are various ways to rewrite (7.4) as a generalized eigenproblem of dimension $2n$, if n is the order of the matrices M, B, K. For example, if we set $v = \begin{pmatrix} \lambda u \\ u \end{pmatrix}$ and assume M to be regular, then

$$\begin{pmatrix} -B & -K \\ I & 0 \end{pmatrix} v = \lambda \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} v, \tag{7.5}$$

which in turn can be transformed into the standard form

$$\begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -B & -K \\ I & 0 \end{pmatrix} v = \lambda v,$$

for the matrix $\begin{pmatrix} -M^{-1}B & -M^{-1}K \\ I & 0 \end{pmatrix}$.

We now suppose that we are interested in the family of problems of the type (7.3) depending on a parameter $b \in \mathbb{C}$ (B becomes bB), that is

$$M\frac{d^2u}{dt^2} + bB\frac{du}{dt} + Ku = f$$

and the associated eigenproblem $(\lambda^2 M + \lambda bB + K)u = 0$, or equivalently

$$\begin{pmatrix} -bM^{-1}B & -M^{-1}K \\ I & 0 \end{pmatrix} v = \lambda v. \tag{7.6}$$

If we set $A = \begin{pmatrix} 0 & -M^{-1}K \\ I & 0 \end{pmatrix}$ and $E = \begin{pmatrix} -M^{-1}B & \\ 0 & 0 \end{pmatrix}$ the eigenproblem (7.6) is that of the family $A(b) = A + bE$, where E is a matrix of order $2n$ of rank $\leq n < 2n$: it is rank deficient and satisfies the necessary condition for the existence of critical points with $\delta = 2$. Therefore the results of the paragraph 6.2 above apply. Similar conclusions can be drawn if M (resp. K) is replaced by mM (resp. kK) with $m, k \in \mathbb{C}$.

Because the homotopy parameter is allowed to vary anywhere in $\mathbb{C}$, homotopic deviation provides a powerful computational tool to study not only regular perturbation, but also singular perturbations for linear differential operators, on their matrix discretized version.

In [11, 12, 13], M. Van Gijzen describes a quadratic eigenproblem arising in Acoustics, where the role of the parameter b is played by $\frac{1}{Z}$, where Z is the coefficient of impedance creating the damping of the sound. It is interesting to remark that a *complex* value for Z is physically meaningful. In addition, the matrix B itself is rank deficient.

M. Van Gijzen [13] has skillfully designed a physical example where the evolution of a boundary condition leads to a discretised problem with a deviation matrix E of low rank $\widehat{r} > 1$, for which (A, E) exhibits criticality at three of the kernel points (ξ, $\bar{\xi}$ and 0).

8 Conclusion

In many areas of classical mathematics, singularities can be forgotten because they are locally non generic and disappear under small perturbation.
However, the view point of the classical theory of singularities may not be appropriate in Inexact Computing, which models situations when the accuracy on the data is intrinsically limited, as in the phenomenological world of Natural Sciences. It is of experience that singularities cannot be ignored in finite precision computation and that their influence can be enormous. And analytic perturbation theory is a useful tool to study the phenomenon locally.

However there may be non local effects. To study such a possibility we developped the theory of Homotopic Deviation which analyses eigenvalues of the matrix family $A(t) = A + tE$, $A, E \in \mathbb{C}^{n\times n}$, where the homotopy parameter $t = re^{i\theta} \in \mathbb{C}$. Classical Homotopic Perturbation corresponds to $r = |t| \in [0, 1]$. In contrast, we obtain unexpected results for the case $r \longrightarrow \infty$, when the deviation matrix $E = UV^H$, with $V^H U$ regular, has rank $< n$. The results correspond to *non local* effects induced by the singularities of A: they may disappear locally (for $|t|$ small enough) but not completely at a global level. The effects become visible for $|t|$ large enough.

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