Hypercomplex division in the presence of zero-divisors on $\mathbb{R}$ and $\mathbb{Z}_2$

F. Chaitin–Chatelin[1], T. Meškauskas[2] and A. N. Zaoui[3]
CERFACS, 42 av. G. Coriolis, 31057 Toulouse cedex, France

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Abstract

We consider in this work the resolution of linear equations of the type $a \times x = b$, $a \neq 0$, in algebras which may have zero-divisors. The case of hypercomplex algebras on $\mathbb{R}$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is our domain of study. In the general case, the hypercomplex division is not always possible on $\mathbb{Z}_2$, and when a solution exists, it may not be unique, either on $\mathbb{R}$ or on $\mathbb{Z}_2$. The work ends with a discussion of the role of hypercomplex multiplication in Physics. Two examples are described: i) special relativity and the quaternions, ii) quantum physics and the complex logic B_1 on $\mathbb{Z}_2$, of dimension 2 (and size $2^2 = 4$). We show how the gate $\sqrt{\textbf{not}}$ realized by quantum interference can be interpreted simply in B_1 with a suitable geometric order.

Keywords: algebra, zero-divisor, hypercomplex division, binary, special relativity, quaternions, quantum interference, geometric order, complex logic of dimension 2 and 4.

1 Introduction

This report continues the exploration of the properties of multiplication in hypercomplex algebras, defined either on $\mathbb{R}$ or on $\mathbb{Z}_2$, which started in 1999 [1, 2]. This work was followed by [3, 4, 5]. In the study to be presented, we focus on the consequences of the possible presence of zero-divisors on hypercomplex division, that is on the solution of linear equations of the type $a \times x = b$ or $y \times a = b$, for $a \neq 0$, in finite dimensional algebras. The reader who is unfamiliar with the subject is referred to the report [4] which contains all the necessary background.

[1] Université Toulouse I and CERFACS. E-mail: chatelin@cerfacs.fr
[2] CERFACS. E-mail: meska@cerfacs.fr
[3] CERFACS. E-mail: zaoui@cerfacs.fr

2 Algebras with zero-divisors

An algebra is said to be *without zero-divisor* iff the equation $x \times y = 0$ is equivalent to $x = 0$ or $y = 0$, for any x, y in V [4, definition 2.7]. If this is not the case, the algebra is said to be *with zero-divisors*.

It is known that the *real hypercomplex* algebras A_k, of dimension 2^k, $k \geq 0$, which are defined recursively by the Cayley–Dickson doubling process from $A_0 = \mathbb{R}$, are without zero divisor for $k = 0, 1, 2$ and 3 only. Similarly the finite *binary* algebras B_k, defined from $B_0 = \mathbb{Z}_2$ are without zero divisor for $k = 0$ only [4].

Another example of algebra with zero-divisors, which is much more familiar to Numerical Analysts, is provided by the algebra of square matrices $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$.

For an algebra V of finite dimension, the two following properties are equivalent [6, p. 192]:

i) V has no zero-divisor,

ii) V is a division algebra.

A *division algebra* V is such that for any a, b in V, $a \neq 0$, the two equations $a \times x = b$ or $y \times a = b$ have unique solutions x and y (definition 2.8 in [4]).

Due to the bilinearity of $\times$, the multiplication map $z \to a \times z$ can be written under matrix form as $z \to M(a)z$, where $M(a)$ is a matrix of order $\dim V$, whose elements are defined by the components of a in a manner prescribed by the chosen law for $\times$. When $\times$ is noncommutative, one should consider the right multiplication $z \times a$, in parallel to the left one $a \times z$.

Therefore the uniqueness of the solution x of $a \times x = b$ is not guaranteed in algebras V with zero-divisors. If x_0 is a particular solution, then $x_0 + z \neq x_0$ is also a solution for any $z \neq 0$ such that $a \times z = 0$, that is iff $\operatorname{Ker} M(a) \neq \{0\}$.

These ideas underlie the theory of *linear* phenomena. Most familiar to Numerical Analysts is the case of the solution of a set of *linear* equations. One can see the resolution of *linear* difference, or differential, equations of a given order in the same light.

We address in the present report the specific case of hypercomplex algebras, whose finite dimension is large enough, so they can be *without division*.

3 Hypercomplex algebras with zero-divisors

We saw in the previous Section that the resolution of

$$a \times x = b, \qquad a \neq 0, \tag{1}$$

requires two steps:

i) find any particular solution x_0 which satisfies (1),

ii) find all the solutions of the associated homogeneous equation

$$a \times z = 0. \tag{2}$$

Note that $a \times z = 0$ implies $z \times a = 0$ in A_k (resp. B_k) because $||a \times z|| = ||z \times a||$ on $\mathbb{R}$ [22] (resp. by commutativity on $\mathbb{Z}_2$).

Given any particular solution x_0 of (1), then $x_0 + z$ is also a solution of (1). We shall address these two problems sequentially.

3.1 Existence of a particular solution x_0 for (1) ?

3.1.1 in A_k on $\mathbb{R}$

For $k = 0, 1, 2$, multiplication $\times$ is associative. Therefore, for any $a \neq 0$, $a^{-1} = \frac{\bar{a}}{||a||^2}$ and $x_0 = a^{-1} \times b$ satisfies (1):

$$a \times (a^{-1} \times b) = (a \times a^{-1}) \times b = b$$

For $k \geq 3$, $\times$ is not classically associative anymore: it is alternative ($k = 3$), or flexible [22] and power associative ($k \geq 4$) [4].
Let $[x, y, z] = (x \times y) \times z - x \times (y \times z)$ be the associator in A_k, $k \geq 3$. It is clear that, if $[a, \bar{a}, b] = 0$, then $x_0 = a^{-1} \times b$ is a solution for (1). We observe that $[a, \bar{a}, b] = -[a, a, b]$.
There are two simple cases where $[a, \bar{a}, b] = 0$ for $a, b \neq 0$:

i) $b \in \mathbb{R}$, in particular $b = 1$, then $x_0 = a^{-1} = \frac{\bar{a}}{||a||^2}$,

ii) a and b colinear: $b = \alpha a, \alpha \in \mathbb{R}$, then $x_0 = \alpha \in \mathbb{R}$.

A vector $a \in A_k$ such that $[a, a, x] = 0$ for any x in A_k is said to be *alternative*. Clearly a alternative $\iff$ $\mathcal{I}m\, a$ alternative. All vectors in A_k, $k \leq 3$, are alternative. The notion is meaningful for $k \geq 4$.
If $a \neq 0$ in A_k is *alternative*, then $\operatorname{Ker} M(a) = \{0\}$ and $a \times x = b$ has the *unique* solution $x = x_0 = a^{-1} \times b$ for any b in A_k [22]. We conclude to the existence and uniqueness of x for $k \leq 3$ for any b and $a \neq 0$ in A_k.

3.1.2 in B_k on $\mathbb{Z}_2$

For $k > 0$, a^{-1} exists in B_k iff $a^2 = 1$, then $a^{-1} = a$ [4]. There is no problem with associativity and $x_0 = a^{-1} \times b = a \times b$. When $a^2 = 0$, a particular solution x_0 may exist for certain b such that $a \times b = 0$, see Section 4.

3.2 Resolution of (2)

(2) has nonzero solutions iff $a \times z = 0$ for nonzero a and z, that is iff $\operatorname{Ker} M(a) \neq \{0\}$. Due to the recursive definition of x by the Cayley–Dickson process, the structure of $M(a)$ is itself highly structured [4, 5, 22]. $\operatorname{Ker} M(a)$ and the set of zero-divisors for A_k which are non trivial only for $k \geq 4$, are studied in [22].
To the best of our knowledge there has been no systematic study of the zero-divisors in the case of *binary* algebras, that is for $B_k, k > 0$.

It is important to remark that, because binary algebras are *finite* and *commutative*, a vast simplification occurs with respect to the characterization of $\operatorname{Ker}_{\mathbb{Z}_2} M(a)$. On $\mathbb{Z}_2$, it is a finite set and all the solutions to (2) are displayed in the (hypercomplex) multiplication table of B_k. We treat the cases $k = 0, 1$ and 2 in Section 4. The general situation when $k \geq 1$ is studied in [23].

3.3 Resolution of (1)

The general solution x of (1) is given by $x = x_0 + \operatorname{Ker} M(a)$ where x_0 is a particular solution of (1), when it exists.

There are significant differences between *real* and *binary* hypercomplex algebras with respect to hypercomplex division:

1) In a *real* algebra A_k, $k \geq 0$, a^{-1} exists iff $a \neq 0$. Therefore $x_0 = a^{-1} \times b$ is a solution for any b such that $[a, a, b] = 0$ when $k \geq 4$. This implies that (1) always has solutions of the form $x = a^{-1} \times b + \operatorname{Ker} M(a)$ for any b ($k \leq 3$), and under the sufficient condition $[a, a, b] = 0\,(k \geq 4)$. Uniqueness is characterized by $\operatorname{Ker} M(a) = \{0\}$. This is satisfied for $k = 0, 1, 2, 3$, and for particular a for larger k. $\operatorname{Ker} M(a)$ has the structure of a linear vector space on $\mathbb{R}$, of finite dimension. The reader is referred to [22] for a detailed analysis of the structure of $\operatorname{Ker} M(a)$, for $a \in A_k, k \geq 4$.

2) In a *binary* algebra B_k, $k \geq 0$, a^{-1} exists iff $a^2 = 1$ and in this case $a^{-1} = a$ (lemma 6.2 in [4]). Therefore $a^2 = 1$ guarantees the existence and uniqueness of the solution $x = a \times b$ for any b. For $a^2 = 0$, there may or may not exist x_0 which satisfy (1), depending on b. The question is answered by inspection of the multiplication table. Similarly, the multiplication table describes $\operatorname{Ker}_{\mathbb{Z}_2} M(a)$.

Consequently, the multiplication table contains all the necessary information to solve (1). Unless $k = 0$, (in which case $\operatorname{Ker} M(1) = \{0\}$), when a solution to (1) exists in B_k, $k > 0$, it may not be unique depending on

$\operatorname{Ker} M(a)$ just like for A_k. But, in striking difference with the first four *real* hypercomplex algebras ($k \leq 3$), the *existence* of a solution is not guaranteed on $\mathbb{Z}_2$ for $k > 0$.

We treat completely the cases $k = 0, 1$ and 2 in the next Section, which are the binary analogues of $A_0 = \mathbb{R}, A_1 = \mathbb{C}$ and $A_2 = \mathbb{H}$.

4 Binary hypercomplex algebras for $k = 0, 1$, and 2

We look at the multiplication tables of the binary algebras B_0, B_1 and B_2 of respective (geometric) dimension 1, 2 and 4 on $\mathbb{Z}_2$. Their number of elements is finite (respectively 2, 4 and 16). To avoid ambiguity between the geometric dimension 2^k and the algebraic dimension $|B_k| = 2^{2^k}$ which represents the number of elements in B_k, we call this latter number the *size* (or cardinality) of the algebra. We recall that $\times$ is commutative and associative in $B_k, k \geq 0$ (lemma 6.1 in [4]).

4.1 $k = 0$

The multiplication table in $B_0 = \{0, 1\}$ is the standard table

$$\begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \tag{3}$$

which is often interpreted in Logic as a truth table. B_0 is a commutative field, therefore it is a division algebra such that $1^2 = 1$. Hence the equation $a \times x = b$ has, for $a = 1 \neq 0$, the unique solution $x = b$ for any $b \in \{0, 1\}$.

4.2 $k = 1$

$B_1 \sim (\mathbb{Z}_2)^2$ is an algebra of 4 elements $\{00, 01, 10, 11\}$.

1) To conveniently represent these elements, one may think of numbering them according to their binary expansion as is classical [8]. This gives $\{0, 1, 2, 3\}$ to represent the above sequence. Therefore one gets the corre-

spondence

$\times$	00	01	10	11
00	00	00	00	00
01	00	10	01	11
10	00	01	10	11
11	00	11	11	00

$\Longrightarrow$

$\times$	0	1	2	3
0	0	0	0	0
1	0	2	1	3
2	0	1	2	3
3	0	3	3	0

from which follows that $3^2 = 0$ and $a^2 = 2$ for $a = 1$ and $a = 2$. The conventional numbering which interprets a binary sequence as the binary expansion of its ordinal does not satisfy the condition $x^2 = 1$ or 0 which is required by the law of hypercomplex multiplication on $\mathbb{Z}_2$. This is a posteriori evident since 10 represents 2 instead of $(1,0) = 1$, as it should!

2) We proposed in [3, 4] another ordering which respects the hypercomplex structure. We choose the notation:

$$00 \to 0, \quad 10 \to 1, \quad 11 \to 2, \quad 01 \to 3,$$

which differs from the previous one by a cyclic permutation of the three *nonzero* elements. The correspondence is now:

$\times$	00	10	11	01
00	00	00	00	00
10	00	10	11	01
11	00	11	00	11
01	00	01	11	10

$\Longrightarrow$

$\times$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

And, in the right hand table, one recognizes the multiplication table mod 4 in $\mathbb{Z}_4$.

Now we read on the table that $a^2 = 1$ for $a = 1$ or 3, and that $2^2 = 0$.

Therefore $a \times x = b$ has a *unique* solution $x = a \times b$ for $a = 1$ or 3, and for any b. For $a = 2$, one reads $2 \times 1 = 2 \times 3 = 2$, therefore the equation $2 \times x = 2$ has *two* solutions: $x = 1$ and $x = 3$. In addition $2 \times x = 0$ for $x = 2$. There is *no* solution when $a = 2$ for $b = 1$ or 3. For $a = 0$, there is *no* solution unless $b = 0$.

The significance of this algebra B_1 for Quantum Physics will be discussed in the next Section.

Because B_1, equipped with the geometric ordering, is isomorphic to $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, one can immediately interpret the above elementary analysis in the light of $(\mathbb{Z}_4)^* = \{1, 3\}$, the group of units of $\mathbb{Z}_4$, that is the group of invertible elements [7]. A primitive root mod 4 is 3, of order 2: $3 \neq 1$, $3^2 = 1$.

4.3 $k = 2$

$B_2 \sim (\mathbb{Z}_2)^4$ is an algebra of 16 elements that we number in a similar fashion, which we call *geometric* as it respects the geometric structure of the hypercomplex multiplication, and is therefore suited for computation. It should be contrasted with the classical numbering based on the binary expansion which is best suited for a descriptive analysis of the *addition* table [8].

The **geometric ordering** is defined as follows:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
				×	×	×	×	×	×	×		×			
			×	×	×	×		×			×		×	×	
		×	×	×	×				×			×	×		×
	×	×	×	×				×		×		×		×	

"*blank*" $= 0$, "×" $= 1$

i. e. we denote $0000 \to 0$, $1000 \to 1$, $1100 \to 2$, and so on until $0100 \to 15$.

4.3.1 Multiplication table in B_2

We now give the hypercomplex multiplication table in B_2 with this ordering (the nontrivial zero values are highlighted in bold font)

×	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	**0**	6	**0**	2	**0**	6	2	4	4	6	6	4	4	2
3	0	3	6	1	4	7	2	5	11	14	10	8	15	13	9	12
4	0	4	**0**	4	**0**	4	**0**	4	4	**0**	**0**	4	4	**0**	**0**	4
5	0	5	2	7	4	1	6	3	15	9	10	12	11	13	14	8
6	0	6	**0**	2	**0**	6	**0**	2	6	4	4	2	2	4	4	6
7	0	7	6	5	4	3	2	1	12	14	10	15	8	13	9	11
8	0	8	2	11	4	15	6	12	1	14	13	3	7	10	9	5
9	0	9	4	14	**0**	9	4	14	14	**0**	4	9	9	4	**0**	14
10	0	10	4	10	**0**	10	4	10	13	4	**0**	13	13	**0**	4	13
11	0	11	6	8	4	12	2	15	3	9	13	1	5	10	14	7
12	0	12	6	15	4	11	2	8	7	9	13	5	1	10	14	3
13	0	13	4	13	**0**	13	4	13	10	4	**0**	10	10	**0**	4	10
14	0	14	4	9	**0**	14	4	9	9	**0**	4	14	14	4	**0**	9
15	0	15	2	12	4	8	6	11	5	14	13	7	3	10	9	1

Table 4.1. Multiplication table in B_2 with the geometric ordering.

We first comment that the 8×8 upper left corner differs from the multiplication table mod 8 in $\mathbb{Z}_8$ only in the two diagonal terms 2^2 and 6^2 and the 2 off diagonal terms $(2,6)$ and $(6,2)$. Their common value $4 (\text{mod } 8)$ is set to 0 by Table 4.1.

Next we read in Table 4.1 the following information:

a) list of the 8 values of a such that $a^2 = 1$:

$$a = \{1, 3, 5, 7, 8, 11, 12, 15\};$$

b) table of a, z such that $a^2 = 0$ and $a \times z = 0$ in B_2 for $a \neq 0$ and $z \neq 0$ to construct $\operatorname{Ker} M(a) - \{0\}$. There are altogether 16 different pairs (a, z), $a, z \neq 0$ (out of 25 occurences) such that $a \times z = z \times a = 0$.

a	2	4	6	9	10	13	14
z	2	2	2	4	4	4	4
	4	4	4	9	10	10	9
	6	6	6	14	13	13	14
		9					
		10					
		13					
		14					

From this information we deduce immediately that $a \times x = b$ with $a^2 = 1$ has a *unique* solution $x = a \times b$ for any b.

In addition there are solutions of $a \times x = b$ for a such that $a^2 = 0$, for certain $b \neq 0$ satisfying the necessary condition $a \times b = 0$. We list them below for $a = 2$:

x	1	3	5	7	8	9	10	11	12	13	14	15
b	2	6	2	6	2	4	4	6	6	4	4	2

In other words $2 \times x = b$ has 1 or several solutions x for $b \in \{2, 4, 6\}$. We give now the list $a \neq 0$, $b \neq 0$ for which $a^2 = 0$ and $a \times x = b$ has 1 or several solutions:

a	2	4	6	9	10	13	14
b	2	4	2	4	4	4	4
	4		4	9	10	10	9
	6		6	14	13	13	14

For the other values of b (such that $a \times b = 0$), equation (1) has *no* solution: this is an *absolute logical impossibility.*

4.3.2 Resolution if (1) for $a^2 = 1$ and $a^2 = 0$

1) For $a \in \mathbb{O} = \{1, 3, 5, 7, 8, 11, 12, 15\} \subset B$, $a^2 = 1$ and $a \times x = b$ has the *unique* solution $x = a \times b$, for any $b \in B_2$.
2) For $a \in \mathbb{E}^* = \{2, 4, 6, 9, 10, 13, 14\}$, $a^2 = 0$ with $a \neq 0$. Therefore the existence of a solution to (1) is not guaranteed for all $b \in B_2$. Indeed $a \times x = b$ and $a^2 = 0$ imply $a \times b = 0$, which can be satisfied for $b \in \mathbb{E}$, but not for $b \in \mathbb{O}$ (see Table 4.1).

We remark that $B_2 = \{0\} \cup \mathbb{E}^* \cup \mathbb{O} = \mathbb{E} \cup \mathbb{O}$. Direct inspection of Table 4.1. yields the solutions x for $a \in \mathbb{E}$, and for the possible values of b in $\mathbb{E}^*$, which are listed in Table 4.2.

a	b	x	x^2
2	2	1, 5, 8, 15	1
	4	9, 10, 13, 14	0
	6	3, 7, 11, 12	1
4	4	1, 3, 5, 7, 8, 11, 12, 15	1
6	2	3, 7, 11, 12	1
	4	9, 10, 13, 14	0
	6	1, 5, 8, 15	1
9	4	2, 6, 10, 13	0
	9	1, 5, 11, 12	1
	14	3, 7, 8, 15	1
10	4	2, 6, 9, 14	0
	10	1, 3, 5, 7	1
	13	8, 11, 12, 15	1
13	4	2, 6, 9, 14	0
	10	8, 11, 12, 15	1
	13	1, 3, 5, 7	1
14	4	2, 6, 10, 13	0
	9	3, 7, 8, 15	1
	14	1, 5, 11, 12	1

Table 4.2. Resolution of $a \times x = b$ for $a^2 = 0$, $a \neq 0$, $b \neq 0$, in B_2.

A close look at Table 4.2. reveals that:
i) for $a = 4$ there is only *one* value $b = 4$ such that $a \times x = b$ has solutions: they are the *eight* elements in $\mathbb{O}$.
ii) for $a \neq 4$, $a \in \mathbb{E}$, there are *three* values for b denoted $(4, \alpha, \beta)$, where $\alpha, \beta \in \mathbb{E}^*$, such that $a \times x = b$ has solution. At each value of b correspond *form* solutions for x. One notices that for $b = \alpha$ and β, the $4+4 = 8$ solutions

generate the set $\mathbb{O}$, whereas for $b = 4$, the 4 corresponding solutions, together with $\{4, \alpha, \beta\}$, generate the set $\mathbb{E}^*$.

Obviously, the number 4 which represents the sequence (1111) plays a special role for B_2. One remarks that such a special role was played by $2 = (11)$ in B_1.

4.3.3 Multiplication table in $(B_2)^*$

The structure of the group $(B_2)^* = \{1, 3, 5, 7, 8, 11, 12, 15\}$ of the 8 units of B_2 is not cyclic anymore. However its multiplication table displays a remarkable recursive structure

×	1	3	5	7	8	11	12	15
1	1	3	5	7	8	11	12	15
3	3	1	7	5	11	8	15	12
5	5	7	1	3	15	12	11	8
7	7	5	3	1	12	15	8	11
8	8	11	15	12	1	3	7	5
11	11	8	12	15	3	1	5	7
12	12	15	11	8	7	5	1	3
15	15	12	8	11	5	7	3	1

=

(1, 3)	(5, 7)	(8, 11)	(12, 15)
(5, 7)	(1, 3)	(15, 12)	(11, 8)
(8, 11)	(15, 12)	(1, 3)	(7, 5)
(12, 15)	(11, 8)	(7, 5)	(1, 3)

Table 4.3. Multiplication table of $(B_2)^*$.

where the notation (a, b) represents the 2×2 matrix $aI + bL$ with

$$I_2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The table can be further partitioned into $\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}$, with $T_2 = T_3^T$. The 4×4 blocks can be described by means of the 5 basic matrices of order 4:

$$I_4 = I \otimes I, \quad L \otimes L, \quad I \otimes L, \quad L \otimes I \quad \text{and} \quad K = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & L \end{array}, \right).$$

For example $T_1 = I_4 + 3\, I \otimes L + 5\, L \otimes I + 7\, L \otimes L$ where $\otimes$ denotes the Kronecker product of 2 matrices. The matrix K is used to express $T_2 = T_3^T$.

The qualitative analysis of Table 4.2 goes as follows: there are 4 ordered pairs $\alpha = (1, 3)$, $\beta = (5, 7)$, $\gamma = (8, 11)$ and $\delta = (12, 15)$ and 3 pairs in reverse order $\bar{\beta} = (7, 5)$, $\bar{\gamma} = (11, 8)$ and $\bar{\delta} = (15, 12)$. The pair α is invariant on the diagonal αI_4. The couple β, $\bar{\beta}$ appears also on the 2 diagonal blocks T_1 and T_4. On the offdiagonal block $T_2 = T_3^T$ appear the 2 couples γ, $\bar{\gamma}$ and δ, $\bar{\delta}$.

4.4 Some implications for hypercomputation

The above detailed analysis of the three binary algebras B_0, B_1 and B_2, with respect to solving (1) brings into fuller light the important difference between hypercomputation based on $\mathbb{R}$ or based on $\mathbb{Z}_2$. Real algebras such that $k \leq 3$ always provide a unique solution $x = a^{-1} \times b$ to (1) because a^{-1} exists iff $a \neq 0$, whereas in binary algebras, (1) may have no solution for $k \geq 1$ for particular $b \neq 0$. Hypercomplex algebras on $\mathbb{Z}_2 = \{0, 1\}$ lead to absolute **logical impossibilities** for $k \geq 1$: for certain a and b, there exists no solution x. This is *never* the case with hypercomplex algebras of dimension $1, 2, 4$ and 8 over the real numbers $\mathbb{R}$.

Broadly speaking, hypercomputation based on $\mathbb{R}$ leads to the vast domain of Mathematics described as Analysis. And hypercomplex computation based on $\mathbb{Z}_2$ underlies various domains which appear superficially as unconnected, such as Logic, elementary Arithmetic and Algebra done in the spirit of Diophantus, on $\mathbb{Q}$. We therefore call *complex logic* the binary algebras on $\mathbb{Z}_2$ such as B_1 and B_2. The algebra B_0, of dimension 1, describes the classical logic of Aristotle. The algebra B_1, of dimension 2, will be used in the next Section to give an elegant interpretation to quantum interference.

5 Hypercomplex Computation in Physics

The above analysis brings forth immediately the question: "Does Nature use hypercomplex multiplication in its computation?" Or is this operation, with all its interesting properties, a "meta product" of human intelligence?

One of us (F. Chaitin–Chatelin) has argued in many talks about Computation that the first alternative seems to be, indeed, the case [9, 10, 11, 12, 13].

We review below the evidence offered by Nature about the computing role of the real algebra A_2 of quaternions in Electromagnetism, and of the binary algebra B_1 in Quantum Physics. Note that the quaternions form a division algebra, but B_1 does not.

5.1 Special Relativity in Electromagnetism and the quaternions $\mathbb{H} = A_2$

Poincaré [14] was the first scientist to realize that the impossibility to detect the absolute movement of the Earth by optical or electromagnetic means was a law of Nature, and not a temporary experimental difficulty. He taught the correct relativistic interpretation of Fresnel's formula for the astronomical aberration, during his Spring course on the "Théorie mathématique de la lumière" in 1888, at the Sorbonne (University of Paris) for his debut as a Professor. Later in 1904–1905, his intuition was confirmed by his proof of the invariance of Maxwell's equations under the action of the Lorentz group [15, 16].

His proofs are both elegant and far-reaching. The reason why History has decided to attribute all the credit of the discovery of special relativity to Einstein seems to reside in the visionary aspects of Poincaré's reasoning.

A century ago, the scientific minds were not prepared to look at the Universe as if it were a giant Computer. So the algebraic reasons of Poincaré (based implicitly on quaternionic multiplication and explicitly on group theory) seemed unconvincing even to his fellow mathematicians. The scientific community, in its vast majority, decided that only the 1905 paper of Einstein, was compelling enough, because it used familiar and, therefore, "more physical" concepts, such as rods and clocks to measure local distance and time. The conceptual difficulties raised by the use of such "physical objects" were dismissed as irrelevant.

However, time has vindicated the vision of Poincaré. Almost a century has passed which gave birth to the theory of Computation and Information, and to the industry of Computers. No scientist now shies away from the idea that computation is ubiquitous. And in the most advanced corners of Theoretical Physics, elementary particles are defined only by their symmetry group structure.

The main ingredients of Maxwell's equations, in its quaternionic version [17] are, for 4D-vectors:

a) two *antisymmetric* functions:

  - the conjugator $x \to [x] = \frac{1}{2}(x - \bar{x}) = \mathcal{I}m\, x$, which is a measure of non reality,
  - the commutator $(x, y) \to [x, y] = \frac{1}{2}(x \times y - y \times x) = \mathcal{I}m\, x \wedge \mathcal{I}m\, y$, which is a measure of non commutativity. "$\wedge$" denotes the usual vector product in $\mathbb{R}^3$,

b) their two *symmetric* counterparts:

  - $x \to \{x\} = \frac{1}{2}(x + \bar{x}) = \mathcal{R}e\, x$,
  - $(x, y) \to \{x, y\} = \frac{1}{2}(x \times y + y \times x)$.

It has been found by Maxwell himself that his equations can be displayed conveniently in the 4D space (representing time t + space (xyz)) with a quaternionic structure. The quaternionic symbol

$$\Diamond = \left(\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

plays a fundamental role in deriving the partial differential equations. In the computational equivalent of Maxwell's equations, the role of $\Diamond$ is played by the quaternion $(1, 1, 1, 1)$ with all components equal to 1 [18].

5.2 Quantum Physics and the binary algebra B_1

The second revolution of the past century has been the revolution induced in our worldview by Quantum Physics [19, 20]. The breakthrough of Feynman in the 1940s to explain Quantum Electrodynamics was to make full use of what is known, amongst physicists, as amplitudes of probabilities, that is, generalized probabilities which are two dimensional vectors, rather than positive numbers like in the classical theory of probabilities. These generalized probabilities behave computationnaly like complex numbers.

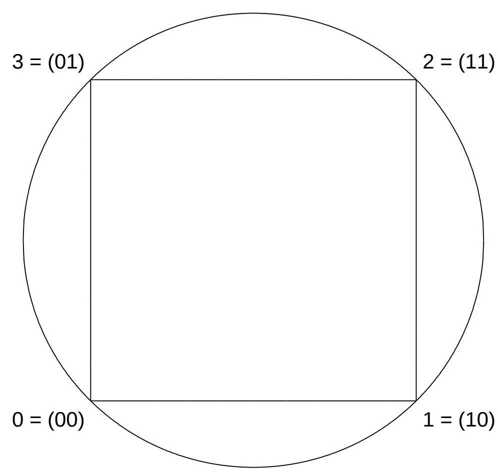

Figure 5.1. Geometric numbering on a circle and a square.

In the last two decades of the 20$^{\text{th}}$ Century another breakthrough occurred with the conjunction of Computation and Quantum Physics, leading to a quantum theory of Computation [21]. The logical gate $\sqrt{\mathbf{not}}$ was first imagined, then experimentally realized: two physical devices, each representing the logical gate $\sqrt{\mathbf{not}}$, and acting one after the other independently, implement the logical operation **not** by quantum interference: $\sqrt{\mathbf{not}}(\sqrt{\mathbf{not}}) = \mathbf{not}$ [21].

This can receive a very elegant interpretation in B_1. We consider the four states $\{00, 01, 10, 11\}$ introduced in [21]: they are the four elements of B_1. We can place them on the circle of radius $1/\sqrt{2}$, centered at $(1/2, 1/2)$, after numbering them $\{0, 1, 2, 3\}$. We consider successively the *two* numberings introduced in Section 4, § 4.2.

1) *Binary expansion* numbering:

$$S_b = \{00, 01, 10, 11\} = \{0, 1, 2, 3\}.$$

The **not** operation transforms this binary sequence into

$$\mathbf{not}(S_b) = S_b + \{11, 11, 11, 11\} = \{11, 10, 01, 00\} = \{3, 2, 1, 0\},$$

which is the original sequence in *reverse* order.

2) *Geometric* numbering:

$$S_g = \{00, 10, 11, 01\} = \{0, 1, 2, 3\}.$$

The **not** operation gives

$$\mathbf{not}(S_g) = U = S_g + \{11, 11, 11, 11\} = \{11, 01, 00, 10\} = \{2, 3, 0, 1\},$$

which is a *cyclic permutation* of $S_g(0 \to 2)$.

With the binary numbering, it is not clear how to define the "square root" of the reverse order. But with the geometric numbering, there is a simple interpretation with rotations on a circle centered at $(\frac{1}{2}, \frac{1}{2})$ (see Figure 5.1).

The sequence U is a rotation of angle π of the original sequence S_g. We define

$$T = \{10, 11, 01, 00\} = \{1, 2, 3, 0\}$$

as the rotation of $\frac{\pi}{2}$ of S_g, and U is the rotation of $\frac{\pi}{2}$ of T.

Therefore $\sqrt{\mathbf{not}}$ is the result of a rotation of $\frac{\pi}{2}$ on the sequence S_g which lists the four possible states in the *geometric* order. We remark in passing that

a) the 4 states could also be thought of as the 4 corners, numbered in the anticlockwise order, of the unit square $[0, 1]^2$, as shown on Figure 5.1,

b) the sequence

$$E = \{11, 11, 11, 11\} = \{2, 2, 2, 2\}$$

represents 4 times the same state $(11) = 2$: this sequence is associated with an event of probability 1.

From the computational point of view of the complex logic B_1, equipped with the *geometric* order, there is no mystery to the possibility of $\sqrt{\mathbf{not}}$. This simply reflects a logical-geometrical property of the algebra B_1 of dimension 2, which, by construction, has a richer structure than the classical (logical only) algebra $B_0 = \{0, 1\}$ of lesser geometric dimension 1.

6 Conclusion

We have reviewed the resolution of $a \times x = b$ in algebras with zero-divisors, with a special focus on hypercomplex algebras on $\mathbb{R}$ and $\mathbb{Z}_2$.

The case of the binary hypercomplex algebras B_k, $k = 0, 1, 2$ has been treated fully by means of the (hypercomplex) multiplication table, together with a *geometric* ordering which, as we show, is better suited for computation than the ordering based on the binary expansion of the ordinal. Finally, we reviewed two examples of the use of hypercomplex multiplication by Nature, in Special Relativity and Electromagnetism (the quaternions A_2), and in Quantum Physics (the binary algebra B_1).

The importance of the *geometric* order on the 4 elements of B_1 is exemplified by the illuminating interpretation that it provides for the logical/physical gate $\sqrt{\mathbf{not}}$ realized by quantum interference.

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