Elements of Hypercomputation on $\mathbb{R}$ and $\mathbb{Z}_2$ with the Dickson-Albert inductive process

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Abstract

This report is a follow-up for [3, 4, 5, 6, 14, 15]. It reviews the properties of the real and binary algebras of numbers generated by the Dickson-Albert inductive process. Properties with regard to commutativity, associativity, alternativity and zero divisors on $\mathbb{R}$ and $\mathbb{Z}_2$ are contrasted. It is shown how noncommutativity, non alternativity and zero divisors for higher dimensional hypercomplex algebras on $\mathbb{R}$ play an essential role in the emergence of anisotropy in real euclidean spaces of dimension $2^k - 1$, $k \geq 4$, as well as a discrete aspect for Life's Computation related to Pythagorean triples.

Key words: Hypercomplex algebra, Dickson-Albert inductive process, quadratic algebra, conjugator, commutator, associator, alternativity, flexibility, power associativity, zero divisors, anisotropy, additive versus multiplicative representation of Numbers, exponential, Pythagorean triples.

1. Introduction

In this report, we review the properties of hypercomplex algebras **generated by the Dickson-Albert doubling process** vis à vis basic notions in Computing, such as commutativity, associativity, alternativity and zero divisors. We contrast the properties of real hypercomplex algebras A_k on $\mathbb{R}$, with that of binary hypercomplex algebras B_k on $\mathbb{Z}_2$, $k \in \mathbb{N}$. A remarkable duality of behaviour emerges from this analysis. Properties which are *rare*, or sporadic, in one family (that is, valid for few small values of k) tend often to be *ubiquitous* (valid for all $k \in \mathbb{N}$) in the other family. This is summarised in Tables 2.1 and 2.2. Section 3 studies the subalgebras generated by one element x of A_k or B_k; again a duality of behaviour with respect to the question of $\lim_{n\to\infty} x^n$ emerges.

Section 4 reviews the linear equation $a \times x = b$, and extensions of the Fundamental Theorem of Algebra into A_k, $k \geq 2$. And Section 5 goes back to the central question, for Scientific Computing, of additive versus multiplicative representation of Numbers. Since its first statement [12], more than a decade ago, the question has been addressed from a variety of viewpoints, see [6, 13, 14, 15] for example. In this report, we explore the same question by means of the exponential function, either real or complex, which is known to transform multiplication into addition of exponents. How much of this can be extended for $k \geq 2$? What is the meaning for Hypercomputation? This is discussed in conclusion, with an emphasis on the emergence of the discrete from the continuous through hypercomplex exponentiation, by means of Pythagorean triples.

2. Hypercomputation on $\mathbb{R}$ and $\mathbb{Z}_2$

A denotes a generic hypercomplex algebra defined on the basis field K by the **Dickson-Albert inductive process** [1, 7] (see Remark 2.1 below). More specifically A_k denotes such an algebra of dimension 2^k, $k \in \mathbb{N}$. The basis field K for A_k is assumed to be unspecified, or $K = \mathbb{R}$. In the particular case where K is specified to be $K = \mathbb{Z}_2$, the corresponding binary algebra is denoted B_k, $k \in \mathbb{N}$.

For $k \in \mathbb{N}^*$, an element $x \in A_k$ will be written as $x = \alpha + X$, where $\alpha = \alpha 1 \in K$ is the *real part* of x, $\alpha = \Re x$, and $X \in \Im A_k$, a subspace of A_k of dimension $2^k - 1$, is the *vector* (or *imaginary*) *part* of x.

2.1. Three basic operations to define A_k

Three basic operations are defined on A_k, namely:

i) addition $+$: componentwise for $k \geq 1$,

ii) conjugation: $x = \alpha + X \;\rightarrow\; \bar{x} = \alpha - X$,

iii) multiplication $\times$: by induction on k for $k \in \mathbb{N}^*$, via the *Dickson-Albert doubling process* [4, 7] deriving from the natural multiplication on $A_0 = K$.

The addition in A_k is the conventional addition in the linear vector space A_k. But the conjugation and the multiplication are little known, beyond the algebra of complex numbers: $\mathbb{C} = A_1$ on $K = \mathbb{R}$.

It is easy to check that for $K = \mathbb{Z}_2$, conjugation reduces to identity in $B_k,\ k > 0$.

Remark 2.1 L. E. Dickson [7] proposed to define inductively an unbounded sequence of algebras A_k of dimension $2^k,\ k \in \mathbb{N}$, on the basis field $\mathbb{R}$, by a doubling process which uses addition, conjugation and multiplication at the preceeding level $k-1$:

$$(a, b) \times (c, d) = (ac - \bar{d}b,\ da + b\bar{c}) \tag{1}$$

where ac denotes $a \times c$ in A_{k-1}, starting from $A_0 = \mathbb{R}$.

A. A. Albert [1] extended the idea to a general basis field K, including K of characteristic 2.

Against historical evidence, tradition amongst Algebraists attributes the formula (1) to Cayley, (or to Cayley-Dickson [9, 16]). We depart from this trend, which does not do justice to the algorithmic insight of Dickson. It seems that the constructive power of the induction (1) is still underappreciated, almost a century after its first publication in 1919. Such a lack of appreciation is witnessed in the conventional introduction of octonions in Algebra textbooks. Their introduction follows Cayley closely (usually with no reference to their true inventor Graves) by giving a multiplication table for the 7 imaginary basis vectors. Almost never is any hint provided that the octonions represent one instance of an endless algorithmic process starting with the complex numbers and the quaternions [8].

Finally, remark that for $B_0 = \mathbb{Z}_2$, the formula (1) becomes, in $B_k = B_{k-1} \times B_{k-1}$,

$$(a, b) \times (c, d) = (ac + bd,\ ad + bc) \tag{2}$$

because conjugation reduces to identity, and $\times$ is commutative [4].

2.2. Basic definitions in A

We introduce the following definitions for a hypercomplex algebra A on K [8]:

Definition 2.1 *An algebra A is quadratic if for any $x \in A$, there exist $a, b \in K$ such that $x^2 = a + bx$.*

Definition 2.2 *An algebra A is power associative iff for all $x \in A$ and any $m, n \in \mathbb{N}^*$, then*

$$x^m \times x^n = x^{m+n}.$$

Lemma 2.1 *An algebra A_k, $k \in \mathbb{N}$, on $K = \mathbb{R}$ is quadratic and power associative*

Proof An easy calculation shows that for $x = \alpha + X$, then

$$x^2 \;=\; -(\alpha^2 + \|X\|^2) + 2\alpha x \;=\; -\|x\|^2 + 2\alpha x,$$

where $2\alpha = x + \bar{x}$ defines $b \in \mathbb{R}$ and $\|x\|^2 = x\bar{x} = N(x)$ defines $-a \in \mathbb{R}^+$. The power associativity follows by induction on k [4]. $\square$

Observe that $\|x\|^2$ represents the square of the euclidean norm, for $x \in A_k$.

As a consequence, $x^2 = 0 \Leftrightarrow x = 0$ in A_k (on $\mathbb{R}$). The imaginary vectors $X \in \Im A_k$ satisfy $x^2 = X^2 = -\|X\|^2$, since $\alpha = \Re x = 0$: their squares X^2 generate the real negative axis $\mathbb{R}^-$.

Lemma 2.2 *An algebra B_k, $k \in \mathbb{N}$ on $K = \mathbb{Z}_2$ is quadratic, associative and commutative.*

Proof Lemma 2.2 is satisfied for $B_0 = \mathbb{Z}_2$. For $x \in B_k$ (on $\mathbb{Z}_2$), $k \in \mathbb{N}^*$, $x^2 = 0$ iff x has an *even* number of zerocomponents, therefore x need not be 0 [4, 5]. Lemma 6.2 in [4] proves that $x^2 \in K = \mathbb{Z}_2$, that is $x^2 = 0$ or 1. B_k is quadratic with $a = 0$ or 1, and $b = 0 \in \mathbb{Z}_2$. It is also associative and commutative [4]. $\Box$

Next, we introduce the

Definition 2.3 *Given x, y, z in the algebra A on K, we define the three following vectors in $\Im A$:*

i) the conjugator: $x \longrightarrow [x] = x - \bar{x} = 2\Im x$,

ii) the commutator: $x, y \longrightarrow [x, y] = x \times y - y \times x$,

iii) the associator: $x, y, z \longrightarrow [x, y, z] = (x \times y) \times z - x \times (y \times z)$.

It is clear that the first two vectors belong to $\Im A$, for the third one, see Proposition 2.5 below. Also the definition is trivial for the algebras B_k (on $\mathbb{Z}_2$) which are selfconjugate, commutative and associative. The 3 vectors are therefore $\equiv 0$ for all $k \in \mathbb{N}$. This is not always the case in A_k (on $\mathbb{R}$). The conjugator (resp. commutator, associator) is identically zero for $k = 0$ (resp. $k \in \{0, 1\}$, $k \in \{0, 1, 2\}$). When the vector is nonzero, it indicates the departure from reality (resp. commutativity, associativity). Observe that each vector is zero whenever one entry is real; it depends only on the imaginary part of each entry.

The following two skew-symmetry properties hold:

i) $[\bar{x}] \ = \ -[x]$,

ii) $[x, y] \ = \ -[y, x]$.

2.3. Commutativity in A, $K = \mathbb{R}$

When $K = \mathbb{R}$, the commutator for $k = 2$ and 3 is related to the vector product in $\mathbb{R}^3$ and $\mathbb{R}^7$ in the following way:

$$x, y \in \Im A_k \ \Rightarrow \ [x, y] \ = \ x \times y - y \times x \ = \ 2(x \wedge y) \tag{3}$$

for $k = 2, 3$, where $\wedge$ denotes the *vector product* in $\Im A_k$ [8, Chapter 10, p. 288–292].

The following relation with the (euclidean) scalar product in $\Im A_k$

$$\langle x, y\rangle = \Re(x \times \bar{y}) = -\Re(x \times y) \in \mathbb{R}$$

is valid for $k \in \mathbb{N}^*$, and $x, y \in \Im A_k$:

$$x \times y \;=\; -\langle x, y\rangle + \frac{1}{2}[x, y], \;\text{ in }\; A_k. \tag{4}$$

This results from the identities in $\Im A_k$:

$$x \times y \;=\; \frac{1}{2}(x \times y + y \times x) \;+\; \frac{1}{2}(x \times y - y \times x) \;=\; \Re(x \times y) \;+\; \Im(x \times y),$$

and

$$\overline{x \times y} \;=\; \bar{y} \times \bar{x} \;=\; y \times x.$$

Therefore $\Re(x \times y) \;=\; -\langle x, y\rangle$ and $\Im(x \times y) \;=\; \frac{1}{2}[x, y]$.

Remark 2.2 The vector (resp. scalar) product is called *cross* (resp. *dot*) product in the american literature. The alert reader should keep in mind the correspondence between the continental notation: $x \times y$, $\langle x, y\rangle$, $x \wedge y$ [8] and the american notation: xy, $x \cdot y$, $x \times y$ [2] to refer respectively to the hypermultiplication, the scalar (dot) product ($k \in \mathbb{N}$) and to the vector (cross) product ($k \in \{1, 2, 3\}$).

An important case of non commutativity for the pair $\{x, y\}$ in $\Im A_k$, $k \geq 2$ corresponds to *anticommutativity*:

$$x \times y = -y \times x \quad \Leftrightarrow \quad [x, y] = 2(x \times y) \quad \Leftrightarrow \quad \langle x, y\rangle = 0.$$

Anticommutativity is equivalent to orthogonality in $\Im A_k$.

Let $x = \alpha + X$ be given, with $X \neq 0$ in A_k, $\;k \geq 2$. Set $I_X = X/\|X\|$. The vector x lies in the plane $\mathrm{lin}(1, I_X)$. In the orthonormal frame $\{1, I_X\}$, x can be represented as

$$x \quad = \quad \|x\| \;\left(\cos\xi + \sin\xi \; I_X\right),$$

with $\cos\xi = \alpha/\|x\|$ in $[-1, 1]$ and $\sin\xi = \|X\|/\|x\|$, $0 \leq \sin\xi \leq 1$. The angle $\xi = \measuredangle(1, x)$ is such that $0 \leq \xi \leq \pi$: it is defined mod π. We observe that, when $k = 1$, $\mathbb{C} = \text{lin}(1, i)$ where i is *independent* of x. Therefore ξ is classically defined mod 2π.

We now prove the

Lemma 2.3 *Let X, Y be given in $\Im A_k$. For any $k \in \mathbb{N}$, $X = tY$ for $t \in \mathbb{R}$ implies $[X, Y] = 0$. The reciprocal is true for $k \in \{0, 1, 2, 3\}$.*

Proof

1) We suppose that $X = tY$, $t \in \mathbb{R}$. Then $X \times Y = tY^2 = Y \times X$ and $[X, Y] = 0$.

2) To prove the reciprocal for $0 \leq k \leq 3$, we use the formula valid in $\Im A_k$; see (3) and (4):
$$X \times Y = X \wedge Y - \langle X, Y \rangle.$$
Therefore $[X, Y] = 2(X \wedge Y) = 0 \quad \Leftrightarrow$
$$\|X \times Y\| = \|X\| \, \|Y\| = |\langle X, Y \rangle|.$$
By the Cauchy-Schwarz inequality, this is equivalent to $X = tY$, $t \in \mathbb{R}$.

3) For $k = 4$, the hexadecanions $A_4 = \mathbb{G} \times \mathbb{G}$ provide an easy counter example where $[X, Y] = 0$ but $X \neq tY$, $t \in \mathbb{R}$ [4, 10].

   We set
$$X = (e_7, e_8) = (00000010, 00000001)$$
   and
$$Y = (e_5, e_6) = (00001000, 00000100)$$
   where $\{e_i\}$, $i = 1$ to 8 is the canonical basis for $A_3 = \mathbb{G}$. An easy calculation with (1) shows that $X \times Y = Y \times X = 0$, however $X \neq tY$ [4]. Observe that $[X, X, Y] \neq 0$ (Proposition 2.8). $\square$

The notion of *centralizer* C_X for $X \in \Im A_k$ is also useful:
$$C_X = \{Y \in \Im A_k; \; X \times Y = Y \times X\}, \;\; \text{for } 0 \neq X \in \Im A_k.$$

It can easily be shown, by linearity of the map $L_X : y \to X \times y$, $y \in A_k$, [5, 18] that

$$C_X = \text{lin}\, X \oplus \ \text{Ker}\, L_X,$$

where Ker L_X is the set of zerodivisors for X. See Section 4, paragraph 4.1, and Lemma 4.1. Because of (4):

$$X \times Y = -\langle X, Y \rangle + \frac{1}{2}[X, Y],$$

it is clear that Ker $L_X \subset \{X\}^\perp$. ($X \times Y = 0 \iff \langle X, Y \rangle$ and $[X, Y] = 0$.) The above decomposition for C_X is orthogonal. We denote by Zer(X) the set of zerodivisors Ker L_X for X in $\Im A_k$, $k \geq 4$.

Clearly, for $k \geq 2$, the *centre* for A_k is

$$\mathbb{R}1 \ = \ \{z \in A_k;\ z \times x = x \times z \text{ for all } x \text{ in } A_k\}.$$

For $k \geq 4$, a necessary condition for Zer$(x) \neq \{0\}$, $x \in A_k$ is that $\Re x = 0$, that is $x = X \ \in \ \Im A_k$. This follows easily from:

$$x \times y \ = \ (\alpha + X) \times (\beta + Y) \ = \ \alpha\beta - \langle X, Y \rangle + \beta X + \alpha Y + \frac{1}{2}[X, Y].$$

For x given in A_k,

$$\text{Zer}(x) \ = \ \left\{ y;\ \langle X, Y \rangle = \alpha\beta \ \text{ and } \ \frac{1}{2}[X, Y] + \beta X + \alpha Y = 0 \right\}.$$

Now, by Lemma 5.2 in Section 5, $\langle [X, Y],\, Y \rangle = 0$ therefore

$$\beta\langle X, Y \rangle + \alpha \|Y\|^2 \ = \ \alpha(\beta^2 + \|Y\|^2) \ = \ 0$$

implies that $\alpha = 0$ or $y = 0$.

Proposition 2.4 *Let $u, v \in A_k$. The product $u \times v$ is commutative iff the imaginary parts are colinear, when $k \in \{0, 1, 2, 3\}$. For $k \geq 4$, this condition is sufficient but not necessary. Commutativity in $\Im A_k$, $k \geq 4$, is equivalent to $Y = tX + Z$, $Z \in$ Zer(X), $t \in \mathbb{R}$.*

Proof Let $u = \alpha + X,\ \ v = \beta + Y,\ \ X, Y \in \Im A_k$.

$$u \times v - X \times Y = \alpha\beta + \beta X + \alpha Y$$

shows that $[u,v]=0 \;\Leftrightarrow\; [X,Y]=0$. For X in $\Im A_k$, $k \geq 4$, the structure $C_X = \operatorname{lin} X \oplus \operatorname{Zer}(X)$ implies that any $Y \in \Im A_k$ which commutes with X can be written as $Y = tX + Z$, with $t \in \mathbb{R}$ and $Z \in \operatorname{Zer}(X)$. $\square$

We observe that for $k \geq 4$, A_k is not a division algebra [8]: the existence of *zero divisors* opens the possibility for X, Y to commute ($X \times Y = Y \times X = 0$) with $X \neq tY$. For zero divisors X and Y such that

$$X \times Y = 0 = \pm Y \times X,$$

commutativity and anticommutativity are equivalent, and $\langle X, Y\rangle = 0$.

The set of zero divisors is nongeneric: it is closed and nowhere dense in A_k [19]. It has increasing relative dimension in A_k, as k increases from 4. For example, in the hexadecanions A_4 of dimension 16 the set of zero divisors has dimension 14. It is the exceptional Lie Group G_2 of rank 2 (Moreno, 1998), which itself is the automorphism group of $A_3 = \mathbb{G}$ (Cartan, 1914), see [16, 2].

We also remark that for $k \in \{0,1\}$, the imaginary parts are always colinear. Hence the commutativity of the product $\times$ holds in $\mathbb{R}$ and $\mathbb{C}$, but not necessarily in the quaternions $A_2 = \mathbb{H}$ or the octonions $A_3 = \mathbb{G}$ (or A_k, $k \geq 4$). The geometric notion of *colinearity* in $\Im A_k$, $k \geq 4$, appears to be stronger than the algebraic notion of *commutativity* in A_k.

A necessary and sufficient condition for colinearity in $\Im A_k$, $k \geq 4$, is given in paragraph 2.5 (Proposition 2.8).

2.4. Associativity in A, $K = \mathbb{R}$

The *classical notion of associativity* on A can be defined by $[x,y,z]=0$ for any $x,y,z \in A$.

Again, an important case for non associativity of the *ordered* triple $\{x,y,z\}$ in $\Im A_k$, $k \geq 3$ corresponds to *antiassociativity*:

$$(x \times y) \times z = -x \times (y \times z) \;\;\Leftrightarrow\;\; [x,y,z] = 2(x \times y) \times z.$$

Antiassociative triples play a role in the construction of zero divisors [16].

It is useful to weaken the classical definition of associativity into:

i) *alternativity*: for any $x, y, z \in A$,

$$[x, y, z] = [z, x, y] = -[x, z, y] = [y, z, x] = -[y, x, z].$$

That is, the associator is an alternating function of its three arguments, This implies

$$[x, y, x] = [x, y, y] = 0$$

for any $x, y \in A$.

ii) *power associativity* (see Definition 2.2).

As we know, the octonions $\mathbb{G} = A_3$ on $\mathbb{R}$ are alternative, and $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$, $A_2 = \mathbb{H}$ are classically associative.

These three levels of associativity capture respectively the sort of associativity enjoyed by an algebra which is such that the subalgebra generated by any 3 (resp. 2, or 1) elements is itself associative [2]. Associativity at one level implies associativity at the subsequent weaker levels.

A fourth level of associativity has been introduced by Algebraists [1], between alternativity and power associativity:

Definition 2.4 *An algebra A is flexible iff*

$$[x, y, x] = 0$$

for any $x, y \in A$.

It is not difficult to prove that all hypercomplex algebras built inductively by the Dickson-Albert doubling process are *flexible* [9].

An equivalent definition of flexibility uses 3 vectors: A is flexible iff the condition $[X, y, Z] = 0$ for any X, Z in $\Im A$ and any y in A, implies that $X = tZ$, $t \in \mathbb{R}$ [17].

It is clear that $X = tZ$ and $[X, y, Z] = 0$ implies that A is flexible. The converse, known as Yiu's conjecture among experts, was established elegantly by Moreno in 2000 (Theorem 2.3 in [17]).

The associator enjoys the properties [16] given in

Proposition 2.5 *For any* $x, y, z \in A_k$, $k \geq 3$

a) $-[x,y,z] = [\bar{x},y,z] = [x,\bar{y},z] = [x,y,\bar{z}]$,

b) $\Re[x,y,z] = 0$.

Proof

a) We write $x = \alpha + X$, $X \in \Im A_k$, $\alpha \in \mathbb{R}$. $[\bar{x},y,z] = [\alpha - X, y, z]$ and $[\alpha,y,z] = 0$ for $\alpha \in \mathbb{R}$. Therefore $[\alpha - X, y, z] = [-\alpha - X, y, z] = -[x,y,z]$. $[x,\bar{y},z]$ and $[x,y,\bar{z}]$ can be treated in a similar fashion.

b) $[x,y,z] = (x \times y) \times z - x \times (y \times z)$, and

$$\begin{aligned} \overline{[x,y,z]} &= \overline{(x \times y) \times z} - \overline{x \times (y \times z)} \;=\; \bar{z} \times (\bar{y} \times \bar{x}) - (\bar{z} \times \bar{y}) \times \bar{x} \\ &= -[\bar{z},\bar{y},\bar{x}] \;=\; [\bar{z},y,\bar{x}] \;=\; -[z,y,\bar{x}] \;=\; [\bar{x},y,z] \\ &= -[x,y,z], \end{aligned}$$

by *a*) and by flexibility, since

$$[x,y,x] = 0 \quad \Leftrightarrow \quad [x,y,z] \;=\; -[z,y,x]$$

for any x,y,z. □

Corollary 2.6 *For any $x,y,z \in A_k$, $k \geq 1$*

a) $\langle x,\ y \times z\rangle = \langle x \times \bar{z},\ y\rangle = \langle \bar{y} \times x,\ z\rangle$,

b) $\|x \times y\| = \|\bar{x} \times y\| = \|x \times \bar{y}\| = \|y \times x\|$.

Proof

a) $\langle x,\ y \times z\rangle = \Re(x \times (\bar{z} \times \bar{y})) = \Re((x \times \bar{z}) \times \bar{y}) = \langle x \times \bar{z},\ y\rangle$ because of Proposition 2.5, *b*). The rest follows in a similar fashion.

b) $\|x \times y\|^2 = \langle x \times y,\ x \times y\rangle$. By *a*),

$$\langle x \times y,\ x \times y\rangle = \langle \bar{x} \times (x \times y),\ y\rangle = \langle \bar{x} \times (x \times y) \times \bar{y},\ 1\rangle = \Re(\bar{x} \times (x \times y) \times \bar{y}).$$

And similarly,

$$\|\bar{x} \times y\|^2 = \langle \bar{x} \times y,\ \bar{x} \times y\rangle = \langle x \times (\bar{x} \times y) \times \bar{y},\ 1\rangle = \Re(x \times (\bar{x} \times y) \times \bar{y}).$$

Hence

$$\begin{aligned}\|x \times y\|^2 - \|\bar{x} \times y\|^2 &= \Re\left(\bar{x} \times (x \times y) \times \bar{y} - x \times (\bar{x} \times y) \times \bar{y}\right) \\ &= \Re\left\{-[\bar{x}, x, y] + (\bar{x}x)y + [x, \bar{x}, y] - (x\bar{x})y\right\} \times \bar{y} \\ &= \Re\left\{[x, x, y] + [x, \bar{x}, y] + (\|x\|^2 - \|x\|^2)y\right\} \times \bar{y} \\ &= 0.\end{aligned}$$

The rest follows by $\|x\| = \|\bar{x}\|$. □

In summary, for $k \geq 4$, A_k still enjoys two weaker properties with respect to associativity, namely flexibility and power associativity. However, vectors in A_k may enjoy stronger properties with respect to certain or all other vectors [17]. This is the subject of the next paragraph.

2.5. Alternative vectors in A_k, $k \geq 4$, $K = \mathbb{R}$

We introduce the

Definition 2.5 *A nonzero vector $a \in A$ flexible is alternative iff $[a, a, x] = 0$ for any $x \in A$, that is $x \times (a \times x) = a^2 \times x$.*

We denote by $\mathcal{A}_k$ the set of alternative vectors in A_k. The canonical basis vectors in A_k are alternative [9]. a is alternative iff $\Im a$ is alternative, because the real vectors are alternative.

It is clear that, for $k \leq 3$, all vectors are alternative and $\mathcal{A}_k = A_k$. The inclusion $\mathcal{A}_k \subset A_k$ is strict for $k \geq 4$.

The alternative vectors in A_{k+1} are constructed from their counterparts in A_k according to the characterization [17, theorem 3.3] for $a, b \in \Im A_k$:

$$(a, b) \in \mathcal{A}_{k+1} \quad \Leftrightarrow \quad \begin{cases} i) & a \text{ and } b \text{ are in } \mathcal{A}_k, \text{ and} \\ ii) & a \text{ and } b \text{ are dependent.} \end{cases}$$

Therefore, all alternative vectors in $\Im\mathcal{A}_k$ originate from the same $\Im\mathcal{A}_3 = \Im A_3 = \Im\,\mathbb{G}$. They become relatively rare as k increases from the value 4.

The property "a alternative" is *global* since $[a, a, b] = 0$ holds for any $b \in A$. We turn to the weaker notion for a couple (a, b) where b is fixed as well as a: this expresses a *particular* alternativity between a and b.

Definition 2.6 *Let $a, b \in A_k$, $k \geq 4$. a is b-alternative iff $[a, a, b] = 0$.*

Note that this definition is *not* symmetric in a and b. Again, an interesting metric characterisation [17] holds in $\Im A_k$:

$$[a,a,b]=0 \quad \Leftrightarrow \quad \left\{ \begin{array}{llcl} i) & \|a \times b\| & = & \|a\|\,\|b\|, \quad \text{and} \\ ii) & \|a \times (a \times b)\| & = & \|a\|^2\,\|b\|. \end{array} \right.$$

If one remembers that alternativity and isometry hold universally in A_k for $k \leq 3$ only, one realises that the notion of an alternative vector a (resp. a alternates with b) is fruitful to regain for a (resp. (a, b)) some of the lost universal properties. This will be confirmed by Sections 4 and 5.

Definition 2.7 *a and b multiply isometrically in A_k, $k \geq 4$ iff*

$$\|a \times b\| \ = \ \|a\|\,\|b\|.$$

This definition is due to Moreno[1], under the name "weak-alternativity." Observe that for $k \leq 3$, all vectors in A_k multiply isometrically.

Proposition 2.7 *a and b multiply isometrically in A_k, $k \geq 4$ iff*

$$\langle [\bar{a}, a, b],\ b\rangle \ = \ \langle [a, a, b],\ b\rangle \ = \ 0.$$

Proof Recall that $[\bar{a}, a, b] \ = \ -[a, a, b]$. Apply Corollary 2.6 to compute:

$$\begin{array}{rcl} \langle [\bar{a}, a, b],\ b\rangle & = & \|a\|^2\,\|b\|^2 \ - \ \langle \bar{a} \times (a \times b),\ b\rangle \\ & = & \|a\|^2\,\|b\|^2 \ - \ \|a \times b\|^2. \end{array} \qquad \square$$

This proposition is remarkable: it transforms the geometric condition of orthogonality into a metric one. It follows from it that couples of alternating

[1] G. Moreno (June 1, 2004). Private communication.

vectors (a, b) multiply isometrically. Indeed, $\{a, b, a \times b\}$ multiply isometrically. It also follows that the conditions

$$\langle [a, a, b],\ b\rangle \quad = \quad \langle [a,\ a,\ a \times b],\ a \times b\rangle \quad = \quad 0$$

are equivalent to

$$[a, a, b] \quad = \quad [a,\ a,\ a \times b] \quad = \quad 0$$

for $a, b \in A_k$.

We turn to colinearity in $\Im A_k$, for $k \geq 4$.

Proposition 2.8 *Let $X, Y \in \Im A_k$, for $k \geq 4$. The following five conditions are equivalent:*

i) *X and Y are colinear.*

ii) $[X, Y] = [X, X, Y] = 0$

iii) $[X, Y] = [Y, Y, X] = 0$

iv) $[X, Y] = [X, X, Y] + [Y, Y, X] = 0$

v) *X and Y commute and multiply isometrically:*

$$[X, Y] = \langle [X, X, Y], Y\rangle = 0.$$

Proof The necessary part is easy: $i) \Rightarrow \{ii), iii), iv), v)\}$. We prove the reciprocal.

$ii) \Rightarrow i)$: we suppose $X \neq 0$, $X \times X = -\|X\|^2$.

$$[X, Y] = 0 \iff X \times Y = -\langle X, Y\rangle = -\sigma \in \mathbb{R}$$

$$[X, X, Y] \ = \ -\|X\|^2 Y + \sigma X \ = \ 0$$

implies

$$Y \ = \ \frac{\sigma}{\|X\|^2} X,$$

for $\sigma \neq 0$, and $Y = 0$ for $\sigma = 0$.

$iii) \Rightarrow i)$: exchange X and Y, and suppose $Y \neq 0$.

iv) $\Rightarrow$ *i*):

$$\begin{aligned}
[X,X,Y]+[Y,Y,X] &= -\|X\|^2Y+\sigma X-\|Y\|^2X+\sigma Y \\
&= (\sigma-\|Y\|^2)X+(\sigma-\|X\|^2)Y \\
&= 0.
\end{aligned}$$

$\sigma = 0$ implies X and Y colinear, hence $X = Y = 0$. When $\sigma \neq 0$, X and Y are non zero, and the above condition can be written as $\alpha X + \beta Y = 0$ with

$$\alpha = \frac{\sigma}{\|X\|\,\|Y\|} - \frac{\|Y\|}{\|X\|}, \qquad \beta = \frac{\sigma}{\|X\|\,\|Y\|} - \frac{\|X\|}{\|Y\|}.$$

Now $\frac{\sigma}{\|X\|\,\|Y\|} \leq 1$, with equality iff $X = tY$. When α and β are nonzero, then X and Y are colinear. We suppose $\|Y\| \leq \|X\|$, then $\alpha = 0$ is possible. If $\beta \neq 0$, then $\beta Y = 0 \;\Rightarrow\; Y = 0$ and $\sigma = 0$, which is impossible. Therefore one gets the option $\alpha = \beta = 0$ which is realised when $\|X\| = \|Y\|$ and $\sigma = \|X\|\,\|Y\| \neq 0$, hence X and Y are colinear.

v) $\Rightarrow$ *i*): immediate by Cauchy-Schwarz on $\langle [X,X,Y],Y\rangle = [X,Y] = 0$. Therefore $X \times Y = -\langle X,Y\rangle$ and

$$\langle [X,X,Y],Y\rangle = -N(X)N(Y) + \langle X,Y\rangle^2 = 0 \qquad \Longleftrightarrow \qquad X = tY, \;\; t \in \mathbb{R}.$$

$\square$

For $k \geq 4$, the second condition on X, Y, added to $[X,Y] = 0$ to get colinearity, serves the purpose of eliminating all nontrivial zero divisors in A_k.

Proposition 2.8 leads naturally to the

Definition 2.8 *$x, y \in A$ are imaginary dependent ($\Im$-dependent) iff their imaginary parts X, Y in $\Im A$ are linearly dependent: $X = tY$, $t \in \mathbb{R} = K$.*

The same notion is proposed in [18] under the name "$\mathbb{C}$-dependence."

The $\Im$-dependence in a flexible algebra A can be characterised in two ways:

a) in $\Im A$, X and Y are dependent iff they commute *and* multiply isometrically each other (Proposition 2.8), *b*) in A, x and y are $\Im$-dependent iff the associator $[x, z, y] = 0$ for any z in A (Theorem 2.3 in [17]).

In other words, two $\Im$-dependent vectors in A_k generate a 2 dimensional subalgebra isomorphic to $\mathbb{C}$.

We can now state the following four remarkable geometric interpretations for the role of the algebraic multilinear maps (conjugator, commutator, associator) which were introduced in Definition 2.3, when $K = \mathbb{R}$:

i) $x^2 = \|x\|^2$ iff $[x] = 0$,

ii) for $X, Y \in \Im A$, $\langle X, Y\rangle = 0$ iff
$[X, Y] \;=\; 2X \times Y \;=\; -2Y \times X$,

iii) for $X, Y \in \Im A$, X and Y are dependent iff
$[X, Y] = 0$ and $[X, X, Y] = 0$,

iv) x and y are $\Im$-dependent in A iff $[x, z, y] = 0$ for any z in A.

The first two interpretations *i*) and *ii*) are well known [8]. By comparison, *iii*) and *iv*) are more recent (resp. this report and [17]). They provide an unsuspected link between the familiar notion of linear dependence and the seemingly foreign notion of associativity for the multiplication of 3 vectors. This link appears only for $k \geq 4$, in algebras A_k possessing non trivial zerodivisors.

An equivalent way to state *iii*) is the following: x and y are $\Im$-dependent in A_k iff:

- for $k \leq 3$, $x \times y = y \times x$ (Proposition 2.4),
- for $k \geq 4$, the multiplication is not only commutative but isometric (Moreno, 2004).

This result specifies how the the property of $\Im$-dependence is more demanding in A_k, $k \geq 4$, than commutativity: the isometry of the product: $x, y \to x \times y = y \times x$ rules out the presence of non trivial zerodivisors predicted by Proposition 2.4.

2.6. The metric qualification of multiplication for $k \geq 4$

When isometry is not the rule for multiplication, we introduce the map

$$x, y \quad \longmapsto \quad N(x \times y) - N(x)\, N(y) \quad = \quad \delta(x, y) \quad \in \quad \mathbb{R}$$

with $N(x) = x\bar{x} = \|x\|^2$, $\delta(x, y) = \delta(y, x) \not\equiv 0$ for $k \geq 4$. Observe that

$$\langle [x, x, y],\, y \rangle \quad = \quad \langle [y, y, x],\, x \rangle \quad = \quad \delta(x, y)$$

by Proposition 2.7. Let $x = \alpha + X$, $y = \beta + Y$ be given in A_k, $k \geq 4$, where X, Y are nonzero vectors in $\Im A_k$, $C = \frac{1}{2}[X, Y]$.

Lemma 2.9 $\delta(x, y) = \delta(X, Y) = \langle X, Y \rangle^2 - \|X\|^2 \|Y\|^2 + \|C\|^2$.

Proof Simple calculation from

$$x \times y \quad = \quad \alpha\beta - \langle X, Y \rangle + \alpha Y + \beta X + C$$

where we use the fact that the vectors $1, C$ and $\alpha Y + \beta X$ are mutually orthogonal (Lemma 5.2).

$$\begin{aligned} N(x, y) \quad &= \quad \langle \alpha\beta - \langle X, Y \rangle + \alpha Y + \beta X + C, \quad \alpha\beta - \langle X, Y \rangle + \alpha Y + \beta X + C \rangle \\ &= \quad (\alpha\beta - \langle X, Y \rangle)^2 \;+\; \langle \alpha Y + \beta X,\; \alpha Y + \beta X \rangle \;+\; \langle C, C \rangle \\ &= \quad \alpha^2\beta^2 \;+\; \langle X, Y \rangle^2 \;+\; \alpha^2 N(Y) \;+\; \beta^2 N(X) \;+\; N(C). \end{aligned}$$

$$\begin{aligned} N(x)N(y) \quad &= \quad \big(\alpha^2 + N(X)\big)\big(\beta^2 + N(Y)\big) \\ &= \quad \alpha^2\beta^2 \;+\; \beta^2 N(X) \;+\; \alpha^2 N(Y) \;+\; N(X)N(Y). \end{aligned}$$

The result follows. □

This establishes the identity:

$$\langle X, Y \rangle^2 \;+\; \frac{1}{4} N([X, Y]) \quad = \quad N(X)N(Y) \;+\; \langle [X, X, Y], Y \rangle,$$

valid for any X, Y in $\Im A_k$.

Let θ be the angle (mod 2π) between X and Y. We set $I_X = X/\|X\|$, $I_Y = Y/\|Y\|$, $C_n = \frac{1}{2}[I_X, I_Y]$ where the subscript n stands for "normalised".

Because $\delta(x,y)$ does not depend on $\alpha = \Re x$, $\beta = \Re y$, we normalise it also by $(\|X\|\|Y\|)^2$:

$$\delta_n(x,y) \quad = \quad \frac{\delta(x,y)}{\|X\|^2\|Y\|^2} \quad = \quad \cos^2\theta - 1 + \|C_n\|^2 \quad = \quad \|C_n\|^2 - \sin^2\theta.$$

Lemma 2.10 *δ_n is a real number such that $\delta_n \geq -1$. $\delta_n = -1$ iff $y \in \mathrm{Zer}(X) \neq \{0\}$.*

Proof The minimum value for $\delta(x,y)$ is obtained when $x \times y = 0$. By normalisation, this yields $\delta_n \geq -1$, where the value -1 is obtained for $C_n = 0$, $\theta = \pm\pi/2$, $y \in \mathrm{Zer}(X) \neq \{0\}$. □

Definition 2.9 *The sign of the real number $\delta_n \geq -1$ provides a metric qualification for the product $(x,y) \mapsto x \times y$, when x and y are non real vectors $\neq 0$ in A_k, $k \geq 4$.*

When δ_n is negative: $-1 \leq \delta_n < 0$, the multiplication is *contractive*; $\delta_n = -1$ iff $x \times y = 0$. When δ_n is positive, $\delta_n > 0$, the multiplication is *expansive*. When $\delta_n = 0$, the multiplication is *isometric*: x and y multiply *isometrically* (Definition 2.7).

The latter case $\delta_n = 0$ is the only possibility for $k \leq 3$. Indeed $C_n = I_X \wedge I_Y$ and $\|C_n\| = |\sin\theta|$ for any θ is equivalent to $\delta_n = 0$ for any X, Y.

For $k \geq 4$, we define $D_n = [I_X, I_X, I_Y] = -I_Y - I_X \times (I_X \times I_Y)$. Therefore $\delta_n = 0 \iff \langle D_n, I_Y\rangle = 0 \iff \|C_n\| = 1$ when $\theta = \pm\pi/2$ and $C_n = I_X \times I_Y$.

Let be given an orthogonal pair (X,Y) in $\Im A_k$. For $k \geq 4$, $C_n = I_X \times I_Y$ represents the *vector product* $I_X \wedge I_Y$ when $\|C_n\| = 1$, or equivalently D_n is orthogonal to Y. This is not satisfied by an arbitrary pair of orthogonal vectors unless X or Y is alternative. When neither X nor Y are alternative, then $\delta_n = 0$ is clearly satisfied by any pair (X,Y) such that X is Y-alternative, or Y is X-alternative. For $k \geq 4$, the product $X \times Y$ can be interpreted as a vector product for any pair of orthogonal vectors X and Y which multiply isometrically.

The relation $\|C_n\|^2 = \sin^2\theta + \delta_n$ shows that if any two of the quantities C_n, θ and δ_n is zero, this implies that the third is zero. This confirms the equivalence $i) \Leftrightarrow v)$ in Proposition 2.8.

Commutativity ($C_n = 0$) implies *non expansion* ($\delta_n \leq 0$) for $k \geq 4$. For $k \leq 3$, $C_n = I_X \wedge I_Y$ and $\|C_n\| = |\sin\theta|$ depends on θ alone and is maximum for $\theta = \pm\frac{\pi}{2}$. For $k \geq 4$, an expansive product is always *non*commutative ($\|C_n\|^2 \geq \delta_n > 0$). $C_n = \frac{1}{2}[I_X, I_Y]$ and the relation $\|C_n\|^2 = \sin^2\theta + \delta_n$ shows that $\|C_n\|$ may depend on $\delta_n \not\equiv 0$ when X and Y do not multiply isometrically ($D_n \neq 0$ not orthogonal to Y).

Lemma 2.11 *In the 2D-plane* $\mathrm{lin}(X, Y)$, $\|C_n\|$ *is maximum for* $\theta = \pm\pi/2$, *hence* $C_n = I_X \times I_Y$.

Proof Clear from $X \times Y = -\langle X, Y\rangle + C$ with C orthogonal to 1. Therefore $\|C\| \leq \|X \times Y\|$ with equality for $\langle X, Y\rangle = 0$. □

We assume that X is not alternative, so that $\mathrm{Zer}(X)$ can be nonzero (Lemma 4.1).

Proposition 2.12 *Let X be nonalternative. When Y varies in* $\{X\}^\perp \cap \Im A_k$, $k \geq 4$, *the product* $(X, Y) \mapsto X \times Y$ *is expansive (resp. contractive, isometric) iff* $\|I_X \times I_Y\| > 1$ *(resp.* $< 1, = 1$*).* $I_X \times I_Y = 0$ *is realised by any* Y *in* $\mathrm{Zer}(X) \neq \{0\}$.

Proof Because $\langle X, Y\rangle = 0$, $C_n = \frac{1}{2}[I_X, I_Y] = I_X \times I_Y$. □

Mutiplication in $A_k \sim \mathbb{R}^{2^k}$ makes the imaginary subspace $\Im A_k \sim \mathbb{R}^{2^k-1}$ be **anisotropic** for the euclidean norm when $k \geq 4$. This is an unexpected geometric property for the euclidean space $\mathbb{R}^n$ which emerges with $n = 2^k - 1 \geq 15$, $k \geq 4$. By comparison, $\mathbb{R}$, $\mathbb{R}^3$ and $\mathbb{R}^7$ are all isotropic (the multiplication is universally isometric for $k \leq 3$).

2.7. Real versus binary algebras

We list in Table 2.1 the respective domain of validity in $\mathbb{N}$, of various properties in A_k (on $\mathbb{R}$) and B_k (on $\mathbb{Z}_2$). The duality between sporadic and ubiquitous properties appears clearly.

Property	A_k *on* $\mathbb{R}$	B_k *on* $\mathbb{Z}_2$
conjugation $x \to \bar{x}$	identity for $k = 0$ $\neq$ identity for $k \in \mathbb{N}^*$	identity for $k \in \mathbb{N}$
commutativity of $\times$ associativity alternativity	$k \in \{0, 1\}$ $k \in \{0, 1, 2\}$ $k \in \{0, 1, 2, 3\}$	$k \in \mathbb{N}$ $k \in \mathbb{N}$ $k \in \mathbb{N}$
$x^2 = 0 \Leftrightarrow x = 0$	$k \in \mathbb{N}$	$k = 0$
isometric division algebra	$k \in \{0, 1, 2, 3\}$	$k = 0$

Table 2.1 *Real versus binary algebras.*

Table 2.2 shows another interesting difference between A_k and B_k with respect to *inversion*: it is possible in A_k (resp. B_k) iff $x \neq 0$ (resp. $x^2 = 1$). This has consequences for the resolution of $a \times x = b$ in the presence of nonzero zerodivisors for a. The study performed in [5, 6] reveals a similar duality of pattern. See also Section 4.

$x \times x^{-1} = 1$	A_k *on* $\mathbb{R}$	B_k *on* $\mathbb{Z}_2$
inversion $x \to x^{-1}$	$k \in \mathbb{N}$ iff $x \neq 0$ $x^{-1} = \frac{\bar{x}}{x\bar{x}} + \mathrm{Zer}(x)$	$k \in \mathbb{N}$ iff $x^2 = 1$ $x^{-1} = x$

Table 2.2 *Inversion in* A_k *versus* B_k.

3. The subalgebra σ_x generated by $x \in A,\ x \neq 0$

Because of the power associativity of $\times$ in A, the power x^n is defined for $n \in \mathbb{N}$, and the subalgebra σ_x, generated in A by $x \neq 0$, is well-defined. We study its properties on $\mathbb{R}$ and on $\mathbb{Z}_2$ successively.

3.1 The subalgebra σ_x on $\mathbb{R}$

We set $x = \alpha + X$, $\alpha = \Re x = \frac{1}{2}(x + \bar{x})$, and $\|x\|^2 = x\bar{x} = \alpha^2 + \|X\|^2$. We suppose first x non real: $X = \Im x \neq 0$. The linear map $y \to x \times y$ can be represented in the basis $\{1, x\}$ by the 2×2 matrix

$$J_x = \begin{pmatrix} 0 & -\|x\|^2 \\ 1 & 2\alpha \end{pmatrix},$$

for $X \neq 0$.

J_x has the two complex conjugate eigenvalues λ, and $\bar{\lambda}$, with $\lambda = \alpha + i\|X\|$; its spectral radius is $\rho(J_x) = |\lambda| = \|x\|$. J_x is diagonalisable for any non real λ, i.e. $\lambda^2 \neq 1$.

The behaviour of J_x^n, as $n \to \infty$, is ruled by its eigenstructure and by $\|x\| = |\lambda|$:

a) $\|x\| = |\lambda| < 1$ (resp. > 1) implies convergence to 0 (resp. divergence to infinity),

b) when $\|x\| = |\lambda| = 1$, we set $e^{i\theta} = \lambda$, with $\alpha = \cos\theta = \Re x$. The behaviour of λ^n depends whether θ is commensurate with 2π, or not:

  i) If $\theta/2\pi$ is irrational, λ^n describes all the unit circle.

  ii) If $\theta/2\pi$ is rational, $\neq 0$, the behaviour is cyclic. The period is determined by the ratio $\theta/2\pi$. As an example, the following 4 values for $\alpha = \Re x = \cos\theta$ give a cyclic behaviour:

α	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\theta = \text{Arccos}\,\alpha$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0
period	4	6	8	12	1

If we choose, for example, x to be an imaginary unit $x = X/\|X\| = I_X$, such that $\|x\| = \|I_X\| = 1$ and $\alpha = \Re x = 0$, then $\theta = \pi/2 = 2\pi/4$: the sequence J_x^n is 4-cyclic. In the basis $\{1, I_X\}$, the subalgebra σ_x is isomorphic to $\mathbb{C}$.

$$J_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has the eigenvalues $\pm\, i$. $I_X \in A$ is the multidimensional analogue of the 2D vector $i \in \mathbb{C}$. Notice that J_x is orthogonal ($J_x\, J_x^T = I$) iff $x = I_X$ (hence $\|J_x\|_2 = 1$).

When $\|x\| = 1$ with $\alpha \neq 0$, then $1 < \|J_x\|_2 \leq \sqrt{3 + 2\sqrt{2}}$, the largest singular value for J_x.

Another remarkable value for $\alpha = \Re x$ is $\alpha = 1/2$: it yields $J_x^3 = -I$ and $J_x^6 = I$. Therefore, although J_x is not orthogonal ($\alpha \neq 0$ and $\|J_x\|_2 = \sqrt{\frac{3}{2} + \sqrt{3}}$), all its powers of the form $3n$, $n \in \mathbb{Z}$, are orthogonal and therefore isometric.

A similar property is valid for any $\alpha = \cos\theta$ with $\frac{\theta}{2\pi} \in \mathbb{Q}$.

The value $\alpha = 1$ is exceptional because $X = 0$ and $x = 1$ is the real unit ($\theta = 0 \bmod 2\pi$). Therefore $x^n = 1$ for $n \in \mathbb{N}$: 1 is a fixed point.

More generally, when x is real ($x = \alpha$, $X = 0$), the linear map $y \to x \times y$ can be represented by the *scalar* x in the basis $\{1\}$: the 2×2 matrix J_x becomes 1×1. We are back to the classical problem of the convergence of $x^n,\ x \in \mathbb{R}$.

The real units $x = \pm 1$ yield $\lambda = \pm 1$ respectively as *double* eigenvalues. The corresponding matrices are:

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad J_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.$$

The respective Jordan decompositions are

$$\begin{aligned} J_1 &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \\ J_{-1} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

For any non real $\lambda = e^{i\theta}$, $\lambda^2 \neq 1$, one has the *eigen* decomposition associated with the pair $(\lambda, \bar{\lambda} = \frac{1}{\lambda})$

$$J_x = \frac{1}{1 - \lambda^2} \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

for x normalised and non real.

It is interesting to observe that the real units ± 1 (which can classically be treated in $\mathbb{R}$, that is as scalars with 1D) lead to a *Jordan* form, rather than a *simple* form, when they are treated in 2D. When their potential geometric structure in A_k, $k \geq 1$, is taken into account, they express a "resistance at being treated as scalars".

3.2 The subalgebra σ_x on $\mathbb{Z}_2$

The situation simplifies drastically when $x \in B_k$ (on $\mathbb{Z}_2$) for $k \in \mathbb{N}^*$, because x^2 can only take the two values 0 or 1 for any $x \neq 0$. Therefore

i) either $x^2 = 1$, and the sequence x^n alternates between 1 and x for $x \neq 1$, *ii)* or $x^2 = 0$, and x^n stabilises at 0 at the second iteration, when $x \neq 0$.

It follows that, for $x \neq 0, 1$, σ_x can have one of the two structures:

i) $\sigma_x = \{1, x\}$: successive multiplication cycles with period 2, or

ii) $\sigma_x = \{0, x\}$: multiplication converges to 0 in 1 iteration.

When $x = 0$ or 1, we get obvious fixed points. If we consider the even/odd partition $B = E \cup O$ introduced in [6], we get

i) $\sigma_x = \{1, x\} \Leftrightarrow x \in O \Leftrightarrow x^2 = 1$,

ii) $\sigma_x = \{0, x\} \Leftrightarrow x \in E \Leftrightarrow x^2 = 0$.

The divergence to infinity, which occurs on $\mathbb{R}$ when $\|x\| > 1$, has no analogue in the context of $\mathbb{Z}_2$: we get a fixed point at 0 with at most one transient step (for $x \in E$), or a 2-cycle (for $x \in O$) in general.

4. Algebraic equations in A_k and B_k, $k \geq 1$

We first look at linear equations and review the resolution of $a \times x = b$ for $a \neq 0$, b given in A_k or B_k.

4.1. Resolution of $a \times x = b$ in A_k (on $\mathbb{R}$)

Associated with left (resp. right) multiplication by $a \neq 0$, we define the linear map L_a (resp. R_a): $A_k \to A_k$ by

$$L_a : x \to a \times x \ \ (\text{resp. } R_a : x \to x \times a).$$

L_a and R_a are skew-symmetric for $a \in \Im A_k$, $k \geq 1$, and $L_a \neq R_a$ for $k \geq 2$. However Ker L_a = Ker R_a for $k \geq 0$ by Corollary 2.6. We know that $a \neq 0 \Leftrightarrow$ Zer(a) = Ker $L_a = \{0\}$ for $k \leq 3$. What about $k \geq 4$? We briefly survey results given in [5].

Lemma 4.1 *If a is alternative, $a \neq 0$ in A_k, then Ker $L_a = \{0\}$, for $k \geq 4$.*

Proof $[a, a, x] = -\|a\|^2 x - a \times (a \times x) = 0$ for any x. Therefore $a \times (a \times x) = -\|a\|^2 x$, and $a \times x = 0$ implies $x = 0$, i.e. $\mathrm{Zer}(a) = \mathrm{Ker}\ L_a = \{0\}$. $\square$

Proposition 4.2 *Let $a \neq 0$ be given in A_k, $k \geq 4$. If a is b-alternative, $b \neq 0$, $x = a^{-1} \times b$ is a particular solution for $a \times x = b$. If a is alternative, x is the unique solution for any b, and satisfies $\|x\| = \|b\|/\|a\|$.*

Proof We recall that $\mathcal{A}_k \subset A_k$ for $k \geq 4$. $[a, a, b] = -[a, \bar{a}, b] = 0$, therefore $x = a^{-1} \times b = \frac{\bar{a}}{\|a\|^2} \times b$ satisfies $a \times (\frac{\bar{a}}{\|a\|^2} \times b) = 1 \times b = b$, for $a \neq 0$. By linearity, all the solutions are of the type $x = a^{-1} \times b + z$, $z \in \mathrm{Zer}(a)$, that is of the type $x = \bar{a} \times b/\|a\|^2 + \mathrm{Zer}(a)$ [5]. If a is alternative, x is unique by Lemma 4.1. And $\|x\| = \frac{1}{\|a\|^2}\|a\|\|b\| = \frac{1}{\|a\|}\|b\|$. $\square$

Corollary 4.3

1) *For $k \leq 3$, $a \neq 0$ is the necessary and sufficient condition for the existence and uniqueness of the solution for $a \times x = b$.*

2) *For $k \geq 4$, a is b-alternative is a sufficient condition for the existence of a solution for $a \times x = b$. If a is alternative, $x = \frac{\bar{a}}{\|a\|^2} \times b$ is the unique solution for any b.*

Proof Clear. $\square$

The role of zerodivisors in $\Im A_k$, $k \geq 4$, should not be underestimated. For example, $I_a \times x = I_a$ for $a \in \Im A_k$ has the unique solution $x = 1$ for $k \leq 3$. For $k \geq 4$, however, the solutions are $1 + \mathrm{Zer}(a)$, with $\mathrm{Zer}(a) \subset \{I_a\}^\perp \cap \Im A_k$ which may not be $\{0\}$ when a is not alternative.

It is clear from above that, amongst the unit imaginary vectors in $\Im A_k$, the ones in $\mathcal{A}_k$ enjoy special properties. We describe some of them below.

Let $S_k = S^{2^k - 2} = S(\Im A_k)$ represent the unit imaginary sphere in $\Im A_k \sim \mathbb{R}^{2^k - 1}$ for $k \geq 1$. We look at the vectors on S_k which are alternative. $I = I_k$ denotes the identity map in A_k.

Proposition 4.4 *Let $a \in S_k$ be alternative. Then L_a is orthogonal and $L_a^2 = -I_k$. The powers L_a^n form a 4-cycle sequence, $n \in \mathbb{Z}$.*

Proof We already know that L_a is a bijection, and is skew-symmetric. We show that L_a is isometric. From

$$\begin{aligned} 0 &= \langle [a, a, x], x\rangle = \langle -x - a \times (a \times x),\ x\rangle \\ &= -\|x\|^2 - \langle a \times (a \times x),\ x\rangle = -\|x\|^2 + \|a \times x\|^2, \end{aligned}$$

we get $\|L_a x\| = \|x\|$ for any $x \in A_k$. Therefore $L_a^{-1} = L_a^T = -L_a$, and $L_a^2 = -I_k$. Iterating with L_a defines the 4-cycle $\{L_a\,, -I_k,\ -L_a,\ I_k\}$ for a alternative (resp. arbitrary) in S_k, when $k \geq 4$ (resp. $k = 1, 2, 3$). When a is not alternative, but alternates with x, the action on $x \in A_k$ defines the sequence $\{x,\ L_a x,\ -x,\ -L_a x\}$.

If we choose $x = a$ on S_k, we can drop the assumption that a and x alternate.

See paragraph 3.1: L_a has the 2×2 representation $J_a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the basis $\{1, a\}$ with $a = I_a$. The sequence $\{a,\ -1,\ -a,\ 1\}$ oscillates between $\Im A_k$ and $\mathbb{R}$ for $k \geq 2$ for $a \in S_k$.

For $k = 1$, $S_1 = S(i\mathbb{R})$ is reduced to $\{\pm i\}$. There are two possible orders on the unique set $\{\pm i,\ \pm 1\}$ of four complex points on the unit circle. $\square$

Let $a, b \in S_k$ be *alternative* and *orthogonal* ($\langle a, b\rangle = 0$). The subalgebra generated by the pair (a, b) is the 4D-vector space $\mathrm{lin}(1, a, b, a \times b)$, which is isomorphic to $A_2 = \mathbb{H}$ [16].

The alternative vectors are also useful to study the inductive construction of the set of zero divisors in A_k, for $k \geq 4$, because $A_3 = \mathbb{G}$ contains only alternative vectors. For a thorough study of the set of zero divisors in A_k, $k \geq 4$, the reader is referred to [16, 17, 19]. They form in $\mathbb{R}^{2^k-2} \subset \Im A_k$ a non generic set which is real algebraic [19].

4.2. Resolution of $a \times x = b$ in B_k (on $\mathbb{Z}_2$)

As is customary [5, 6], the situation is much simpler in the finite commutative and associative algebras B_k (on $\mathbb{Z}_2$) than in the finite dimensional nonassociative algebras A_k (on $\mathbb{R}$).

Proposition 4.5 *For $k \geq 1$, $a \times x = b$ has the unique solution $x = a \times b$ iff $a^2 = 1$. When $a^2 = 0$, $a \times b = 0$ is a necessary condition for the existence of solutions which are never unique, when they exist.*

Proof

1) $a^2 = 1 \;\; \Rightarrow \;\; a^{-1} = a$ and $x = a \times b$, and $a \times x = 0$ implies $x = 0$.

2) $a^2 = 0 \;\; \Rightarrow \;\; a \times (a \times x) = 0 = a \times b$. Therefore a solution x can exist only if $b \in Z(a) = \{y; a \times y = 0\}$, the set of zero divisors for a. See [6] for more. $\square$

For $a = 0$, $Z(0) = B_k$. For any $a \neq 0$, $a \in E_k$ the even part of B_k ($\Leftrightarrow a^2 = 0$ [6]), $Z(a) \subseteq E_k$. The inclusion is strict for $a \neq \mathbf{e}$, the vector of all ones in B_k, and $Z(\mathbf{e}) = E_k$.

4.3. About extensions of the Fundamental Theorem of Algebra, from $A_1 = \mathbb{C}$ to A_k, $k \geq 2$, $K = \mathbb{R}$

The d'Alembert-Gauss theorem plays a pivotal role in Algebra, as well as in real and complex Analysis. The Fundamental Theorem of Algebra (FTA) states that any polynomial in $\mathbb{C}$ of degree $n > 0$ has at least one zero in $\mathbb{C}$. Besides the guarantee of *existence*, it gives the *number* and *form* of all the zeros: they are n complex numbers. Therefore, it serves to prove that $\mathbb{C}$ is the *algebraic* closure of $\mathbb{R}$ (itself the *topological* closure of $\mathbb{Q}$). No purely algebraic proof has been found yet. One elegant proof relies on Liouville's theorem: an holomorphic function in $\mathbb{C}$ which is bounded everywhere is constant. The reader is referred to [8], Chapter 4, pp. 91–117, for an illuminating historical account of the FTA.

Because numbers in $\mathbb{R}$ and $\mathbb{C}$ extend naturally in real hypercomplex algebras, it is quite natural to look at the question of how the roots of polynomials survive the geometric extension from $A_1 = \mathbb{C}$ to A_k, $k \geq 2$.

The first difficulty encountered is **noncommutativity** in $A_2 = \mathbb{H}$. A monomial of degree n can be of the form $x^{n-m} \times c \times x^m$ for any *non real* coefficient c, $m < n$. And there is no more guarantee for existence of solutions:

$$1 + c \times x - x \times c \; = \; 1 + [c, x] \; = \; 0$$

has no solution in A_k, $k \geq 2$, for $c \in A_k$. It is equivalent to the two equations

$$\left\{ \begin{array}{rcl} 1 & = & 0 \quad \text{in } \mathbb{R}, \\ [c,x] & = & 0 \quad \text{in } \Im A_k, \end{array} \right.$$

and the first equation is impossible: $1 \neq 0$.

Two types of extension have been considered for insuring existence:

a) an *algebraic* extension in A_2 with p being the *monic* polynomial: $x \mapsto p(x) = x^n + q(x)$, where q is a polynomial of degree $< n$. See the Eilenberg-Niven extension (1944) of FTA to $A_2 = \mathbb{H}$, in [8], pp. 213–214,

b) a *topological* extension in A_k, $k \geq 2$ with p being the quasi monomial: $x \mapsto p(x) = x^n \times (1 + g(x))$, where $g : A_k \setminus \{0\} \to A_k$, $k \geq 2$ is continuous such that $\|g(x)\| \to 0$ as $\|x\| \to \infty$. See Moreno [18].

In both cases, the proof considers the continuous extension $\hat{p}$ of p into $S^{2^k} = A_k \cup \infty = S(\mathbb{R}^{2^k+1})$ for $k \geq 2$. Therefore $\hat{p}$ is continuous from S^{2^k} to S^{2^k} with $\hat{p}(\infty) = \infty$. It has a topological degree $n \neq 0$, hence it is onto: $\hat{p}(A_k) = A_k$. Therefore there exist roots in A_k. See [8] and [18] for more details.

The particular case of polynomials with *real* coefficients falls into the category of quasimonomials [18].

Lemma 4.5 *The monic polynomial*

$$p(x) = x^n + \sum_{i=0}^{n-1} \beta_i \, x^i, \quad \beta_i \in \mathbb{R}, \quad x = \alpha + X \in A_k, \quad \text{and} \quad X \neq 0, \quad k \geq 2,$$

has exactly n roots in each plane lin $(1, I_X)$, *with* $I_X = \frac{X}{\|X\|}$. *When n is odd, there is at least one real root.*

Proof We saw in Section 3 that in the basis $\{1, I_X\}$, the subalgebra σ_x is isomorphic to $\mathbb{C}$. The FTA in $\mathbb{C}$ applies readily to $p(x)$ with real coefficients. The real axis is a symmetry axis for the roots: the non real roots come in pairs $(x = \alpha + X, \ \bar{x} = \alpha - X)$. When n is odd, there is at least one real root $x = \alpha$ $(X = 0)$. □

When I_X describes $S_k = S(\Im A_k)$ the unit imaginary sphere, the plane lin $(1,\, I_X)$ rotates around the real axis, the center for the algebra A_k. What happens to the roots? The real ones remain invariant. The pairs of non real roots $(x,\, \bar{x})$ describe the same circle centered at $\alpha = \Re x$ and having radius $\|X\| > 0$.

Proposition 4.6 *The roots $x = \alpha + X$ of the monic polynomial*

$$p(x) = x^n + \sum_{i=0}^{n-1} \beta_i \, x^i,$$

with real coefficients $\beta_i \in \mathbb{R}$, $x \in A_k$, $k \geq 2$ belong to the two categories:

i) *real roots x, with $x = \alpha \in \mathbb{R}$,*

ii) *conjugate pairs $(x,\, \bar{x})$ which describe in A_k the circle $(\alpha,\, \|X\|)$ for $x = \alpha + X$.*

When n is odd, there is always at least one real root.

Proof Clear. In general, there are an *uncountable infinity* of roots for $k \geq 2$. If the roots are *all real,* then there are n of them.

Observe that the circles in A_k are spheres in the imaginary affine variety passing through $(\alpha,\, 0)$. □

An important example is provided by the equation

$$p(x) \;=\; x^n + 1 \;=\; 0 \quad \text{in} \quad A_k, \quad k \geq 0. \tag{5}$$

All the roots for (5) with $n = 2, 3$, have been listed in [3] for $k = 0, 1, 2, 3$. The general formula for $k \geq 2$ requires to distinguish whether n is even or odd. For $n = 2m + 1$ (resp. $2m$) there are m circles plus the real root -1 (resp. m circles). The m circles in A_k are centered at $\cos \frac{j\pi}{n}\, 1$, and have radius $\sin \frac{j\pi}{n}$, for $j = 1, \ldots, m$.

It is instructive to contrast the characteristics of the solutions of (5) to that of the better known nth roots of 1, that is

$$x^n - 1 \;=\; 0 \quad \text{in} \quad A_k, \quad k \geq 0. \tag{6}$$

The factorisation $x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$ holds in A_k, therefore the real root 1 is *always present.* The real root -1 exists also for n even, $n = 2m$ in addition to $m-1$ circles. For $n = 2m+1$, there are m circles plus $\{1\}$.

We leave for a future report the study of the FTA on B_k.

5. The exponential function in A_k (on $\mathbb{R}$)

5.1. Motivation

By Lemma 2.1, an algebra A_k on $\mathbb{R}$, for $k \in \mathbb{N}$, is power associative: for any $a \in A_k$, and any $m, n \in \mathbb{N}^*$

$$a^m \times a^n = a^n \times a^m = a^{m+n}.$$

We suppose $a \notin \{0,1\}$. If we define $F_a : m \to a^m$ as a function $\mathbb{N}^* \to A_k$, power associativity for a can be rewritten as

$$F_a(m) \times F_a(n) = F_a(n) \times F_a(m) = F_a(m+n) \tag{7}$$

for $m, n \in \mathbb{N}^*$. This is a functional relation between $+$ and $\times$ which is fundamental in Scientific Computing [12, 13, 14, 15].

The relation (7) was already known to Archimedes for $k = 0$ ($a \in \mathbb{R}$). The case $k = 0$ (resp. 1) received a lot of attention during the 17th (resp. 18th) century, when the relation was gradually extended from $m, n \in \mathbb{N}^*$ to $x, y \in \mathbb{R}$. We briefly review this extension in the next two paragraphs.

5.2. The real exponential function

One way to introduce the real exponential function is to look for a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\left\{ \begin{array}{lcl} f(x+y) & = & f(x)\, f(y), \\ f'(0) & = & 1. \end{array} \right. \tag{8}$$

The set of conditions (8) *uniquely* defines the *positive* function $f_e : x \to f(x) = e^x$, where e is the real number such that $\ln e = 1$, or $\int_1^e dx/x = 1$. Observe that $e^0 = 1$.

1) More generally, the condition $f'(0) = l \in \mathbb{R},\ l \neq 0$ defines uniquely the *positive* function $f_a : x \to a^x$, with $a = e^l > 0$. Therefore $f_a(x) = a^x = e^{x \ln a}$ satisfies $f'(0) = \ln a = l \neq 0$.

2) Can we define a^x, with $a < 0$?

Such a definition requires to consider a *complex* multiplication rather than the real one which occurs in (8). We look for $F : \mathbb{R} \to \mathbb{C}$, with $F = f + ig,\ \ f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} F(x+y) & = & F(x) \times F(y), \\ F'(0) & = & i. \end{array} \right. \tag{9}$$

This yields for f, g the 4 conditions:

$$\left\{ \begin{array}{rcl} f(x+y) & = & f(x)\, f(y) - g(x)\, g(y) \\ g(x+y) & = & f(x)\, g(y) + f(y)\, g(x) \\ f'(0) & = & 0 \\ g'(0) & = & 1 \end{array} \right. \tag{10}$$

which characterise uniquely ([11], p. 309) the cosine and sine functions. This yields the celebrated Euler formula:

$$F(x) = f(x) + ig(x) = \cos x + i \sin x = e^{ix}, \quad x \in \mathbb{R}$$

where F has periodicity 2π ($e^{2i\pi} = 1$) and modulus $1 = e^0$. Therefore, the initial condition for $x = 0$ should be understood as $x = 0 \bmod 2\pi$. This is a periodic condition of the form

$$F'(2q\pi) = i, \quad q \in \mathbb{Z}.$$

To define a^x when $a < 0$, we set $a = -|a|$, hence

$$a^x = (-|a|)^x = (e^{i\pi}|a|)^x = e^{i\pi x}|a|^x$$

is a well defined complex function $\mathbb{R} \to \mathbb{C}$ with

$$e^{i\pi x} = \cos \pi x + i \sin \pi x.$$

It agrees with $|a|^x$ for $x = 2q,\ \ q \in \mathbb{Z}$.

5.3. The complex exponential function

It can be shown [21], p. 33, that f *continuous* $\mathbb{C} \to \mathbb{C}$ such that

$$f(z + z') \quad = \quad f(z) \ \times \ f(z')$$

for any $z, z' \in \mathbb{C}$ has the form

$$f(z) \quad = \quad e^{Az+B\bar{z}},$$

with $A, B \in \mathbb{C}$. Observe that $z \mapsto e^{B\bar{z}}$ is not derivable for $B \neq 0$.

In Section 5.2, the variable x is real, and a^x is real positive (resp. complex) for $a \in \mathbb{R}^+$ (resp. $\mathbb{R}^-$). We now turn to $z \to a^z$, with $a, z \in \mathbb{C}$. We still assume $a \notin \{0, 1\}$.

Let be given $a = \rho e^{i\theta}$, with $l = \ln \rho \in \mathbb{R}$. For $z = x + iy,\;\; x, y \in \mathbb{R}$, one gets:

$$a^z = e^{z \ln a} = e^{xl - \theta y}\, e^{i(\theta x + ly)}.$$

The periodic conditions satisfied by $z \to a^z$ is given by the

Lemma 5.1

$$\left(\frac{d}{dz} a^z\right)(z_q) = \ln a \quad \textit{for} \quad z_q = 2i\pi\, \frac{q}{\ln a}, \quad q \in \mathbb{Z}.$$

Proof $\frac{d}{dz} a^z = (\ln a)\, a^z$, with $\ln a = l + i\theta$. Note that $a \neq 1$ ensures $\ln a \neq 0$. We need to look for z such that $a^z = 1$, that is

$$\left\{ \begin{array}{rcl} xl - \theta y & = & 0, \\ \theta x + ly & = & 2\pi q, \quad q \in \mathbb{Z}. \end{array} \right.$$

The obvious solution $z = 0$ corresponds to $q = 0$. For $q \in \mathbb{Z}^*$, one gets

$$\left\{ \begin{array}{l} \frac{x}{\theta} = \frac{y}{l} = \frac{1}{\lambda} \\ \theta^2 + l^2 = 2\pi q \lambda \end{array} \right.$$

where $\lambda \neq 0$ belongs to the denumerable sequence of real numbers

$$\left\{ \lambda_q = \frac{|\ln a|^2}{2\pi q}, \;\; q \in \mathbb{Z}^* \right\}.$$

This implies

$$z_q = x_q + iy_q = \frac{2\pi q}{|\ln a|^2}\,(\theta + il) = \frac{2\pi q}{\theta^2 + l^2}\,(\theta + il) = 2i\pi\, \frac{q}{\ln a}$$

for $q \in \mathbb{Z}$. $\square$

Observe that z_q is real (resp. pure imaginary) if $\ln a$ is pure imaginary (resp. real). In particular, $\ln a = i$ (resp. $\ln a = 1$, $a = e$) yields $z_q = 2\pi q$ real (resp. $z_q = 2\pi i q$ pure imaginary) as expected from Section 5.2.

A classical application of the complex exponential is the representation

$$z \quad = \quad a + ib \quad = \quad \rho\,(\cos\xi + i\sin\xi) \quad = \quad \rho\, e^{ib}$$

where $\rho = \sqrt{a^2+b^2} > 0$, $\xi = \measuredangle(1,z)$, and $\theta = \measuredangle(1,e^{ib})$ represent respectively the arguments in $\mathbb{C}$ of z and e^{ib} which are defined mod 2π: $\theta \equiv b \pmod{2\pi}$.

The Euler formula yields the identification $\theta = \xi$. The two values $\theta = b \pmod{2\pi}$, and ξ which satisfies $\sin\xi = b/\rho$ and $\cos\xi = a/\rho$, are identified computationally. What are the possible exceptions to this identification?

1. When $\alpha = 0$, $\xi = \pm\pi/2$ for any $b \neq 0$ and $\sin\xi = b/|b|$ is independent of θ. The identification $e^{i\theta} = \pm i$ imposes $b \equiv \pi/2 \pmod{\pi}$.

2. When $b = 0 \pmod{\pi}$, $\theta = 0$ or π and e^z is real.

For $b \neq 0$, ξ is necessarily $\neq 0$ or π. For $b = 0$ and $a \neq 0$, $\xi = \frac{\pi}{2}(1 - \operatorname{sgn} a) = 0$ or π independently of θ.

The question "Is the computational identification $\xi = \theta$ logically required?" has 3 possible answers listed in Table 5.1.

Condition	Answer
$a \neq 0$ and $b \not\equiv 0 \pmod{\pi}$	yes
$a = 0$ and $b \not\equiv 0 \pmod{\pi}$	no, unless $b \equiv \pi/2 \pmod{\pi}$
$b \equiv 0 \pmod{\pi}$	no

Table 5.1 $\xi \equiv ?\, b \pmod{2\pi}$ *for* $z = a + ib$ *in* $\mathbb{C}$

The exceptions to Euler's law are nongeneric. They correspond to $a = 0$ (that is, z pure imaginary with $b \not\equiv \pi/2 \pmod{\pi}$) and to $b \equiv 0 \pmod{\pi}$ (that is, e^z real).

They are not robust to perturbations (such as these created by finite precision computation). They disappear by a simple continuity argument. In other words, continuity enforces Euler's law.

5.4. Formal definition of a^x in A_k, $k \geq 2$

5.4.1. e^x in A_k, $k \geq 2$

We begin by the definition of e^x for $x = \alpha + X$, $X \neq 0$, in A_k. The series expansion $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$ shows that formally

$$e^x = e^\alpha(\cos b + \sin b\, I_X) = e^\alpha\, e^X \tag{11}$$

where $b = \|X\| \in \mathbb{R}^+$, $I_X = \frac{X}{\|X\|} = \frac{1}{b}X$, and $e^X = e^{bI_X}$ has norm 1 [10, 18]. This is the general form of the Euler formula in A_k, $k \geq 2$. Because of the period 2π for sine and cosine in (11), we define $b = \|X\| = 2n\pi + \theta$, with $n \in \mathbb{N}$ and $0 \leq \theta < 2\pi$, so that $\|X\| \equiv \theta \pmod{2\pi}$ where θ is the *real* positive remainder of the *integral* division $\|X\|/2\pi$.

For $t \in \mathbb{R}$, the formula $e^{tX} = \cos t\theta + \sin t\theta\, I_X$ is easily derived [10, 18]. This relaxes the condition $\theta \in [0, 2\pi]$ since $t\theta$ can describe $\mathbb{R}$.

The identity $e^{\theta I_X} = \cos\theta + \sin\theta I_X = e^X$ shows that the argument of e^X, that is $\measuredangle(1, e^X)$ in $\mathrm{lin}(1, I_X)$ for $X \neq 0$, is specified by $\|X\| \pmod{2\pi}$: θ describes the position of the unit vector e^X.

To study the relationship between $\xi = \measuredangle(1, x)$ and θ in this context, it is useful to introduce $\varphi = \|X\| \pmod{\pi}$, that is $\varphi = \theta$ (resp. $2\pi - \theta$) iff $0 \leq \theta \leq \pi$ (resp. $\pi \leq \theta \leq 2\pi$).

There are 5 cases to consider for $x \neq 0$, depending whether α, φ and X are 0 or not. Note that $\varphi = 0$ iff e^x is real.

Euler formula in $\mathrm{lin}(1, I_X)$ yields the identification $\xi = \varphi$ for α and $\varphi \neq 0$. We review below the 4 cases where a *decoupling* between φ and ξ can take place:

1. $\alpha \neq 0$ and $\varphi = 0$, $X \neq 0$: $(\xi \neq 0) \not\Leftrightarrow (\varphi = 0)$,
2. $\alpha \neq 0$ and $\varphi = 0$ with $X = 0$: $\big(\xi = \frac{\pi}{2}(1 - \mathrm{sgn}\,\alpha)\big) \not\Leftrightarrow (\varphi = 0)$,
3. $\alpha = 0$ and $\varphi \neq 0$: $\xi = \pi/2 \neq \varphi$, unless $\|X\| \equiv \pi/2 \pmod{\pi}$,
4. $\alpha = 0$ and $\varphi = 0$, $X \neq 0$: $(\xi = \pi/2) \not\Leftrightarrow (\varphi = 0)$.

The quantisation $\|X\| \equiv \pi/2 \pmod{\pi}$ restores $\xi = \varphi$ in case (3). In the other cases, decoupling is a logical possibility which is not computationally robust.

The question "Is the computational identification $\xi = \varphi$ logically required?" has again 3 possible answers listed in Table 5.2.

Condition	Answer
$\alpha \neq 0$ and $\varphi \neq 0$	yes
$\alpha = 0$ and $\varphi \neq 0$	no, unless $\varphi = \pi/2$
$\varphi = 0$	no

Table 5.2 $\xi \equiv ?\ \|X\| \pmod{\pi}$ *for* $x = \alpha + X$ *in* $A_k,\ k \geq 2$

Decoupling between ξ and φ takes place, either when e^x is real ($\varphi = 0$), or when $x = X$ is pure imaginary ($\alpha = 0$). The answer $\xi = \varphi$ is the only solution which is robust to perturbation.

The main difference between $\mathbb{C}$ and A_k, $k \geq 2$ is that $b \in \mathbb{R}$ in case $\mathbb{C}$ is replaced by $b = \|X\| \in \mathbb{R}^+$ in case A_k. The important consequence is that $\xi = \|X\| \pmod{\pi}$, and not mod 2π: θ is replaced by φ. In conclusion, Euler's law takes the following form for $x = \alpha + X$ in A_k, $k \geq 0$, $\xi = \measuredangle(1, x)$:

$$\boxed{\begin{array}{l} \xi \text{ is not related to } X = 0 \text{ in } \mathbb{R}, \\ \xi \equiv \|X\| \pmod{2\pi} \text{ in } \mathbb{C}, \\ \xi \equiv \|X\| \pmod{\pi} \text{ in } A_k,\ k \geq 2. \end{array}}$$

This law specifies the robust connection between the unit vectors e^X and $\mathbf{1}_x = x/\|x\|$ in $\text{lin}(1, I_X)$ for $x = \alpha + X$, $X \neq 0$. The two vectors are *identical* in $\mathbb{C}$, but not necessarily in A_k, $k \geq 2$, where $e^X = \mathbf{1}_x$ iff $0 \leq \theta \leq \pi$, and $e^X = \mathbf{1}_{\bar{x}} = \mathbf{1}_{x^{-1}}$ iff $\pi \leq \theta \leq 2\pi$.

When $\alpha = 0$, theory imposes the condition $\|X\| \equiv \pi/2 \pmod{\pi}$ in all A_k, $k \geq 1$. This can be seen as a re-emergence by exponentiation of the discrete from the continuous in $\Im A_k$.

Remark 5.1 $\sin\varphi = |\sin\theta|$ is a continuous function of θ whose derivative is discontinuous for $\theta = 0$ and π (it jumps from -1 to 1). This happens when e^x is real.

5.4.2. a^x in A_k, $k \geq 2$

We turn to $a^x = e^{x \times \ln a}$, with $v = \ln a = l + U \neq 0$. By (4), we write

$$(\alpha + X) \times (l + U) \quad = \quad \alpha l - \langle X, U \rangle + lX + \alpha U + \frac{1}{2}[X, U].$$

We set
$$\begin{array}{rcll} r(x) & = & \alpha l - \langle X, U\rangle & \in \mathbb{R}, \\ L(x) & = & lX + \alpha U & \in \mathrm{lin}(U, X) \subset \Im A_k, \\ C(x) & = & \frac{1}{2}[X, U] & \in \Im A_k. \end{array}$$

$L(x)$ describes the linear part of the product, in $\Im A_k$. $C(x)$ indicates the departure from commutativity, it is always zero for $k = 0, 1$.

Lemma 5.2 *C is orthogonal with X and U, hence with L.*

Proof By Corollary 2.6, $\langle U,\ X \times U\rangle = \langle U \times \bar{U},\ X\rangle = \|U\|^2 \langle 1, X\rangle = 0$. Similarly $\langle X,\ X \times U\rangle = \langle \bar{X} \times X,\ U\rangle = -\|X\|^2 \langle 1, U\rangle = 0$, as well as $\langle U,\ U \times X\rangle = \langle X,\ U \times X\rangle = 0$. As a consequence, $\langle C, L\rangle = 0$ for $L = \alpha U + lX$. □

We can write $a^x = e^{x \times \ln a} = e^{r(x) + L(x) + C(x)} = e^{r(x)}\, e^{L(x) + C(x)}$. This raises the obvious question in $\Im A_k$, $k \geq 2$:

> Does the fundamental property for the exponential function
> $$e^{X+Y} = e^X \times e^Y \tag{12}$$
> which is valid in $\Im A_k$, $k = 0$ and 1, remain valid for arbitrary X and Y in $\Im A_k$, $k \geq 2$?

A necessary condition for (12) is obviously the commutativity property
$$e^X \times e^Y = e^Y \times e^X \tag{13}$$
since $X + Y = Y + X$. Noncommutativity of the multiplication in A_k for $k \geq 2$ makes (13) unlikely to be always satisfied for $X, Y \in \Im A_k$. We study (12) and (13) in the next paragraph.

5.5. e^X for $X \in \Im A_k$, $k \geq 2$

We first look at the necessary condition (13) for (12).

5.5.1. $[e^X, e^Y] = 0$ for $X, Y \in \Im A_k$, $k \geq 2$?

Let X, Y be nonzero vectors in $\Im A_k$. We set $\theta = \|X\|$, $\psi = \|Y\| \pmod{2\pi}$.

Lemma 5.3

$$[e^X, e^Y] \quad = \quad \frac{\sin\theta}{\theta}\,\frac{\sin\psi}{\psi}\,[X, Y].$$

Proof The relations

$$e^X = \cos\theta + \sin\theta\,\frac{X}{\theta}$$

and

$$e^Y = \cos\psi + \sin\psi\,\frac{Y}{\psi}$$

establish the result with

$$[e^X, e^Y] = e^X \times e^Y - e^Y \times e^X.$$

$\square$

Corollary 5.4 *e^X and e^Y commute iff at least one of the following conditions below is satisfied:*

i) $[X, Y] = 0$,

ii) $\sin\theta = 0$ *with* $\theta > 0$,

iii) $\sin\psi = 0$ *with* $\psi > 0$.

Proof Clear from Lemma 5.3, $\sin\theta = 0 \Leftrightarrow \theta = 0 \pmod{\pi}$. Together with $\theta > 0$, this yields $\theta = n\pi$, $n \in \mathbb{N}^*$. We recall that for $k \geq 4$, $[X, Y] = 0 \Leftrightarrow Y = tX + Z$ with $X \times Z = Z \times X = 0$, whereas $Z = 0$ for $k = 2, 3$. $\square$

5.5.2. Sufficient conditions for $e^{X+Y} = e^X \times e^Y$, $\quad X, Y \in \Im A_k$

We distinguish first whether X and Y are dependent or not.

Lemma 5.5 *If X and Y are dependent, then*

$$e^{X+Y} = e^X \times e^Y.$$

Proof The result is obvious if X or $Y = 0$. We suppose that X and Y are nonzero. There exists $t \in \mathbb{R}$, $t \neq 0$ such that $Y = tX$, then

$$e^Y = e^{tX} = \cos t\theta + \sin t\theta \, \frac{X}{\theta}.$$

And an easy calculation shows that

$$e^X \times e^Y = e^{(1+t)X} = e^{X+Y},$$

cf. [10, 18]. $\square$

$Y = tX$ implies $[X, Y] = 0$ by Proposition 2.4. Is it possible to obtain (12) without colinearity for X, Y? Corollary 5.4 suggests that this might be possible, for example, with $\|X\| = n\pi$ or $\|Y\| = m\pi$, $\ n, m \in \mathbb{N}^*$, for which $e^X = (-1)^n$, $e^Y = (-1)^m$. We intend to study the general case in a future work.

Lemma 5.6 *Let X and Y be independent vectors such that*

$$\pi = \frac{\|X\|}{a} = \frac{\|Y\|}{b} = \frac{\|X+Y\|}{c}$$

where $a, b, c \in \mathbb{N}^*$. Then $e^X = (-1)^a$, $e^Y = (-1)^b$ and $e^{X+Y} = (-1)^c$ satisfy $(-1)^c = (-1)^{a+b}$ iff c is even (resp. odd) when a and b have equal (resp. different) parity. If the common ratio is 2π, there is no restriction on a, b, c.

Proof Clear. $\square$

The Lemma is a direct generalisation to the triple $(X,\, Y,\, X+Y)$ in $\Im A_k$, $k \geq 2$ of *Euler's formula* $e^{in\pi} = (-1)^n$, $n \in \mathbb{Z}$, which is valid for the single vector i in $\mathbb{C}$. It is not easy to predict the form of a general triangle in the plane lin (X, Y) so that, by homothety, it can have integral sides.

One obvious case is the *equilateral* triangle with $a = b = c = n,\ n \in \mathbb{N}^*$. Then the common ratio 2π implies $e^X = e^Y = e^{X+Y} = 1$. This corresponds to the angle between lin (X) and lin (Y) being equal to $\pi/3 \pmod{\pi}$.

Another case of importance corresponds to this angle being equal to $\pi/2$ (mod π): X and Y are orthogonal in a *right-angled* triangle. We study this case below.

Lemma 5.7 *If X and Y anticommute (that is, if $\langle X, Y\rangle = 0$) with*

$$\pi = \frac{\|X\|}{n} = \frac{\|Y\|}{m} = \frac{\|X+Y\|}{q},$$

where $n, m, q \in \mathbb{N}^*$ satisfy

$$n^2 + m^2 = q^2, \tag{14}$$

then

$$e^X = (-1)^n,\ \ e^Y = (-1)^m \ \ \textit{and}\ \ e^{X+Y} = (-1)^q,$$

satisfy

$$(-1)^q = (-1)^n(-1)^m = (-1)^{n+m}.$$

Proof X and Y anticommute $\Leftrightarrow [X, Y] = 2X \times Y \Leftrightarrow \langle X, Y\rangle = 0$. The result follows directly from the Pythagorean theorem. $\square$

The solutions of (14) in $\mathbb{N}^*$ are the celebrated *Pythagorean triples*, known since Sumerian times, in the 3rd Millennium B.C. It is a pleasant surprise to see them play a role in the generalization of the exponential function from $\Im\mathbb{C}$ to $\Im A_k$, $k \geq 2$.

In $\Im\mathbb{C}$, $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$ entails a periodicity of 2π expressed by $e^{2ni\pi} = 1$, $n \in \mathbb{Z}$. And $e^{X+Y} = e^X \times e^Y$ for any $X, Y \in \Im\mathbb{C}$. In sharp contrast with $k=1$, in $\Im A_k$ for $k \geq 2$, one does not have commutativity for e^X, e^Y for any independent vectors X, Y in $\Im A_k$ of arbitrary norm, by Corollary 5.4. Moreover, $e^X \times e^Y = e^{X+Y}$ holds for pairs of vectors whose norms satisfy appropriate conditions such as the assumptions of Lemmas 5.6 or 5.7 (X and Y are orthogonal).

We suppose that m, n, q are relatively prime with each other. Therefore q and either m or n are odd integers; the third number is even. We treat the general case in the

Proposition 5.8 *Let (n, m, q) be a Pythagorean triple such that*

$$n^2 + m^2 = q^2, \quad n, m, q \in \mathbb{N}^*.$$

$e^{X+Y} = e^X \times e^Y$ is satisfied for X, Y nonzero in $\Im A_k$, $k \geq 2$, in any one of the four cases:

i) $Y = tX$, $t \in \mathbb{R}$, $t \neq 0$, with $e^{X+Y} = e^{(1+t)X}$,

ii) $\langle X, Y \rangle = 0$ and

$$\pi = \frac{\|X\|}{n} = \frac{\|Y\|}{m} = \frac{\|X+Y\|}{q}, \quad \textit{with} \quad e^{X+Y} = -1,$$

iii) X is a zerodivisor, $Y = tX + Z$, $t \in \mathbb{R}$, and $Z \neq 0$ in $\mathrm{Zer}(X)$, such that:

$$\pi = \begin{cases} \frac{|1+t|\,\|X\|}{n} = \frac{\|Z\|}{m} = \frac{\|X+Y\|}{q} & \textit{for } t \neq -1, \\ \frac{\|X\|}{n} = \frac{\|Y\|}{m} = \frac{\|Z\|}{q} & \textit{for } t = -1, \end{cases}$$

iv) X and Y are neither colinear nor orthogonal, with norms satisfying Lemma 5.6.

For $k = 2, 3$, case iii) vanishes.

Proof Cases i), ii) and iv) are covered by Lemmas 5.5, 5.6 and 5.7. For $t=-1$, $Y = -X = \bar{X}$ and $e^X e^{-X} = e^0 = 1$ real.

To prove iii), observe that Lemma 5.7 applied to X and $Z \in \mathrm{Zer}(X)$ is satisfied. Now we recall that

$$[X, Y] = 0 \Leftrightarrow Y = tX + Z, \quad t \in \mathbb{R}, \quad Z \in \mathrm{Zer}(X),$$

and we write $X+Y=(1+t)X+Z$. If $t=-1$, $X+Y=Z$.

When X is a zerodivisor, the coefficient $\frac{n}{|1+t|}$ can be *real* rather than integral.

For $k=2,3$, $Z=0$ and case *iii*) vanishes. Moreover, for case *ii*) the multiplication is isometric and there are no zero divisors other than 0. Therefore $[X,Y]=2X\wedge Y=2X\times Y$ and $\|[X,Y]\|$ assumes its maximum value $2\|X\|\|Y\|$, which is necessarily nonzero. And $\|[X,Y]\|=2\pi^2nm$, where $2nm$ cannot be a perfect square (Fermat). For $k=1$, cases *ii*) and *iv*) vanish also because X colinear with Y is the only option. For $k=0$, $X=Y=0$ and $e^0=1$. □

We now consider the particular case *iii*) with $t=1$ where the common ratio is 2π:

$$2\pi=\frac{2\|X\|}{n}=\frac{\|Z\|}{m}=\frac{\|X+Y\|}{q},\quad \text{and}\quad e^Z=e^{X+Y}=1.$$

By assumption $X\times Z=0$ and $Y=X+Z$.

Proposition 5.9 *When $\frac{\|X\|}{n}=\pi$, n even (resp. n odd), there are at least three (resp. two) ways to realise $e^X\times e^Y=e^{X+Y}$ for case iii) in Proposition 5.8 by Arithmetic only (in a quantised fashion).*

Proof According to Diophantus, one of the integers n,m is always even. We suppose that $n=2n'$. We consider all the divisors of n, including 1 and n. There are $\delta=d(n)$ such numbers with $\delta\geq 3$.

The smallest Pythagorean triple (3, 4, 5) is special because $\frac{n}{2}=2$ is prime hence $d=4,2$ and 1 are the only divisors. All the other triples yield more than three values for d. For example (5, 12, 13) yields, for $12=2^2\times 3$, the 5 values $d=12,6,3,2$ and 1.

Let d be one of the divisors of n. We define $X_d=\frac{1}{d}X$, which is colinear with X. Then

$$Y \;=\; X+Z \;=\; d\,(\frac{1}{d}X)+Z \;=\; dX_d+Z$$

satisifes $[X_d,\, Y] = 0$. Therefore $[e^{X_d},\, e^Y] = 0$:

$$e^{X_d} \times e^Y \;\; = \;\; e^Y \times e^{X_d}.$$

The conclusion follows since $e^{X_d} = e^X = 1$.

Observe that when n is chosen odd, there can be at least *two* ways, since $\delta = d(n) \geq 2$. There are exactly two ways iff n is a prime number. Summing up, there are at least **five** arithmetic ways to realise $e^X \times e^Y = e^{X+Y}$. $\square$

The exponential function $X \to e^X$, with $X \in \Im A_k, k \geq 2$, can satisfy the fundamental property (12)

$$e^{X+Y} = e^X \times e^Y$$

in two special situations:

i) either X and Y are *colinear,* $Y = tX$, $t \in \mathbb{R}$ and generate a continuous 1-dimensional axis,

ii) or X and Y are *orthogonal* and generate, in a 2-dimensional plane, the infinity of all possible Pythagorean triangles.

The first situation, which is 1D, is essentially *continuous,* whereas the second one is essentially *discrete* in a plane. The continuous situation is the only one existing in $\mathbb{C}$ ($k = 1$). The discrete situation, which appears for $k \geq 2$, is a consequence of the noncommutativity of the multiplication. It is *not* robust to perturbation.

It is important to realise that the *hybrid* case *iii*) which emerges when X is a zerodivisor for $k \geq 4$ is a combination of *i*) together with *ii*) restricted to zero divisors in $\mathrm{Zer}(X)$. The discrete and the continuous are strongly interconnected by the condition $\|X\| = \frac{n\pi}{|1+t|}$, $n \in \mathbb{N}^*$ and $t \in \mathbb{R}$, $t \neq -1$.

Case *iv*) corresponds to X and Y in $\Im A_k$ which are neither colinear nor orthogonal. The situation may have a nonempty intersection with case *iii*) for $k \geq 4$.

To better appreciate property (12), we compare it with the more conventional condition

$$e^{s(X+Y)} = e^{sX} \times e^{sY}$$

for any $s \in \mathbb{R}$ which allows to identify the series expansions of both sides. By looking at coefficients for s^2 and s^3, one gets the conditions $iv)$ of Proposition 2.8, which is equivalent to X and Y colinear.

Proposition 5.10 *The condition*

$$e^{s(X+Y)} = e^{sX} \times e^{sY} \tag{15}$$

for all $s \in \mathbb{R}$ implies that X and Y are colinear in $\Im A_k$. It excludes the possibility that X and Y be independent, in particular that $X \perp Y$.

Proof Clear by Proposition 2.8. $\square$

Remark 5.2 This should be compared with the analogous result valid for the matrix exponential:

$$e^{t(A+B)} = e^{tA}\, e^{tB}$$

for $t \in \mathbb{R}$ (or $\mathbb{C}$) implies that A and B commute, where A and B are square matrices of order n, on $\mathbb{R}$ (or $\mathbb{C}$). This is less demanding than X, Y colinear for (15).

Remark 5.3 The role of the Pythagorean triangles is fundamental for many reasons. Let us review four of them.

1) For $k \geq 4$, even if X and Y are not orthogonal, the relation $Y = tX + Z$ with $Z \in \mathrm{Zer}(X)$ uncovers the role of the triangle (X, Z) with $\langle X, Z\rangle = 0$.

2) For $k \geq 2$, the relation (4): $X \times Y = -\langle X, Y\rangle + \frac{1}{2}[X, Y]$ decomposes the product $X \times Y$ into the sum of its real part $-\langle X, Y\rangle$ and its imaginary part $\frac{1}{2}[X, Y]$ in $\Im A_k$. For $k = 2, 3$, the norm is isometric, hence $\|X \times Y\| = 1$ iff $\|X\|\,\|Y\| = 1$. The condition $[X, Y] = 0$ ensures that $X \times Y$ is real and assumes its maximal absolute value 1. The condition $\langle X, Y\rangle = 0$ appears as the dual condition under which $X \times Y$ is pure imaginary (rather than real) and its norm is maximal at 1.

These extremal properties are not valid for $k \geq 4$ since $\|X \times Y\| - \|X\|\,\|Y\| \not\equiv 0$. As a result, $\frac{1}{2}[X, Y]$ is *not* a vector product for an arbitrary pair (X, Y).

3) With two symmetrical Pythagorean triangles (n, m, q) sharing one side (not the hypotenuse) corresponding to m (resp. n), one constructs an isosceles triangle $(q,q,2n)$ (resp. $(q,q,2m)$) satisfying item *iv*) in Proposition 5.8. This corresponds, in Lemma 5.6, to $a = b = q$, $c = 2n$ (resp. $= 2m$); $(-1)^{2n} = (-1)^{2q} = 1$.

4) The hybrid case *iii*) for $k \geq 4$ allows $\|X\|$ to be an arbitrary real positive number for $t \neq -1$, when X is a zerodivisor.

Comparing Propositions 5.8 and 5.10 tells us that (15) is *more restrictive* than (12): (15) retains only the continuous situation which is robust to perturbation. It excludes the denumerable infinity of cases corresponding to the discrete situation: *ii*) or *iv*) for $k \geq 2$ complemented with the nondenumerable infinity of cases corresponding to the hybrid situation: *iii*) for $k \geq 4$, with nontrivial zero divisors Z and X. Such a discrete situation requires exact arithmetic.

5.6. The resurgence of the discrete from the continuous by hypercomplex exponentiation

The classical axiomatic construction of Numbers starts from the integers $\mathbb{N}$ (defined by the iterated addition: $n \to n+1$) to build the real line $\mathbb{R}$ (by closure of the rational numbers $\mathbb{Q}$), and goes on to construct the complex plane $\mathbb{C}$ (by closure of the algebraic numbers).

In this construction, *addition* is the basic operation. However, Scientific Computing reveals that many aspects of intensive computation are better explained by having *multiplication* as the focus of attention [6, 14, 15]. This complementary viewpoint is the dual of the standard axiomatic approach. Instead of considering the continuous as the limit of the discrete, one can equally well see the discrete as re-emerging from the continuous by exponentiation in Dickson algebras of Numbers of dimension ≥ 2 ($k \geq 1$). Indeed, why stop after the first two steps $k = 0$ and $k = 1$ of an endless inductive process?

A balanced view is possible, where the discrete is not a more primitive notion than the continuous, which is the currently fashionable scientific paradigm. It can be fruitful to consider the pair (discrete/continuous) as a pair of relative notions with a dual explanatory power: far from being contradictory notions of unequal importance, the two concepts of discreteness and continuity reinforce each other by furthering our understanding of both.

This view conveys the dynamics of the creative tension between the discrete and the continuous, always at work in Life's Computation.

6. Epistemological conclusion

There is a growing consensus amongst scientists to say that the world can be seen fruitfully as a giant Computer [5, 20]. However, as it has been argued in [14], the Turing thesis which is the basis for all modern approaches to Computability has epistemological limits. Such limits for an axiomatic approach to knowledge are exemplified by the incompleteness results of Chaitin [20], they follow from considering numbers of dimension 1 only [14]. But it has been known since the Greeks that Geometry provides natural numbers of dimension 2 which are not commensurable with 1D-numbers (the diagonal versus the side of a square).

Therefore, it is most natural to look at the hypercomplex numbers provided by the Dickson-Albert process. Why choose these particular sets of numbers? Precisely because of the *recursivity* of their construction of *multiplication.* And recursivity is essential in Computing. Another essential aspect of Computing is the tension between algebra and analysis, between the discrete and the continuous. This is why we consider real as well as binary algebras of numbers in parallel. And they exhibit a remarkable duality of behaviour. Binary algebras are finite and provide a wealth of algebraic identities. On the other hand, real algebras are continuous: the equations can be continuously modified.

This essential difference between binary (discrete) and real (continuous) algebras has fundamental epistemological consequences, which have been put into full light by the work of Chaitin by means of program-size complexity. Algebraic identities in B_k stand out by themselves. They cannot be continuously interrelated without being imbedded into a larger framework.[2] They are true for no other reason than themselves in B_k. They exemplify the notion of "irreducible mathematical facts" which was uncovered by Chaitin in the 1970's. His celebrated Ω number is the most famous example of such facts [20].

What is the freedom in geometric architecture which is gained by relaxing

[2] This may be done partially by Homotopic Deviation.

the various assumptions for calculation on real numbers? When k increases from 0 to 4, one looses successively reality, commutativity, associativity and alternativity. The trade-off appears clearly on the exponential function.

For $k \geq 2$, the property $e^{x+y} = e^x \times e^y$ which transforms multiplication into addition is not satisfied for arbitrary x and y in $\Im A_k$, because of lack of commutativity in general. The property can be satisfied in various ways. The two basic ways are obtained either i) by imposing commutativity to the product: $x, y \to x \times y$ ($k \geq 2$) together with isometry ($k \geq 4$), or ii) by imposing a geometric condition of orthogonality (i.e. anticommutativity), together with an unexpected metric constraint related to Pythagorean triples. A third way iii) is the combination of i) and ii) which involves zerodivisors appropriately normalized; this provides a synthesis between the discrete and the continuous which appears for $k \geq 4$ only (dimension ≥ 16). A mid-way between dependence and orthogonality is also possible.

Euclidean Geometry in higher dimensional vector spaces appears then as an essential component of Hypercomputation, by providing new concepts which can compensate partially for the severe lack of basic algebraic properties in hypercomplex algebras of dimension ≥ 16. A striking example of the creative power of multiplication is given by the **anisotropy** that it induces in euclidean spaces $\mathbb{R}^{2^k-1}$ of high enough dimension. This gives a rational basis for the existence of the "topological thorn planted in the algebraic flesh," so often encountered by Algebraists, as reported in [8, p. 233]. On the other hand, the algebraic flesh is nourished by classical Greek Arithmetic, as is illustrated in [15] and here by the unsuspected, yet fundamental, role played by the Pythagorean triples in the hypercomplex exponential function. The negative numbers -1 or $-e^r$ are the traces in $\mathbb{R}^-$ of hypercomplex exponentiation in A_k, $k \geq 2$. Pythagorean Arithmetic and zerodivisors for $k \geq 4$ give a clue for the intimate connection between the continuous and the discrete (either denumerable or finite) which is always present in Life's Computation. It is not necessarily always apparent to the naked eye, but it is experienced daily by every scientist practising Scientific Computation.

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