On Lidskii's algorithm to quantify the first order terms in the asymptotics of a defective eigenvalue. Part I

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Abstract

Lidskii's perturbation theory ($\epsilon \to 0$) is presented in algorithmic form. This clarifies the role of a singular step. When such a step is first encountered, the quantification becomes *incomplete* beyond the singularity in a way that is described. An application is made to Homotopic Deviation. In this particular case, the missing information can be provided by direct computation.

Key words: Jordan form, defective eigenvalue, Puiseux order, Schur complement matrix, Schur coefficient, Lidskii's generic condition, Homotopic Deviation

1. Introduction

In this short note, we give an explicit algorithmic formulation to the work of Lidskii [8]. By doing so, we state the precise outcome at a singular step. This question has received a lot of attention [2, 3, 9], but its analysis is not yet complete.

To keep this note short, we do not redefine all our notations: they are those of [9]. The reader unfamiliar with Lidskii's beautiful theory is urged to read the very clear expository paper by Moro, Burke, Overton (1997) [9]. For a complete treatment, see Baumgärtel [4].

2. Lidskii's algorithm

2.1. Definitions

$\lambda(\epsilon)$ is an eigenvalue of

$$J(\epsilon) \quad = \quad J + \epsilon B$$

where J is in Jordan form, B is a given matrix, both in $\mathbb{C}^{n\times n}$, and ϵ, the complex perturbation parameter, tends to 0. As $\epsilon \to 0$,

$$\lambda(\epsilon) \to \lambda \in \sigma(J)$$

with algebraic (resp. geometric) multiplicity m (resp. g). We assume that $\lambda = 0$ is defective ($g < m$).

Let n_j, $j = 1, \ldots, q$, $q \geq 1$ be the *different* sizes of the Jordan blocks for $0 \in \sigma(J)$ ordered by *decreasing* value

$$n_1 > n_2 > \cdots > n_q.$$

We remark that if $n_q = 1$, then $q \geq 2$. Each block of size n_j is repeated r_j times. We define

$$f_j \quad = \quad \sum_{i=1}^{j} r_i$$

with $f_q = g$ and

$$m_j \quad = \quad \sum_{i=1}^{j} n_i r_i$$

with $m_q = m$.

The aim of the algorithm is to determine when possible, for each of the m eigenvalues $\lambda(\epsilon)$ converging to zero, its order of convergence and its nonzero leading coefficient:

$$\lambda(\epsilon) \quad = \quad \xi\epsilon^p + o(\epsilon^p), \quad p \in \mathbb{Q}, \quad 0 \neq \xi \in \mathbb{C},$$

which is the first order term in the asymptotic expansion for $\lambda(\epsilon)$. It is known since Puiseux [4] that, when there is no interaction between the Jordan blocks, the possible exponents are the q rational numbers $1/n_i,\ i = 1, \ldots, q$. These exponents are called *Puiseux exponents*, they are *generic* [7].

2.2. The generic Lidskii process

The Lidskii algorithm constructs a sequence of imbedded matrices ϕ_j of increasing order f_j, starting from ϕ_1 of order $f_1 > 0$, and setting $f_0 = 0$. In a 2×2 block representation, this gives, for $j = 1, \ldots, q$:

$$\phi_j \quad = \quad \left(\begin{array}{c|c} \phi_{j-1} & R \\ \hline L & \Delta \end{array}\right);$$

see [9] for the details of the construction from the knowledge of the right and left *eigenvectors* for $0 \in \sigma(J)$.

Lidskii introduces the assumption:

(L): ϕ_j is nonsingular for all $j = 1, \ldots, q$.

Because ϕ_{j-1} is nonsingular for $j \geq 2$, the Schur complement

$$\Omega_j \quad = \quad \Delta - L\phi_{j-1}^{-1}R$$

of ϕ_{j-1} in ϕ_j is well-defined. And because ϕ_j is nonsingular, Ω_j is itself nonsingular. By a slight abuse of language, we still say that $\Omega_1 = \Delta = \phi_1$ is a Schur complement (it corresponds to ϕ_0 inexistent: $f_0 = 0$).

Under the assumption (L) the Lidskii process asserts that for each step j there are exactly $n_j r_j$ eigenvalues with Puiseux order $1/n_j$ and coefficients deduced from $\sigma(\Omega_j)$ by taking the n_j roots of order n_j of each of the r_j eigenvalues. This follows from the resolution of the equation

(E): $\det \phi_j(t) = 0$ with $t = z^{n_j}$ and $\phi_j(t) = \left(\begin{array}{c|c} \phi_{j-1} & R \\ \hline L & \Delta - tI_{r_j} \end{array}\right)$

for $\det \phi_{j-1} \neq 0$ by the Schur complement formula:

$$\det \phi_j(t) \quad = \quad (\det \phi_{j-1}) \det \left(\Omega_j - tI_{r_j}\right).$$

To the Jordan form J for $0 \in \sigma(J)$, one can associate the continuous *Puiseux-Jordan line* which is a polygonal line built with the segments joining the q vertices (m_j, f_j), $1 \leq j \leq q$, and 0. See Figure 1, which corresponds to

$$q = 4, \qquad n_1 = 5 \ > \ n_2 = 4 \ > \ n_3 = 2 \ > \ n_4 = 1,$$

and

$$r_1 = 1, \quad r_2 = 2, \quad r_3 = 2, \quad r_4 = 3.$$

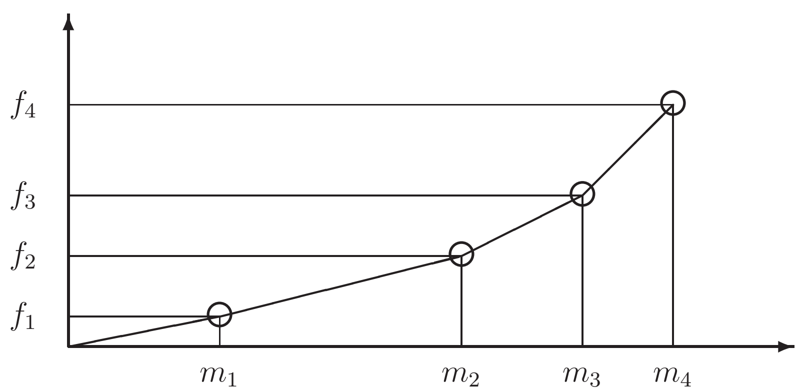

Figure 1. *The Puiseux-Jordan line, generic case.*

This describes $(0^5)(0^4)^2(0^2)^2(0^1)^3$ with $g = f_4 = 8 < m = m_4 = 20$. Each segment has for slope the Puiseux order $1/n_j \leq 1$: the slope of the polygonal line is a strictly increasing function.

This line describes the generic situation where the perturbation matrix B satisfies (L). Each of the r_j Jordan blocks of size n_j creates a Puiseux cycle of order $1/n_j$ [4]. Lidskii's results add that the coefficients are deduced from $\sigma(\Omega_j)$, when $0 \notin \sigma(\Omega_j)$ (hypothesis (L))

According to [9], the q vertices $P_j = (m_j, f_j)$, $j \geq 1$ of the Puiseux-Jordan line in the generic case are the vertices of the Puiseux-Newton diagram [4] associated with the characteristic polynomial $\det(J + \epsilon B - zI) = 0$, in which the coefficient of ϵ^{f_j} is $\pm \det \phi_j$; it is nonzero by assumption (L). They are circled in Figure 1.

The set of matices B that do not satisfy (L) is *nongeneric* in $\mathbb{C}^{n\times n}$ [7]. We call *Schur coefficients* the leading coefficients which are deduced from $\sigma(\Omega_j)\setminus\{0\}$.

2.3. The first nongeneric step where ϕ_j is singular

The challenge to go beyond (L) has been faced in [4] and more directly in [2, 3, 9].

We revisit the analysis presented in [9] under the hypothesis:

(H): there exists j, $1\le j\le q$ such that $\det\phi_j=0$ and $\det\phi_i\neq 0$, $i\neq j$.

What happens when the Lidskii process hits the unique nongeneric step j?

It is shown in [4, p. 306] that the spectral projection $P(\epsilon)$ associated with $0\in\sigma(J)$ can be represented as the sum of two analytic projections in ϵ. The first (resp. second) one concerns the m_{j-1} (resp. $m-m_{j-1}$) eigenvalues $\lambda(\epsilon)$ converging with order $\le 1/n_{j-1}$ (resp. $\ge 1/n_j$). Analyticity cannot be guaranteed any further.

However more algebraic computation can be performed.

The following presentation was inspired by [9]. The degeneracy of step j has an immediate consequence at step j itself, as well as an impact on the next step $j+1$, when $j<q$.

The impact at step j depends on ω_j, the algebraic multiplicity of $0\in\sigma(\Omega_j)$, $1\le\omega_j\le r_j$. There are still $n_j(r_j-\omega_j)$ eigenvalues with exponent $1/n_j$ and Schur coefficients deduced from $\sigma(\Omega_j)\setminus\{0\}$. The classification becomes *incomplete* and the Puiseux-Jordan line is *discontinous.* It stops at the point

$$N_j^- \quad=\quad (m_j-n_j\omega_j,\ f_j-\omega_j)$$

before reaching the Puiseux vertex $P_j=(m_j,f_j)$ which is *degenerate.*

What happens for the remaining $n_j\omega_j$ eigenvalues, which would have been—generically—classified at step j? The case cannot be treated by Lidskii's approach, since it breaks down as soon as Ω_j is singular [4, 9].

All that can be said is that they converge with various unknown orders $>1/n_j$: they converge with *better* rates than the generic Puiseux rate.

2.4. Beyond the first singular step j, $1 < j < q$

By assumption (H), all steps i beyond $j+1$ are generic since Ω_i exists and is nonsingular for $i \geq j+2$. The computing process can be resumed, leading to the classification of $\sum_{i=j+2}^{q} n_i r_i$ eigenvalues. However, there is still a difficulty at step $j+1$: since ϕ_j is singular, one cannot use Ω_{j+1} which does not exist.

The problem we are facing is to solve (with $j \to j+1$) an equation of type (E) where $\det \phi_{j-1} = 0$. Applying Laplace's theorem to compute $\det \phi_j(t)$, we deduce that $\det \phi_j(t)$ is a polynomial in t of degree $< r_j$. This follows from the fact that the term t^{r_j} can appear only in $\det(\Delta - tI_{r_j})$. And the coefficient of $\det(\Delta - tI_{r_j})$ is $\det \phi_{j-1} = 0$.

Going back to step $j+1$, we define $r_{j+1} - \delta_{j+1}$ to be the degree of $\det \phi_{j+1}(t) = 0$, with $t = z^{r_{j+1}}$. Clearly $1 \leq \delta_{j+1} \leq r_{j+1}$ [10]. We cannot get the $n_{j+1} r_{j+1}$ coefficients which would generically follow from $\sigma(\Omega_{j+1})$. We can have access only to $n_{j+1}(r_{j+1} - \delta_{j+1})$ of them. Now, all that can be said about the $n_{j+1}\delta_{j+1}$ eigenvalues not classified is that they converge with unknown orders $< 1/n_{j+1}$: they converge with *worse* rates than the generic Puiseux rate.

Why is this so? The reasoning is similar to the one which proves that a singular matrix pencil necessarily has infinite eigenvalues. Here the polynomial $\det \phi_{j+1}(t)$ should have degree r_{j+1}. Therefore ∞ is a root with multiplicity δ_{j+1}. The change of variable $t \longmapsto z = t^{1/n_{j+1}}$ yields $n_{j+1}\delta_{j+1}$ infinite roots which represent the coefficients for $\epsilon^{1/n_{j+1}}$ in $\lambda(\epsilon)$. This means that the actual orders are $< 1/n_{j+1}$, yielding finite unknown coefficients.

We shall investigate in [10] the use to solve (E) of the dual Schur complement formula:

$$\det \phi_j(t) \quad = \quad \det(\Delta - tI_{r_j}) \; \det\left(\phi_{j-1} - R(\Delta - tI_{r_j})^{-1} L\right)$$

for $t \notin \sigma(\Delta)$.

In summary, a total of

$$\sum_{i \neq j, j+1} n_i r_i \; + \; n_j(r_j - \omega_j) \; + \; n_{j+1}(r_{j+1} - \delta_{j+1})$$

eigenvalues have been classified, out of m, leaving successively $n_j\omega_j$ (resp. $n_{j+1}\delta_{j+1}$) eigenvalues unclassified at step j (resp. $j+1$) with unknown orders larger than $1/n_j$ (resp. less than $1/n_{j+1}$) and unknown coefficients.

Are Puiseux values possible for these unknown orders? Assumption (H) controls this possibility very tightly: some unclassified eigenvalues at step j (resp. $j+1$) may converge with order $= 1/n_{j+1}$ (resp. $1/n_j$) only.

The effect of a nongeneric step j, $1 < j < q$, can be represented as a *discontinuous* Puiseux-Jordan line where the vertex P_j is *isolated*. The *left* segment $P_{j-1}P_j$ is replaced by the smaller segment $P_{j-1}F_j$ containing the segment $P_{j-1}N_j^-$ iff $\omega_j < r_j$. When $\omega_j = r_j$, then $P_{j-1}N_j^-$ is reduced to the point P_{j-1}. The *right* segment P_jP_{j+1} is replaced by a smaller one O_jP_{j+1} containing the segment $N_j^+P_{j+1}$, where N_j^+ is the new vertex

$$(m_j + n_{j+1}\delta_{j+1},\ f_j + \delta_{j+1}).$$

The uncertainty about the endpoint F_j (resp. the origin O_j) comes from the possibility that at least one unclassified eigenvalue at step $j+1$ (resp. j) does converge with order $= 1/n_j$ (resp. $= 1/n_{j+1}$). When this is not the case, $F_j = N_j^{-1}$ (resp. $O_j = N_j^+$) is on the Puiseux-Newton diagram. Observe that $N_j^+ \equiv P_{j+1}$ iff $\delta_{j+1} = r_{j+1}$. It is a possibility that no eigenvalue converges with the Puiseux orders $1/n_{j+1}$ or $1/n_j$.

The discontinuous Puiseux-Jordan line is displayed in Figure 2: it corresponds to $j = 2$ with $\omega_2 = 1 < r_2 = 2$ and $\delta_3 = 1 < r_3 = 2$. The Δ indicates the degenerate vertex P_2 for which $\det \phi_2 = 0$. The circled vertices are on the Puiseux-Newton diagram. The two $\star$ mark the new vertices N_2^- and N_2^+ which result from the accident at step j on the Puiseux-Jordan line.

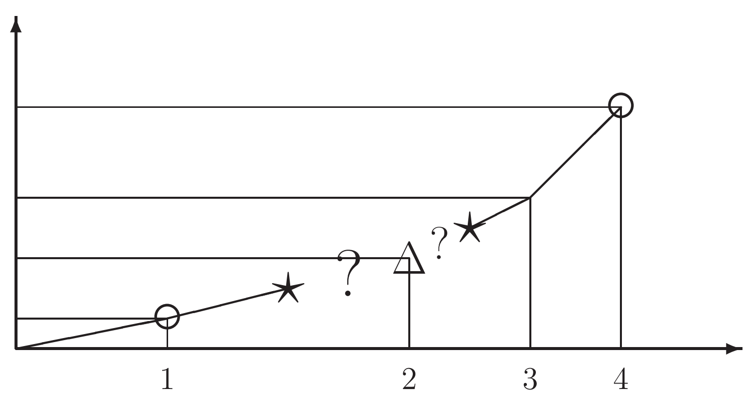

Figure 2. *The discontinuous Puiseux-Jordan line, $\omega_2 = \delta_3 = 1$.*

The two question marks ? indicate that some orders of convergence remain unknown at step j and $j+1$.

2.5. The first or final step is nongeneric, $j = 1$ or q

When $j = 1$ is nongeneric, $\det \phi_1 = 0$ and the origin O plays the role of P_{j-1}. Some eigenvalues converge with orders $> 1/n_1$ and others with orders $< 1/n_2$. The vertex P_1 is disconnected from the Puiseux-Jordan line.

We now look at the case $j = q,\ \det \phi_q = 0,\ \det \phi_i \neq 0,\ i < q$. The new feature is that there are no subsequent steps. At step $q-1$, all m_{q-1} eigenvalues have been classified. When $\det \phi_q = 0$, $n_q(r_q - \omega_q)$ (resp. $n_q \omega_q$) eigenvalues converge with order $= 1/n_q$ (unknown orders $> 1/n_q$), as expected. The Puiseux-Jordan line is discontinuous near the endpoint P_q which is an isolated vertex.

2.6. A comparison with [9]

The above analysis of the degenerate case differs from the one presented in [9] in two aspects:

- It studies the impact of a nongeneric step j on the next step $j+1$, under the assumption (H). In [9], only the step j is considered.
- In the study of step j, the quantity ω_j is the *algebraic* multiplicity of 0 in $\sigma(\Omega_j)$, rather than its *geometric* multiplicity $r_j - \operatorname{rank} \Omega_j = f_j - \operatorname{rank} \phi_j$, denoted β in [9]. When 0 is defective in $\sigma(\Omega_j)$, then $\beta < \omega_j$.

3. An application to Homotopic Deviation

We turn to the even more particular case $j = q$, $n_q = 1$, $q \geq 2$, which arises naturally in the context of Homotopic Deviation [5, 6].

For $A, E \in \mathbb{C}^{n \times n}$, let

$$A(t) \ = \ A + tE$$

represent the coupling of A and E by the parameter

$$t \ \in \ \hat{\mathbb{C}} \ = \ \mathbb{C} \cup \{\infty\}.$$

When rank $E = r < n$, it is possible that a part of $\lim_{|t|\to\infty} \sigma(A(t))$ remains at finite distance, denoted Lim:

$$\lim_{|t|\to\infty} \sigma(A(t)) \quad = \quad \mathrm{Lim} \cup \{\infty\}.$$

We write

$$A(t) \quad = \quad t(E + sA), \qquad s = \frac{1}{t}, \qquad \text{for } t \neq 0.$$

Therefore

$$\lambda(t) \to \xi \in \mathrm{Lim} \quad \text{as} \quad |t| \to \infty, \quad \lambda(t) \in \sigma(A(t))$$

$$\Longleftrightarrow$$

$$\frac{\nu(s)}{s} \to \xi \quad \text{as} \quad s \to 0, \quad \nu(s) \in \sigma(E(s)).$$

This clearly requires that $\nu(s) \to 0$ as $s \to 0$, hence $\nu(0) = 0 \in \sigma(E)$. The eigenvalues which do not escape to ∞ have as limit the coefficient ξ (0 or not) for $\nu(s)/s$: Lim is nonempty iff at least one eigenvalue $\nu(s)$ converges to 0 with order ≥ 1 in s, and

$$\xi \quad = \quad \lim_{s\to\infty} \frac{\nu(s) - \nu(0)}{s} \quad = \quad \nu'(0).$$

The problem can be treated as a special instance of Lidskii's theory applied to $E(s)$ as $s \to 0$, with $q \geq 2$ and $n_q = 1$: there exists at least one trivial Jordan block for $0 \in \sigma(E)$. We find it convenient to distinguish whether $0 \in \sigma(E)$ is semi-simple (Σ) $(q = 1)$, or not [5, 6] $(q \geq 2)$.

3.1. Under (Σ), $q = 1$, $f_1 = g$

$\mathrm{Lim} = \sigma(\Pi)$ where $\Pi = PAP_{\restriction \mathrm{Ker}\ E}$ is easy to prove directly [5, 6]. There are exactly g limit points given by $\sigma(\Pi)$ which may or may not contain 0, or eigenvalues of A.

3.2. $0 \in \sigma(E)$ is defective with $q \geq 2$, $r_q = g' < g = f_q$

The following conditions have been introduced [5]:

(G): *$n_q = 1$ and ϕ_{q-1} is nonsingular* ($\Longrightarrow \Omega_q = \Omega$ of order g' exists).

Then card $\mathrm{Lim} = l_\star \geq g'$ (multiplicities are counted). There are at least g' eigenvalues $\nu(s)$ converging with order ≥ 1. Moreover $\sigma(\Omega) \subset$

Lim. The limit points in Lim which are not in $\sigma(\Omega)$ come from possible previous singular steps which yield orders ≥ 1 for $\nu(s)$. Note that when $0 \in \sigma(\Omega)$, its multiplicity in Lim is $\geq \omega$.

$(\hat{G})$: *$n_q = 1$ and the sequence ϕ_i, $i = 1, \ldots, q-1$ consists of nonsingular matrices.*

Then $l_\star = g'$ and $\sigma(\Omega) = \mathrm{Lim}$. In particular, the limit point 0, when present, has the same multiplicity ω in $\sigma(\Omega)$ and in Lim: ω indicates the number of eigenvalues $\nu(s)$ converging with order > 1.

We observe that $(\hat{G}) \Rightarrow (G)$ for $q > 2$ and $(\hat{G}) \Leftrightarrow (G)$ when $q = 2$.

Now the above analysis of Lidskii's algorithm allows us to relax the assumption (G). Let us consider the equation (E) for $j = q$ and $n_q = 1$, $z = t$. Then it is clear that $\mathrm{Lim} \supset Z(\det \phi_q(z))$, the set of $g' - \delta_q$ zeros in $\mathbb{C}$ for $\det \phi_q(z)$, $0 \leq \delta_q \leq g'$.

The assumption (G) allows the identification

$$Z(\det \phi_q(z)) \quad = \quad \sigma(\Omega_q),$$

which guarantees that $\delta_q = 0$. When $\det \phi_{q-1} = 0$, there are at most $g' - 1$ zeros in $Z(\det \phi_q(z))$, and

$$l_\star \;\; \geq \;\; \mathrm{card}\, Z(\det \phi_q(z)) \;\; = \;\; g' - \delta_q,$$

$1 \leq \delta_q \leq g'$. No more than $m - g' + \delta_q$ eigenvalues escape to ∞.

3.3. $0 \in \sigma(E)$ is defective with $q = 1$

When there is a unique type of Jordan block with $n_1 > 1$, generically the Puiseux exponent is $1/n_1 < 1$, and all eigenvalues $\lambda(t)$ escape to ∞. However $\mathrm{Lim} \neq \phi$ is possible nongenerically.

This is illustrated by Example 3.1 in [1]: $n = 3$, E is a single Jordan block of order 3, and A is the companion matrix for $\pi(z) = z^3 + 1$. $\phi_1 = \Omega = (0)$ and the prediction $\{0\} \subset \mathrm{Lim}$ is incomplete. The Newton diagram for $\det(E(s) - zI_3)$ confirms that $\{0\} = \mathrm{Lim} \neq \phi$.

3.4. An example in which Ω is singular

An illustration of Ω singular is provided by Example 4.1 in Section 7.4 of [1]. In this example E is in Jordan form for $0 \in \sigma(E)$ with $q = 2$, $g' = 8$, $n = 11$.

The condition $(G) = (\hat{G})$ is satisfied, and Ω is such that $0 \in \sigma(\Omega)$ with $\omega = 1$ [1]. The prediction of Lidskii is $\sigma(\Omega) = \text{Lim}$, with $l_\star = g' = 8$.

We turn to the computation of $t \mapsto \sigma(A(t))$ to confirm the prediction. This yields

$$\text{Lim} \;=\; \{-1\} \cup F$$

where F is the set of 7 frontier points predicted theoretically [1]; they are the 7 roots of

$$z(z^2+1)(z^4+1) \;=\; 0.$$

4. Conclusion

The case ϕ_j singular has computational impact at steps j and $j+1$ in Lidskii's algorithm. We have analyzed the situation discussed in [2, 3, 9]: ϕ_j is singular, but ϕ_{j-1}^{-1} exists together with the Schur complement Ω_j of order r_j. This is step j in the algorithm. We also treated the consequence at step $j+1$: ϕ_j is singular and the Schur complement Ω_{j+1} of order r_{j+1} does not exist. This step $j+1$ will be computationally explored in a future work (see Part II [10]). Our analysis so far has already improved significantly our understanding of Homotopic Deviation.

Our main conclusion for Lidskii's algorithm is that a nongeneric step j has a twofold impact: first certain eigenvalues converge better than expected at step j, second some converge slower than expected at step $j+1$. Whereas the first aspect has been partially described [2, 3, 9], the second aspect seems to have remained unnoticed to date.

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