Additive and multiplicative two-level spectral preconditioning for general linear systems

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Abstract

Multigrid methods are among the fastest techniques to solve linear systems arising from the discretization of partial differential equations. The core of the multigrid algorithms is a two-grid procedure that is applied recursively. A two-grid method can be fully defined by the smoother that is applied on the fine grid, the coarse grid and the grid transfer operators to move between the fine and the coarse grid. With these ingredients both additive and multiplicative procedures can be defined. In this work we develop preconditioners for general sparse linear systems that exploit ideas from the two-grid methods. They attempt to improve the convergence rate of a prescribed preconditioner M_1 that is viewed as a smoother, the coarse space is spanned by the eigenvectors associated with the smallest eigenvalues of the preconditioned matrix M_1A. We derive both additive and multiplicative variants of the iterated two-level preconditioners for unsymmetric linear systems that can also be adapted for Hermitian positive definite problems. We show that these two-level preconditioners shift the smallest eigenvalues to one and tend to better cluster around one those that M_1 already succeeded to move in the neighbourhood of one. We illustrate the performance of our method through extensive numerical experiments on a set of general linear systems. Finally, we consider two case studies, one in non-overlapping domain decomposition method in semiconductor device modelling another one from electromagnetism applications.
Keywords: Iterative methods, adaptive preconditioning, additive and multiplicative two-grid cycles, spectral preconditioner.

1 Introduction

Multigrid methods are among the fastest techniques to solve a linear system $Ax = b$ arising from the discretization of partial differential equations. The core of the multigrid algorithms is a two-grid procedure that is applied recursively. A classical two-grid cycle can be shortly described as follows. On the fine grid a few iterations of a smoother are applied that attempts to reduce the high frequencies of the error (i.e. the components of the error in the space spanned by the vectors associated with the largest eigenvalues of A). The residual is then projected on the coarse grid where the low frequencies (i.e. the components associated with the smallest eigenvalues) can be captured and the coarse error equation is solved. The error on the coarse space is prolongated back to the fine grid to update the approximation computed by the pre-smoothing phase and a few more steps of the smoother are applied. Finally, if the new iterate is not accurate enough the two-grid cycle is applied iteratively. In classical multigrid, the coarse space is not defined explicitly through the knowledge of the eigencomponents but by the selection of a space that is expected to capture them. The scheme presented above is a multiplicative algorithm [14] but additive variants [2, 10, 26] also exist. In this work, we apply similar ideas to improve a prescribed

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preconditioner M_1, that is used to define the smoother involved in the two-grid scheme. In many situations such a preconditioner is able to cluster most of the eigenvalues close to one but still leaves a few close to the origin. In that framework we define the coarse space by the span of the eigenvectors V associated with the smallest eigenvalues of $M_1 A$ that are computed explictly (i.e. the components of the error that are not efficiently damped by the smoother). In that context, the prolongation operator is $P = V$, the restriction operator is denoted $R = W^H$ and the matrix involved in the coarse grid error problem is defined by a Galerkin formula $A_c = RAP$.

In Section 2, we describe our approach and the main contribution of this paper. Under the assumption that the initial preconditioner has done a good job of clustering most eigenvalues near one with relatively few outliers near the origin, we use an explicit eigensystem computation for these small eigenvalues to effectively solve the original system in this low dimensional space. We now discuss some background to this before describing our approach in full details.

It is well known that the convergence of Krylov methods for solving the linear system often depends to a large extent on the eigenvalue distribution. In many cases, it is observed that "removing" the smallest eigenvalues can greatly improve the convergence. Several techniques have been proposed in the past few years that attempt to tackle this problem. The proposed approaches can be split into two main families depending on whether the scheme enlarges the generated Krylov space or adaptively updates the preconditioner. For GMRES [25] there are essentially two different approaches for exploiting information related to the smallest eigenvalues. The first idea is to compute a few, k say, approximate eigenvectors of $M_1 A$ corresponding to the k smallest eigenvalues in magnitude, and augment the Krylov subspace with those directions. At each restart, let $u_1, u_2, ..., u_k$ be approximate eigenvectors corresponding to the approximate eigenvalues of $M_1 A$ closest to the origin. This approach is referred to as the *augmented subspace approach* (see [5, 20, 22]). The second idea exploits spectral information gathered during the Arnoldi process to determine an approximation of an invariant subspace of A associated with the eigenvalues nearest the origin, and uses this information to construct a preconditioner or to adapt an existing one. The idea of using exact invariant subspaces to improve the eigenvalue distribution was proposed in [21]. Information from the invariant subspace associated with the smallest eigenvalues and its orthogonal complement are used to construct an adaptive preconditioner in the approach proposed in [1, 9]. A preconditioner for GMRES based on a sequence of rank-one updates that involve the left and right smallest eigenvectors is proposed in [16]. A class of preconditioners only based on low rank updates based on the right smallest eigenvectors is described in [6]. Finally, our multiplicative approach is very close in spirit to the Generalized Global Basis method [28], where the coarse space correction is combined with an already efficient multilevel preconditioner. Results on challenging problems are reported in that latter paper.

In this paper, we follow the second approach and propose a class of preconditioners both for unsymmetric and for symmetric positive definite problems. In the following section, we describe the proposed preconditioners and prove their shifting capabilities on diagonalizable matrices. In Section 3, we illustrate the numerical efficiency of the proposed schemes on a set of unsymmetric and SPD linear systems from Matrix Market [4]. We also show their efficiency on very challenging linear systems that arise in two-dimensional unstructured mixed finite element calculation for semiconductor device modelling as well as for the solution of the dense linear systems arising from electromagnetic applications. Finally, we conclude with some remarks in Section 4.

2 Two-level spectral preconditioning

We consider the solution of the linear system

$$Ax = b, \tag{1}$$

where A is a $n \times n$ nonsingular matrix, and x and b are vectors of size n. The linear system is solved using a preconditioned Krylov solver and we denote by M_1 the left preconditioner, meaning that we solve

$$M_1 Ax = M_1 b. \tag{2}$$

We assume that the preconditioned matrix $M_1 A$ is diagonalizable, that is:

$$M_1 A = V \Lambda V^{-1}, \tag{3}$$

with $\Lambda = diag(\lambda_i)$, where $|\lambda_1| \leq \ldots \leq |\lambda_n|$ are the eigenvalues and $V = (v_i)$ the associated right eigenvectors. Let V_ε be the set of right eigenvectors associated with the set of eigenvalues λ_i with $|\lambda_i| \leq \varepsilon$. In this section we describe two procedures that attempt to improve the eigenvalue distribution of the preconditioned matrix. We formulate the algorithms in the framework of algebraic multilevel methods; the terminology and the design principles are inherited from classical multigrid algorithms. In the next two sections we present these two variants of the algorithm that are either multiplicative or additive.

2.1 Multiplicative two-grid spectral preconditioning

A generic two-level multigrid cycle is illustrated in Algorithm 1, where M_1 is used to define a weighted stationary method that implements the smoother. The algorithm takes as input a vector r that is the residual vector we want to precondition, and returns as output the preconditioned residual vector z. After μ_1 smoothing steps we project using the restriction $R = W^H$ the residual into the coarse subspace and solve the coarse space error equation involving $A_c = W^H A V_\varepsilon$. Finally, we prolongate back the error using V_ε in the original space and smooth again the new approximation. The preconditioner constructed using this scheme depends on A, M_1, μ_1, μ_2, ε and $\omega > 0$ and could be denoted by $M_{Mul}(A, M_1, \varepsilon, \omega, \mu_1, \mu_2)$. For the sake of simplicity of exposure when no confusion is possible we will simply denote it by M_{Mul} or $M_{Mul}(A, M_1)$.

Proposition 1 *Let W be such that $A_c = W^H A V_\varepsilon$ has full rank, the preconditioning operation described in Algorithm 1 can be written in the form $z = M_{Mul}\ r$. For the case $iter = 1$ the preconditioner M_{Mul} has the following expression:*

$$M_{Mul} = A^{-1} - (I - \omega M_1 A)^{\mu_2} (I - M_c A)(I - \omega M_1 A)^{\mu_1} A^{-1}, \tag{4}$$

where $M_c = V_\varepsilon (W^H A V_\varepsilon)^{-1} W^H$.

Proof
Hereby we remove the superscript k in Algorithm 1, as we analyze the case $iter = 1$. The pre-smoothing step generates sequence of vectors of the form $s_1^j = (I - \omega M_1 A) s_1^{j-1} + \omega M_1 r$, that can be written $z_1 = [I - (I - \omega M_1 A)^{\mu_1}] A^{-1} r$ because $s_1^0 = 0$. We also have $z_2 = [I + (M_c A - I)(I - \omega M_1 A)^{\mu_1}] A^{-1} r$. Finally, the post-smoothing generates a vector update that can be written as $z = [I - (I - \omega M_1 A)^{\mu_2} (I - M_c A)(I - \omega M_1 A)^{\mu 1}] A^{-1} r$.

∎

The following proposition describes the eigenvalue distribution of the preconditioned matrix $M_{Mul} A$. It shows that the smallest eigenvalues are shifted to one and that those that were already close to one are clustered closer to this point. On the assumption that the initial preconditioner M_1 has done a good job of clustering most eigenvalues near to one with relatively few outliers near the origin, the effect of M_{Mul} is expected to be very beneficial.

Proposition 2 *The preconditioner M_{Mul} defined by Proposition 1 is such that the preconditioned matrix $M_{Mul} A$ has eigenvalues:*

$$\begin{cases} \eta_i = 1 & \text{if} \quad |\lambda_i| < \varepsilon, \\ \eta_i = 1 - (1 - \omega \lambda_i)^{\mu_1 + \mu_2} & \text{if} \quad |\lambda_i| \geq \varepsilon. \end{cases}$$

Algorithm 1 *: Multiplicative spectral preconditioner*

set $z^1 = 0$

for *k=1, iter* **do**

1. *Pre-smoothing: damp the high frequencies of the error*
   $s_1^0 = z^k$
   for *j=1,* μ_1 **do**
   $s_1^j = s_1^{j-1} + \omega M_1(r - As_1^{j-1})$
   end for
   $z_1^k = s_1^{\mu_1}$
2. *Coarse grid correction*
   $z_2^k = z_1^k + V_\varepsilon A_c^{-1} W^H (r - Az_1^k)$
3. *Post-smoothing: damp again the high frequencies of the error*
   $s_2^0 = z_2^k$
   for *j=1,* μ_2 **do**
   $s_2^j = s_2^{j-1} + \omega M_1(r - As_2^{j-1})$
   end for
4. *Update the solution*
   $z^{k+1} = s_2^{\mu_2}$

end for

$z = z^{iter}$

Proof
We will show that the preconditioned is similar to a matrix whose eigenvalues are those indicated. Let $V = (V_\varepsilon, V_{\bar{\varepsilon}})$, where $V_{\bar{\varepsilon}}$ is a set of $(n-k)$ right eigenvectors associated with eigenvalues $|\lambda_i| > \varepsilon$ of $M_1 A$. Let $D_\varepsilon = diag(\lambda_i)$ with $|\lambda_i| \leq \varepsilon$, and $D_{\bar{\varepsilon}} = diag(\lambda_j)$ with $|\lambda_j| > \varepsilon$. Thus $M_1 A V_\varepsilon = V_\varepsilon D_\varepsilon$, $M_1 A V_{\bar{\varepsilon}} = V_{\bar{\varepsilon}} D_{\bar{\varepsilon}}$ and $M_c A V_\varepsilon = V_\varepsilon$. In addition, it is easy to show by induction on j that $(I - \omega M_1 A)^j V_\varepsilon = V_\varepsilon (I - \omega D_\varepsilon)^j$ and $(I - \omega M_1 A)^j V_{\bar{\varepsilon}} = V_{\bar{\varepsilon}} (I - \omega D_{\bar{\varepsilon}})^j$.

Then the following relations hold: $M_{Mul} A V_\varepsilon = V_\varepsilon$, and $M_{Mul} A V_{\bar{\varepsilon}} = V_{\bar{\varepsilon}}(I - (I - \omega D_{\bar{\varepsilon}})^{\mu_2 + \mu_1}) + V_\varepsilon C$ with $C_{Mul} = (I - \omega D_\varepsilon)^{\mu_2} A_c^{-1} W^H A V_{\bar{\varepsilon}} (I - \omega D_{\bar{\varepsilon}})^{\mu_1}$. This can be written:

$$M_{Mul} A V = V \begin{pmatrix} I & C_{Mul} \\ 0 & I - (I - \omega D_{\bar{\varepsilon}})^{\mu_1 + \mu_2} \end{pmatrix}.$$

■

Remark 1 *It can be noticed that the spectrum of the preconditioned matrix does not depend on the selection of* W^H*, that would play the role of the restriction operator in the multiplicative 2-grid method.*

2.2 Additive two-grid spectral preconditioning

In the additive algorithm, the coarse grid correction and the smoothing operation are decoupled. Each process generates an approximation of the preconditioned residual vector in complementary subspaces. The coarse grid correction computes only components in the space spanned by the eigenvectors associated with the few selected small eigenvalues, while at the end of the smoothing step the preconditioned residual is filtered so that only components in the complementary subspace are retained. These two contributions are summed together for the solution update. A simple additive two-level multigrid is illustrated in Algorithm 2. In this algorithm we follow [10] and select the procedure advocated in [26]; we define the filtering operators using the grid transfer operators as $(I - V_\varepsilon W^H)$. This operator is supposed to remove all the components in the V_ε directions. A natural choice is to select W^H so that $(I - V_\varepsilon W^H)V_\varepsilon = 0$ (i.e. $W^H V_\varepsilon = I$).

Algorithm 2 *:* Additive spectral preconditioner

set $z^1 = 0$

for *k=1, iter* **do**

1. *Compute the residual*
   $s^k = r - Az^k$
2. *Compute the high and low frequency corrections*
   (a) *High frequency correction*
   % Damp all the frequencies of the error
   $e_1^{k,0} = 0$
   for *j=1,* $\mu_1 + \mu_2$ **do**
   $e_1^{k,j} = e_1^{k,j-1} + \omega M_1(s^k - Ae_1^{k,j-1})$
   end for
   % Filter the high fequencies of the correction
   $c_1^k = (I - VW^H)e_1^{k,\mu_1+\mu_2}$
   (b) *Low frequency correction*
   $c_2^k = VA_c^{-1}W^H s^k$
3. *Update the solution*
   $z^{k+1} = z^k + c_1^k + c_2^k$

end for

$z = z^{iter}$

Proposition 3 *Let W be such that $A_c = W^H A V_\varepsilon$ has full rank and satisfies $(I - V_\varepsilon W^H)V_\varepsilon = 0$, the preconditioning operation described in Algorithm 2 can be written the form $z = M_{Add}\, r$. In the case $iter = 1$ the preconditioner M_{Add} has the following expression:*

$$M_{Add} = V_\varepsilon A_c^{-1} W^H + (I - V_\varepsilon W^H)(I - (I - \omega M_1)^{\mu_1+\mu_2})A^{-1}. \quad (5)$$

Proof
Similar arguments as for the proof of Proposition 1.

■

Proposition 4 *The preconditioner M_{Add} defined by Proposition 3 is such that the preconditioned matrix $M_{Add}A$ has eigenvalues:*

$$\begin{cases} \eta_i = 1 & \text{if} \quad |\lambda_i| \le \varepsilon, \\ \eta_i = 1 - (1 - \omega\lambda_i)^{\mu_1+\mu_2} & \text{if} \quad |\lambda_i| > \varepsilon. \end{cases}$$

Proof
Let $V = (V_\varepsilon, V_{\bar{\varepsilon}})$, where $V_{\bar{\varepsilon}}$ is the set of $(n-k)$ right eigenvectors associated with eigenvalues $|\lambda_i| > \varepsilon$ of $M_1 A$. Let $D_\varepsilon = diag(\lambda_i)$ with $|\lambda_i| \le \varepsilon$ and $D_{\bar{\varepsilon}} = diag(\lambda_j)$ with $|\lambda_j| > \varepsilon$. Then the following relations hold: $M_{Add} A V_\varepsilon = V_\varepsilon$, and $M_{Add} A V_{\bar{\varepsilon}} = V_{\bar{\varepsilon}}(I - (I - \omega D_{\bar{\varepsilon}}))^{\mu_2+\mu_1} + V_\varepsilon C_{Add}$ with $C_{Add} = A_c^{-1} W^H A V_{\bar{\varepsilon}} - W^H V_{\bar{\varepsilon}}(I - (I - \omega D_{\bar{\varepsilon}})^{\mu_1+\mu_2}$. This can be written:

$$M_{Add} A V = V \begin{pmatrix} I & C_{Add} \\ 0 & I - (I - \omega D_{\bar{\varepsilon}})^{\mu_1+\mu_2} \end{pmatrix}$$

■

Proposition 5 *If we consider $M_1 = I$ and select $W = U_\varepsilon$ where the k columns of W are the left eigenvectors of A associated with the smallest eigenvalues normalized such that $w_i^H v_i = 1$, then $M_{Mul} = M_{Add}$.*

Proof
The set of vectors W complies with the assumption $W^H V_\varepsilon = I$ because of the normalization of the right eigenvectors $w_i^H v_i = 1$ so the assumptions of Proposition 3 hold.
Finally, the propositions 1 and 3 lead to $C_{Add} = C_{Mul} = 0$ because $W^H V_{\bar{\varepsilon}} = 0$.

■

Remark 2 *A situation where the assumptions of Proposition 5 hold is for instance when either $M_{Mul}(M_1 A, I)$ or $M_{Add}(M_1 A, I)$ is considered (i.e. the two-grid preconditioners are applied to $M_1 A x = M_1 b$).*

Remark 3 *Let B and C be two square matrices, it is known that the eigenvalues of AB are the same as those of BA. Consequently using M_{Add} either as a left or as a right preconditioner will give the same spectrum for the preconditioned matrix. This observation is obviously also valid for M_{Mul}.*

2.3 The Hermitian positive definite linear systems

For Hermitian positive definite (HPD) linear systems a desirable property for M_1 is to be Hermitian positive definite; we denote $M_1 = L_1 L_1^H$ its Cholesky decomposition. We note that in many situations the preconditioner M_1 is given in this factorized form as in the incomplete factorization [19], AINV [3] or FSAI [17]. In the following of this section we review situations where M_{Mul} and M_{Add} are HPD.

Proposition 6 *The preconditioner M_{Mul} with $W = V_\varepsilon$ is HPD if $\mu_1 + \mu_2$ is odd.*

Proof
We do not give the details of the calculation, but using simple matrix manipulations it can be shown that

$$M_{Mul}(A, M_1) = L_1 M_{Mul}(L_1^H A L_1, I) L_1^H. \tag{6}$$

We first show that $M_{Mul}(L_1^H A L_1, I)$ is Hermitian. Because $L_1^H A L_1$ is HPD we have $L_1^H A L_1 = \tilde{V} D \tilde{V}^H$ and

$$M_{Mul}(L_1^H A L_1, I) = \tilde{V} \begin{pmatrix} D_\varepsilon^{-1} & 0 \\ 0 & D_{\bar{\varepsilon}}^{-1}(I - (I - \omega D_{\bar{\varepsilon}})^{\mu_1+\mu_2}) \end{pmatrix} \tilde{V}^H = \tilde{V} \hat{D} \tilde{V}^H. \tag{7}$$

Because $\mu_1+\mu_2$ is odd, all diagonal entries of $\hat{D}$ are positive (the function $f(x) = 1-(1-\omega x)^k$ is positive $\forall x \in (0, \infty)$ for k odd, see also Figure 1 (b)). Furthermore, equality (6) leads to $M_{Mul} = (L_1\tilde{V})\hat{D}(L_1\tilde{V})^H$. The Sylvester law of inertia implies that all the eigenvalues of M_{Mul} are positive.

■

Proposition 7 *If $\mu_1+\mu_2$ is even, the preconditioner M_{Mul} with $W = V_\varepsilon$ is HPD iff $\omega < 2\lambda_{max}^{-1}(M_1A)$.*

Proof
Similar as for Proposition 6.

■

Proposition 8 *Under the assumptions of Proposition 6 or 7, CG, the conjugate gradient method [15] preconditioned by $M_{Mul}(A, M_1)$ and $M_{Mul}(L_1^HAL_1, I)$ generates the same iterates assuming suitable initial guesses are chosen.*

Proof
Because the preconditioners are HPD we consider their Cholesky decompositions, that are $M_{Mul} = L_ML_M^H$ and $M_{Mul}(L_1^HAL_1, I) = \tilde{L}_M\tilde{L}_M^H$. From (6), we have $L_ML_M^T = L_1\tilde{L}_M\tilde{L}_M^HL_1^H$ that shows that $L_M = L_1\tilde{L}_M$ (unicity of the Cholesky decomposition). We now use the result on CG iterates from [12, 24] that states that solving $By = g$ with CG preconditioned by LL^T generates iterates y^k that can also be generated by unpreconditioned CG on $L^TBL\tilde{y} = L^Tg$ provided that the initial guesses are such that $y_0 = L\tilde{y}_0$. We apply this result in the following sequence: CG on A preconditioned by $L_ML_M^T$ generates the same iterates as unpreconditioned CG on $L_M^TAL_M = \tilde{L}_M^HL_1^HAL_1\tilde{L}_M$; these iterates are the same as those generated by CG on $L_1^HAL_1$ preconditioned by $\tilde{L}_M\tilde{L}_M^H$.

■

The additive preconditioner is in general not HPD even when A and M_1 are. One way to still use this preconditioner for a M_1 given in a symmetric factorized form $M = L_1L_1^H$ is to consider $M_{Add}(L_1^HAL_1, I)$ applied to $L_1^HAL_1$. In that case we have $W = V$ and Proposition 5 shows that this preconditioner is the same as $M_{Mul}(L_1^HAL_1, I)$. Furthermore, Proposition 8 shows that then CG generates the same iterates as M_{Mul}. This shows that for HPD linear systems all these preconditioners generate the same sequence of CG iterates and the choice between the two can be made on floating-point arithmetic complexity as described below.

2.4 Some computational considerations

The spectral properties of M_{Mul} and M_{Add} have been derived from Algorithm 1 and 2 where only one outer iteration is considered. With similar calculations as those used so far, it can be shown that implementing "*iter*" outer steps leads to a spectral transformation for the two preconditioners that is defined by the following proposition.

Proposition 9 *The preconditioner M_{Mul} defined by Algorithm 1 with $iter \geq 1$ is such that the preconditioned matrix $M_{Mul}A$ has eigenvalues:*

$$\begin{cases} \eta_i = 1 & \text{if} \quad |\lambda_i| \leq \varepsilon, \\ \eta_i = 1-(1-\omega\lambda_i)^{iter\times(\mu_1+\mu_2)} & \text{if} \quad |\lambda_i| > \varepsilon. \end{cases}$$

Similar proposition holds with M_{Add} defined by Algorithm 2.

From a computational point of view applying M_{Mul} or M_{Add} does not require the same computational effort. In term of application of M_1 both need $(\mu_1 + \mu_2 - 1) + (iter - 1) \times (\mu_1 + \mu_2)$ matrix-vector products. The number of matrix-vector products involving A is one less for M_{Add} than for M_{Mul} per cycle. For M_{Add} the number of products by A is

$$(\mu_1 + \mu_2 - 1) + (iter - 1) \times (\mu_1 + \mu_2). \quad (8)$$

Under the assumption that the convergence of Krylov solver is mainly governed by the eigenvalue distributions (i.e. $iter \times (\mu_1 + \mu_2)$ given) Proposition 9 and the number of matrix-vector products given by (8) indicates that it is more efficient to use only one two-grid cycle with $(iter \times (\mu_1 + \mu_2))$ smoothing steps rather than $iter$ two-grid cycles with $(\mu_1 + \mu_2)$ smoothing steps per cycle.

3 Numerical experiments

In this section we illustrate the numerical behaviour of the two variants of the spectral algorithm described in the earlier section. We consider both unsymmetric and SPD systems. For all the numerical experiments, the preconditioner M_1 is an incomplete factorization, that is the standard $ILU(t)$ preconditioner with threshold t for unsymmetric matrices [23], and the incomplete Cholesky factorization $IC(t)$ for SPD matrices [19]. We report on results using the GMRES and the Bi-CGStab [27] solvers for unsymmetric problems, and the CG method for SPD systems. The experiments are carried out in MATLAB using left preconditioning, and the initial guess for the solution is the zero vector. Because we compare different preconditioners, we choose as stopping criterion the reduction of the normalized unpreconditioned residual by 10^{-6}, so that the stopping criterion is independent of the preconditioner. We explicitly compute the true unpreconditioned residual at each iteration. For the experiments with M_{Add} we use $W = QR^{-1}$ where $V_\varepsilon = QR$ to ensure that $W^H V_\varepsilon = I$. The eigenvectors V_ε are computed in a preprocessing phase using the Matlab function `eigs` that calls ARPACK [18].

3.1 General test problems

In a first stage we consider test matrices from Matrix Market. In Table 1 we give the list of matrices with their characteristics that we used for the experiments on SPD problems. Similarly in Table 2, we display the unsymmetric matrices list. In this section, we only report a few results representative of the general trends. For a complete description of the numerical experiments we refer the reader to the tables in the appendix whose references are given in the last columns of the two tables for each of the test problems.

For SPD matrices, we display the eigenvalue transformation operated by the preconditioners. These are the curves $f(x) = 1 - (1 - \omega x)^k$ $\forall x > 0$ as the eigenvalues of M_1A are real positive. It illustrates that the preconditioned matrices remain SPD for $(\mu_1 + \mu_2)$ odd while ω should be setup such that $\omega < 2\lambda_{max}^{-1}(M_1A)$ for $(\mu_1 + \mu_2)$ even.

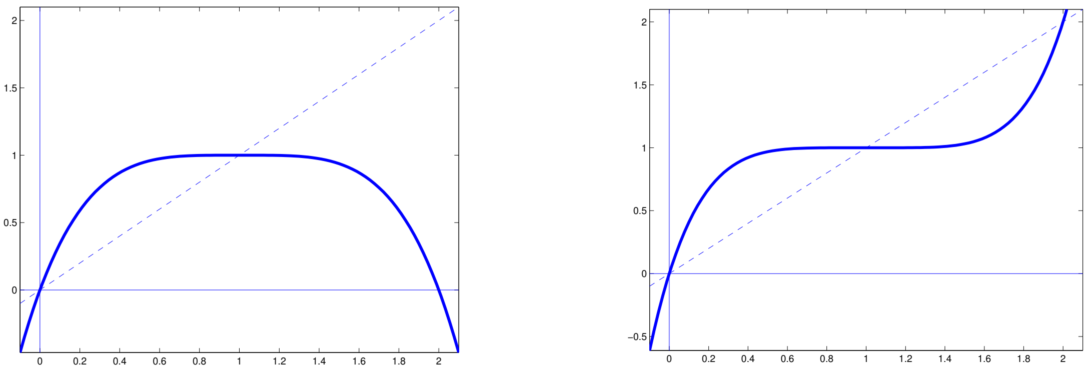
In Figure 2 we depict the spectrum of the preconditioned matrix M_1A for six of the unsymmetric test examples. In Figure 3, Figure 4, we depict the spectrum of the matrices preconditioned by either M_{Mul} or M_{Add} using $\omega = 1.0$ and $\mu_1 = 1$ and $\mu_2 = 1$, $\mu_1 = 1$ and $\mu_2 = 2$ respectively. In these figures, it can be seen that the spectral preconditioners make a good job in clustering close to one most of the eigenvalues. More precisely for $\mu_1 + \mu_2 > 1$ all the eigenvalues lying in the open disk of radius one centered in (1,0) are contracted towards (1,0), the ones out of this disk are moved away from (1,0). When $\mu_1 + \mu_2$ is odd those that are real and larger than 2 become negative whereas for $\mu_1 + \mu_2$ even they remain positive. This is clearly illustrated by the examples of the matrices BFW398A and FS5414.

Name	Size	Source	Table
685BUS	685	Power system networks	Tab. 9
1138BUS	1138	Power system networks	Tab. 10
BCSSTK27	1224	Dynamic analyses in structural engineering - Buckling analysis	Tab. 11
BCSSTK14	1806	Static analyses in structural engineering - Roof of the Omni Coliseum, Atlanta	Tab. 12
BCSSTK15	3948	Static analyses in structural engineering - Georgia Institute of Technology	Tab 13.
BCSSTK16	4884	Static analyses in structural engineering - U.S. Army Corps of Engineers dam	Tab 14.
S1RMQ4M1	5489	Structural mechanics - Cylindrical shell	Tab 15.

Table 1: Set of SPD test matrices.

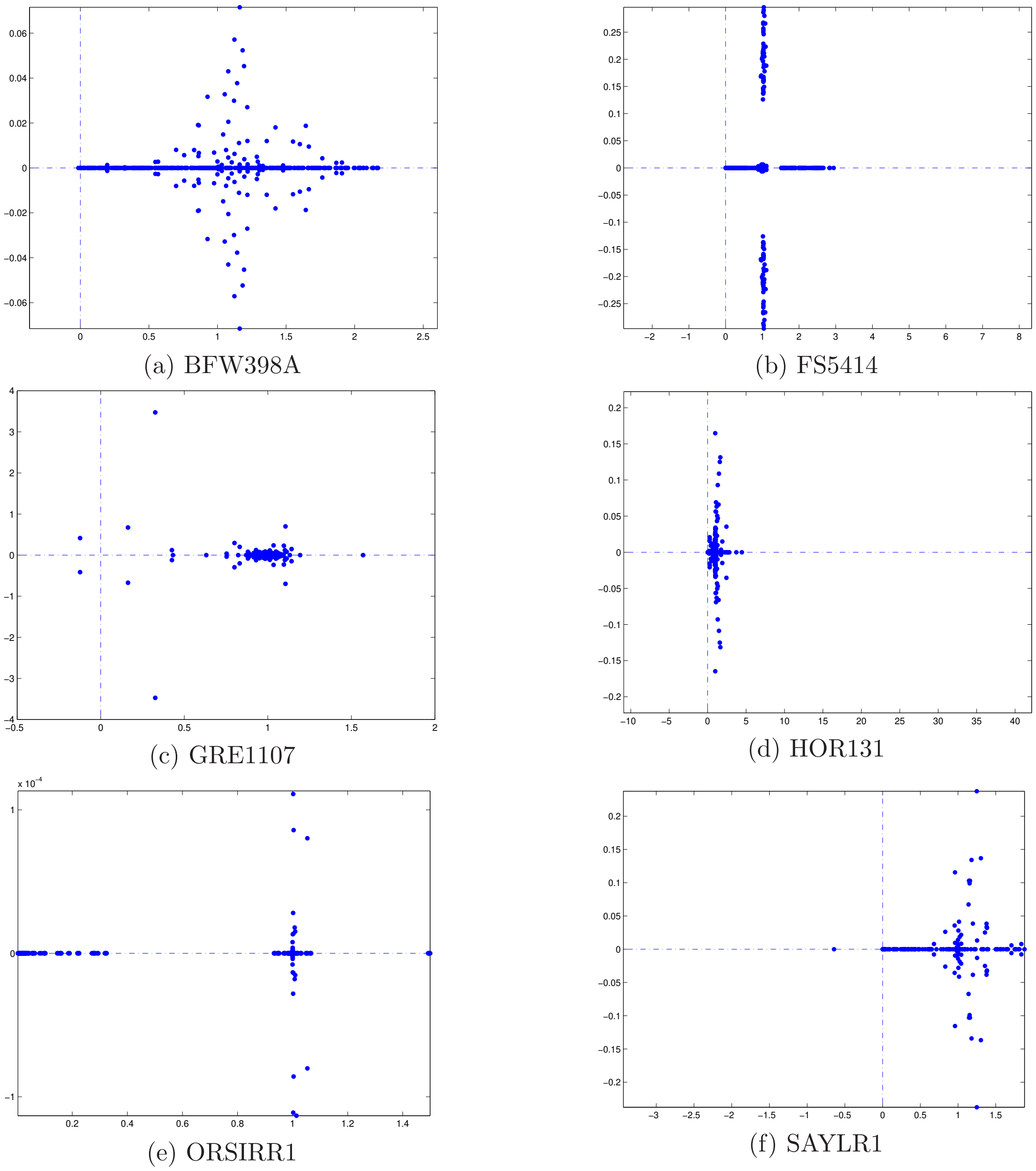
Name	Size	Source	Table
BFW398A	398	Bounded finline dielectric waveguide	Tab. 16 and 27
BWM200	200	Chemical engineering	Tab. 17 and 28
BWM2000	2000	Chemical engineering	Tab. 18 and 29
FS5414	541	one stage of FACSIMILE stiff ODE	Tab. 19 and 30
GRE1107	1107	Simulation studies in computer systems	Tab. 20 and 31
HOR131	434	Flow in networks	Tab. 21 and 32
ORSIRR1	1030	Oil reservoir simulation	Tab. 22 and 33
PORES3	532	reservoir modeling	Tab. 23 and 34
RDB2048	2048	Chemical engineering	Tab. 24 and 35
SAYLR1	238	2D reservoir simulation based on field data	Tab. 25 and 36
SAYLR4	3564	2D reservoir simulation based on field data	Tab. 26 and 37

Table 2: Set of unsymmetric test matrices.



(a) $(\mu_1 + \mu_2)$ even (=4) (b) $(\mu_1 + \mu_2)$ odd (=5)

Figure 1: Shape of the polynomial that governs the eigenvalues distribution of the two-grid spectral preconditioner for $\omega = 1$.

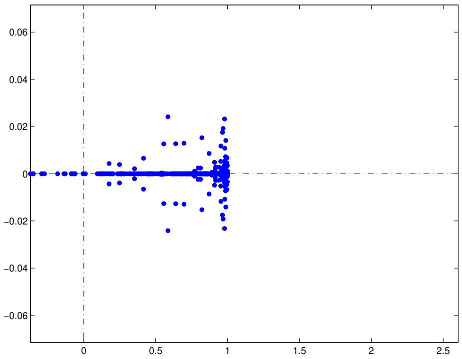


(a) BFW398A (b) FS5414

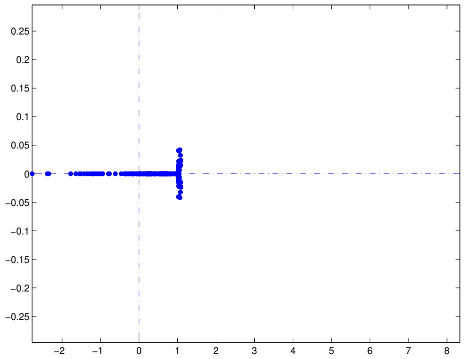
(c) GRE1107 (d) HOR131

(e) ORSIRR1 (f) SAYLR1

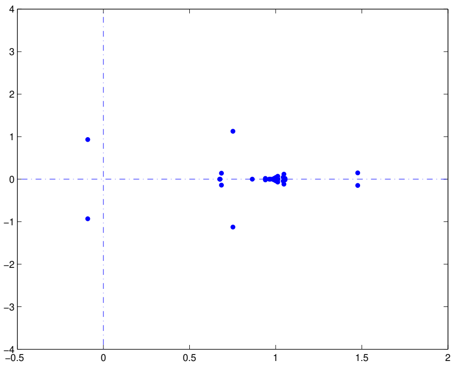
Figure 2: The spectrum of the preconditioned matrices.



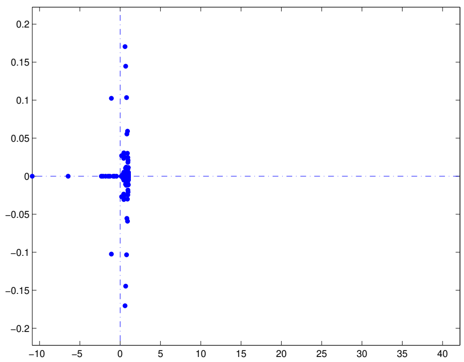
(a) BFW398A



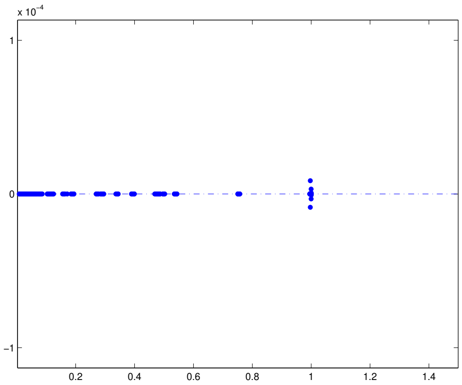
(b) FS5414



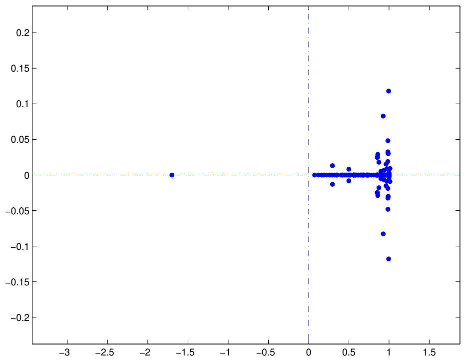
(c) GRE1107



(d) HOR131

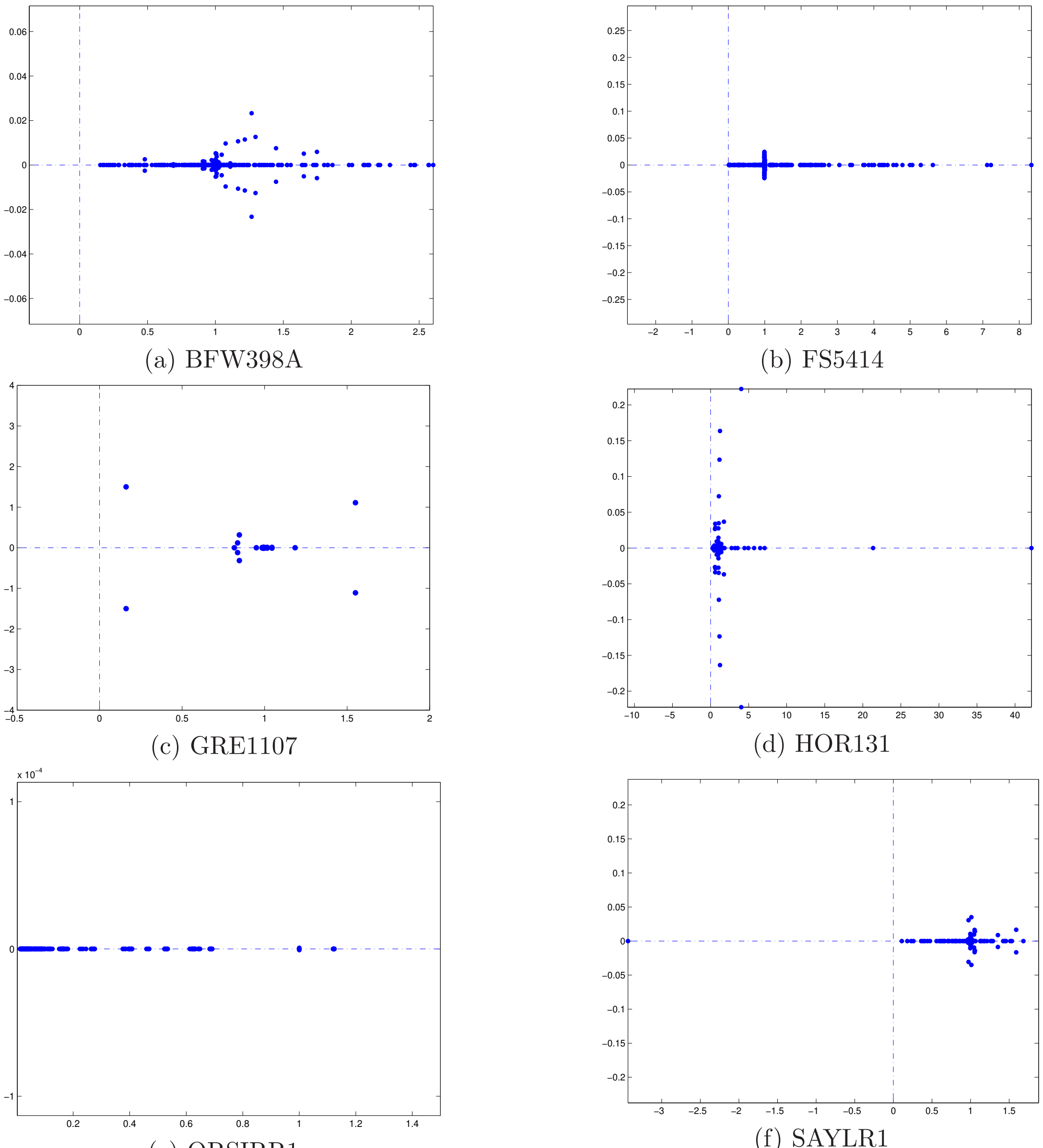


(e) ORSIRR1



(f) SAYLR1

Figure 3: The spectrum of the matrices preconditioned by the two-grid spectral preconditioner with $\mu_1 = \mu_2 = 1$ and $\omega = 1$.



(a) BFW398A (b) FS5414

(c) GRE1107 (d) HOR131

(e) ORSIRR1 (f) SAYLR1

Figure 4: The spectrum of the matrices preconditioned by the two-grid spectral preconditioner with $\mu_1 = 1$ and $\mu_2 = 2$ and $\omega = 1$.

In the following subsections we consider the numerical behaviour of the proposed preconditioners. The efficiency in terms of computational cost is very problem depend and is addressed in Section 3.2 where two real-world applications are presented.

3.1.1 Numerical behaviour of M_{Mul} v.s. M_{add}

The two preconditioners M_{Add} and M_{Mul} give rise to preconditioned systems that have the same eigenvalues distribution but possibly different eigenspaces. In Table 3 we report on the number of iterations of the Krylov solvers for different matrices and different values of μ_1 and μ_2 when the size of the coarse space is varied. The symbol "–" for the dimension of the coarse space means that no coarse space correction and only one smoothing is applied; in that case the preconditioner reduces to the standard incomplete factorization. On all the numerical experiments we have performed M_{Add} and M_{Mul} exhibit a very similar numerical behaviour as illustrated in Table 3. Small differences appear with Bi-CGStab, diminish with restarted GMRES and vanish with full GMRES. However, from a numerical point of view none of the two preconditioners appears superior to the other. When the size of the coarse space is increased it can be seen that the two-level preconditioners monotonically improve the convergence of full GMRES. This is no longer strictly true for restarted GMRES and Bi-CGStab but the trend is still to observe less iterations when the size of the coarse space is increased. Generally the larger the coarse space, the faster the convergence. One can see, on ORSIRR or SAYLR1 matrices for instance, that even without coarse space correction few steps of smoothers can significantly improve the converge of the Krylov solvers. On this latter example, the combination of the smoothing steps and the coarse grid correction is the only way to ensure the convergence of GMRES(50).

Because M_{Add} and M_{Mul} have a similar behaviour and because M_{Mul} is naturally defined for SPD problems we only consider M_{Mul} for the numerical experiments reported in the next three sections For complementary results, we refer the reader to the appendix where the results of the intensive experimentation are reported.

3.1.2 Effect of the number of smoothing steps

In Table 4 we display the number of iterations when the number of smoothing steps is varied from 1 to 3. The symbol "–" indicates that the convergence was not observed in less than 1000 iterations. It can be seen that increasing the number of smoother iterations improves the convergence for all the test examples but for the matrix FS5414. For this latter matrix, the convergence using M_{Mul} is worse than using simply M_1. This poor numerical behaviour is probably due to the fact that on that matrix the preconditioned system M_1A has eigenvalues close to 2. The even number of smoothing iterations move those eigenvalues near zero which dramatically affect the numerical behaviour of restarted GMRES. Full GMRES succeeds to converge but still using 2 steps of smoothing gives a larger number of iterations than just one step. When we move to 3 steps we reduce the number of iterations compared to 2 but also compared to one step of smoothing.

Even though no numerical experiments are reported to illustrate this phenomenon, we observed that only the sum $\mu_1 + \mu_2$ plays a role for the convergence of M_{Mul} whatever these smoothing steps are used only as pre-smoother, post-smoother or split between the pre and post-smoothing steps.

Furthermore, we observe that the relative improvement of using more smoothing steps tends to be larger for small dimension of the coarse space.

3.1.3 Effect of the relaxation parameter ω

In Table 5 we display the number of iterations required by CG, restarted and full GMRES on a set of matrices when the relaxation parameter ω is varied. For SPD matrices, when $\mu_1 + \mu_2$ is odd, no damping is required for the smoother to ensure that the preconditioner is SPD as illustrated for the matrices 1138BUS and BCSSTK27. It can be seen that $\omega = 1.0$ leads to worse iteration counts

HOR131 - $t = 2 \cdot 10^{-1}$ - $\mu_1 = 1$, $\mu_2 = 0$ - $\omega = 1.0$.													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
Bi-CGStab	M_{Add}	42	42	35	30	30	26	25	27	23	21	20	18
	M_{Mul}	42	42	30	36	30	28	25	26	23	19	22	17
GMRES(30)	M_{Add}	93	93	62	55	54	52	52	46	44	40	40	35
	M_{Mul}	93	93	60	56	53	52	51	47	45	40	39	36
GMRES(∞)	M_{Add}	66	66	54	50	49	47	47	43	40	36	37	32
	M_{Mul}	66	66	54	51	49	47	47	43	40	36	37	32

ORSIRR - $t = 3 \cdot 10^{-1}$ - $\mu_1 = 2$, $\mu_2 = 1$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
Bi-CGStab	M_{Add}	441	58	55	50	45	47	53	43	49	37	44	37
	M_{Mul}	441	58	69	47	41	42	51	38	42	43	35	35
GMRES(30)	M_{Add}	160	83	74	73	69	69	65	63	60	54	54	54
	M_{Mul}	160	83	74	74	69	68	66	63	57	54	53	54
GMRES(∞)	M_{Add}	121	73	71	67	66	66	61	60	55	53	52	49
	M_{Mul}	121	73	71	67	66	66	59	58	54	53	52	49

SAYLR1 - $t = 3 \cdot 10^{-1}$ - $\mu_1 = 2$, $\mu_2 = 1$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
Bi-CGStab	M_{Add}	150	77	64	54	43	27	21	19	18	16	14	14
	M_{Mul}	150	77	63	49	40	28	22	19	17	19	13	14
GMRES(50)	M_{Add}	–	–	388	142	56	47	43	36	36	32	29	26
	M_{Mul}	–	–	441	141	56	47	40	39	36	33	29	26
GMRES(∞)	M_{Add}	116	72	63	56	49	43	36	33	33	30	26	23
	M_{Mul}	116	72	63	56	49	42	36	35	33	30	26	23

BFW398a - $t = 5 \cdot 10^{-1}$ - $\mu_1 = 2$, $\mu_2 = 1$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
Bi-CGStab	M_{Add}	112	73	48	44	33	28	23	23	16	14	14	14
	M_{Mul}	112	73	50	43	33	29	23	23	15	14	14	14
GMRES(50)	M_{Add}	286	106	80	52	45	40	34	30	27	25	23	22
	M_{Mul}	286	106	79	52	45	40	34	30	27	25	23	22
GMRES(∞)	M_{Add}	106	66	57	51	45	40	34	30	27	25	23	22
	M_{Mul}	106	66	57	51	45	40	34	30	27	25	23	22

Table 3: Number of iterations with M_{Mul} and M_{Add} .

than the incomplete factorization alone, i.e. the two space cycles slow down the convergence of CG. This might be due to the fact that when the number of smoothing steps is increased, they tend to spread the right part of the spectrum if the largest eigenvalues are bigger than two. For this reason, using a damping parameter ensures that the smoother is a contraction that better clusters the right part of the spectrum around one. Nevertheless this contraction may not have a positive effect. For instance if there were a cluster beyond two it would be spread by the smoothing iterations which would possibly penalize the convergence of CG. On the other hand an isolated large eigenvalue would not affect CG, but a scaling to move it below two might shrink the complete spectrum and create clusters near the origin which might have a negative affect on CG convergence.

For unsymmetric problems, the numerical experiments reveal that the damping also plays a role. Often using a damping parameter improves the convergence of GMRES especially when the number of smoothing steps is odd. However no general trends on the suitable choice of this parameter has been revealed by the experiments.

S1RMQ4M1 - $t = 1 \cdot 10^{-1}$ - $\omega = \lambda_{\max}^{-1}(M_1 A)$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
CG	$\mu_1 + \mu_2 = 1$	204	204	134	114	104	90	89	89	83	82	78	77
	$\mu_1 + \mu_2 = 2$	204	150	97	83	76	65	64	64	60	60	56	56
	$\mu_1 + \mu_2 = 3$	204	123	80	68	62	53	53	53	49	49	46	46

BWM200 - $t = 6 \cdot 10^{-1}$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
GMRES(∞)	$\mu_1 + \mu_2 = 1$	100	100	99	94	95	94	86	77	77	77	69	61
	$\mu_1 + \mu_2 = 2$	100	93	83	79	82	79	75	69	69	69	64	62
	$\mu_1 + \mu_2 = 3$	100	65	60	56	59	56	50	45	45	45	40	36

BWM2000 - $t = 3 \cdot 10^{-1}$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
GMRES(20)	$\mu_1 + \mu_2 = 1$	66	66	56	36	19	16	12	11	11	11	11	10
	$\mu_1 + \mu_2 = 2$	66	18	16	13	11	9	8	7	7	7	7	7
	$\mu_1 + \mu_2 = 3$	66	15	13	11	9	7	6	6	6	6	6	6

FS-541-4 - $t = 8 \cdot 10^{-1}$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
GMRES(100)	$\mu_1 + \mu_2 = 1$	126	126	125	126	124	125	125	99	99	99	99	99
	$\mu_1 + \mu_2 = 2$	126	–	–	–	–	–	–	–	–	–	–	–
	$\mu_1 + \mu_2 = 3$	126	87	86	86	84	84	84	72	72	72	72	72
GMRES(∞)	$\mu_1 + \mu_2 = 1$	106	106	103	106	102	103	103	99	99	99	99	99
	$\mu_1 + \mu_2 = 2$	106	168	163	163	163	163	163	167	167	175	167	167
	$\mu_1 + \mu_2 = 3$	106	78	77	77	77	77	77	71	71	71	70	69

GRE1107 - $t = 1 \cdot 10^{-2}$ - $\omega = 1.0$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
GMRES(40)	$\mu_1 + \mu_2 = 1$	–	–	80	40	37	36	32	31	27	26	24	23
	$\mu_1 + \mu_2 = 2$	–	34	31	28	25	24	22	21	19	18	17	16
	$\mu_1 + \mu_2 = 3$	–	28	26	24	23	21	20	20	18	18	16	14

Table 4: Number of iterations with M_{Mul} when the number of smoothing steps is varied.

3.1.4 Sensitivity to the accuracy of the eigencomputation

As mentioned in the previous section, the eigenvalue calculation is performed in a pre-processing phase using ARPACK on the preconditioned matrix. In order to investigate the sensitivity of our algorithm to the eigencomputation accuracy we would like to have a similar backward error on each eigenpair and to vary it. To do this, we compute the eigenpairs of a slightly perturbed matrix, $(M_1 A + E)$, with $\frac{||E||}{||M_1 A||} = \eta$, and we use these eigenvectors to build our preconditioners and compute the backward error of these eigenvectors as if they were eigenvectors of $M_1 A$. By varying η, we can monitor the level of the backward error that becomes comparable for each eigenvector.

In Table 6, we give the number of iterations of the Krylov solvers when varying the backward error of the computed eigenvectors. As we have one backward error per eigenvector, we give the average of them in the table. It can be seen that, in general, there is no need for very high accuracy in the computation of the eigenvectors. However, if some of the eigenvectors are ill-conditioned, even a small backward error might imply a large forward error and lead us to make a correction in the wrong space. Such a behaviour can be observed on the GRE1107 matrix.

Furthermore, it seems that when the number of smoothing steps is increased the accuracy of

	ω	-	0	1	2	3	4	5	6	7	8	9	10
1138BUS - $t = 4 \cdot 10^{-1}$ - $\mu_1 + \mu_2 = 3$													
			Dimension of the coarse space										
CG	1.0	295	502	361	273	224	181	174	156	156	146	132	132
	$2\lambda_{\max}^{-1}(M_1A)$	295	221	161	120	97	81	73	69	69	62	58	58
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	299	232	166	128	104	86	81	78	77	66	61	58
	$\lambda_{\max}^{-1}(M_1A)$	295	251	185	137	111	91	83	79	77	74	69	66
BCSSTK27 - $t = 1 \cdot 10^{-1}$ - $\mu_1 + \mu_2 = 2$													
			Dimension of the coarse space										
	ω	-	0	1	2	3	4	5	6	7	8	9	10
CG	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	191	139	122	107	94	93	85	75	68	67	67	67
	$\lambda_{\max}^{-1}(M_1A)$	191	159	141	123	108	106	96	85	77	76	76	76
$\mu_1 + \mu_2 = 3$													
CG	1.0	191	1011	896	810	702	688	621	563	505	503	503	501
	$2\lambda_{\max}^{-1}(M_1A)$	191	110	98	85	74	73	65	59	53	53	53	53
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	191	119	105	91	80	79	71	63	56	56	56	56
	$\lambda_{\max}^{-1}(M_1A)$	191	138	121	106	93	91	82	74	67	67	67	67
FS-541-4 - $t = 8 \cdot 10^{-1}$ - $\mu_1 + \mu_2 = 2$													
			Dimension of the coarse space										
	ω	-	0	1	2	3	4	5	6	7	8	9	10
GMRES(100)	1.0	126	–	–	–	–	–	–	–	–	–	–	–
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	126	80	80	81	80	80	80	70	70	70	69	69
	$\lambda_{\max}^{-1}(M_1A)$	126	90	88	88	87	87	87	80	80	80	75	75
$\mu_1 + \mu_2 = 3$													
GMRES(∞)	1.0	106	87	86	86	84	84	84	72	72	72	72	72
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	106	78	77	77	77	77	77	71	71	71	70	69
	$\lambda_{\max}^{-1}(M_1A)$	106	67	67	67	65	65	67	59	59	59	58	58
PORES3 - $t = 1 \cdot 10^{-1}$ - $\mu_1 + \mu_2 = 3$													
			Dimension of the coarse space										
	ω	-	0	1	2	3	4	5	6	7	8	9	10
GMRES(10)	1.0	266	120	33	24	19	16	14	13	10	10	9	9
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	266	100	28	19	17	13	11	10	9	8	7	7
	$\lambda_{\max}^{-1}(M_1A)$	266	80	32	25	19	16	14	12	10	9	9	8
RDB2048 - $t = 4 \cdot 10^{-2}$ - $\mu_1 + \mu_2 = 3$													
			Dimension of the coarse space										
	ω	-	0	1	2	3	4	5	6	7	8	9	10
GMRES(30)	1.0	–	110	86	81	69	58	49	30	29	28	26	25
	$\frac{3}{2}\lambda_{\max}^{-1}(M_1A)$	–	327	268	265	180	152	172	90	81	57	48	48
	$\lambda_{\max}^{-1}(M_1A)$	–	507	–	–	437	360	319	131	111	83	57	57

Table 5: Number of iterations with M_{Add} when the relaxation parameter is varied.

the eigencalculation has a smaller impact on the efficiency of the preconditioner.

3.2 Implementation in applications

In the previous sections we have illustrated the potential benefit of using a few smoothing iterations to improve the convergence of the preconditioned Krylov solver. In some situations it is the only way to get convergence and the benefit is clear. In other cases it improves the convergence but we might question about:

1. Is there a final benefit in term of computational time because of the extra cost introduced by the calculation of the residual at each iteration of the smoother ?

Backward error	-	0	1	2	3	4	5	6	7	8	9	10
685BUS - IC($4 \cdot 10^{-1}$) - CG - $\mu_1 + \mu_2 = 1$												
		Dimension of the small dimensional correction space										
$\approx 1 \cdot 10^{-15}$	123	122	83	83	70	57	51	43	40	36	36	34
$\approx 1 \cdot 10^{-6}$	123	123	86	86	72	57	53	43	41	37	37	35
$\approx 1 \cdot 10^{-4}$	123	122	91	90	76	61	57	46	43	39	38	36
$\approx 1 \cdot 10^{-3}$	123	122	127	127	108	94	81	67	59	49	49	44
$\approx 1 \cdot 10^{-2}$	123	122	134	130	290	298	261	257	250	207	210	209
$\mu_1 + \mu_2 = 3$												
$\approx 1 \cdot 10^{-15}$	123	60	43	44	35	28	28	22	20	19	19	17
$\approx 1 \cdot 10^{-6}$	123	60	46	46	38	32	29	24	22	19	18	18
$\approx 1 \cdot 10^{-4}$	123	60	49	49	41	34	25	25	24	21	20	19
$\approx 1 \cdot 10^{-3}$	123	60	64	59	49	48	39	30	30	24	24	22
$\approx 1 \cdot 10^{-2}$	123	60	81	74	76	75	79	63	71	68	67	65

Backward error	-	0	1	2	3	4	5	6	7	8	9	10
HOR131 - ILU($2 \cdot 10^{-1}$) - GMRES(30) - $\mu_1 + \mu_2 = 1$												
		Dimension of the small dimensional correction space										
$\approx 1 \cdot 10^{-14}$	93	93	63	57	55	53	51	47	45	39	39	36
$\approx 1 \cdot 10^{-4}$	93	93	63	57	54	52	52	47	45	40	39	35
$\approx 1 \cdot 10^{-3}$	93	93	57	58	58	54	54	48	47	40	41	37
$\approx 1 \cdot 10^{-2}$	93	93	106	114	134	114	150	103	139	137	143	149
$\mu_1 + \mu_2 = 3$												
$\approx 1 \cdot 10^{-14}$	93	56	37	38	32	32	32	26	25	23	22	21
$\approx 1 \cdot 10^{-4}$	93	56	37	38	35	32	32	28	24	22	23	21
$\approx 1 \cdot 10^{-3}$	93	56	36	35	32	31	33	27	27	24	24	21
$\approx 1 \cdot 10^{-2}$	93	56	53	57	68	63	73	53	79	59	58	82

Backward error	-	0	1	2	3	4	5	6	7	8	9	10
GRE1107 - ILU($1 \cdot 10^{-2}$) - GMRES(30) - $\mu_1 + \mu_2 = 1$												
		Dimension of the small dimensional correction space										
$\approx 1 \cdot 10^{-14}$	–	–	80	76	37	34	32	31	27	26	24	23
$\approx 1 \cdot 10^{-8}$	–	–	95	80	40	36	33	34	31	30	29	28
$\approx 1 \cdot 10^{-7}$	–	–	117	105	77	40	38	38	34	33	33	25
$\approx 1 \cdot 10^{-6}$	–	–	440	80	79	78	77	67	40	74	39	38
$\mu_1 + \mu_2 = 3$												
$\approx 1 \cdot 10^{-14}$	–	28	26	25	22	21	19	18	17	16	15	14
$\approx 1 \cdot 10^{-8}$	–	28	27	26	24	23	21	21	19	19	18	18
$\approx 1 \cdot 10^{-7}$	–	28	27	27	26	24	23	24	21	21	21	20
$\approx 1 \cdot 10^{-6}$	–	28	29	27	27	27	26	25	26	27	25	25

Table 6: Sensitivity of M_{Mul} efficiency versus the accuracy of the eigencomputation.

2. Is there a way to alleviate this residual calculation cost ?

The answer to the first question is very problem and machine dependent. On parallel distributed computer, the dot product calculation in CG and the orthogonalization process in GMRES are the main bottleneck for performance; reducing the number of iterations is an obvious way to alleviate this cost. When the preconditioner enables a significant reduction of iterations of the Krylov solvers, the extra cost of the matrix-vector products (that usually involves only neighbour to neighbour communication) can eventually be compensated by the decrease of the cost of the dot products (that involves global communications).

One route to reduce this extra cost, is to implement a cheaper but approximated matrix-vector product. Such a possibility exists for instance when the fast multipole techniques are implemented to compute the matrix vector product in electromagnetics application. Another example arises in non-overlapping domain decomposition and is further investigated in the next section.

3.2.1 A case study in semiconductor device simulation

The numerical simulation of 2D semiconductor devices is extremely demanding in term of computational time because it involves complex embedded numerical schemes. At the kernel of these schemes is the solution of very ill-conditioned large linear systems. Such problems are challenging because they are large, very ill-conditioned, extremely badly scaled. In that respect, robust and efficient linear solvers should be selected in order to reduce the elapsed time required to successfully perform a complete simulation. In this section we present some numerical experiments obtained by using the spectral two-level preconditioners for the solution of the Schur complement systems resulting from a non-overlapping domain decomposition approach. For that example the smoother is defined by an additive Schwarz preconditioner for the Schur complement system, we refer the reader to [11] for more details on this application.

In this implementation of the iterative substructuring approach the Schur complement matrix that is sparse with dense blocks is computed explicitly. In order to reduce the computational cost of the residual calculation involved in each step of the smoother, we consider a sparsified Schur complement to perform this operation. This sparse approximation of the Schur complement is obtained by dropping all the entries that are smaller than a prescribed threshold. Even though we have not considered other variants to get this approximation we mention that other possibilities exist. For instance if the Schur complement is not explicitly formed we might have considered some approximations computed using incomplete factorizations for the local subproblems or using a probing technique [8].

In Table 7 we report numerical experiments observed on one of the SPD linear systems that has to be solved in our semiconductor device simulation. It corresponds to a partitioning of the domain into eight sub-domains as shown in Figure 5. The complete domain is discretized by 155 000 degrees of freedom (dof) and the size of the interface between the subdomains (i.e. the size of the associated Schur complement matrix) is 1607. In this table, we vary the sparsity of $\tilde{S}$ the sparse approximation of the Schur complement matrix and report the number of CG iterations required to obtain a reduction of the normalized unpreconditioned residual by 10^{-11}. This very small threshold for the stopping criterion of the linear solver was required to ensure the convergence of the nonlinear scheme. It can be seen that when the size of the coarse space is varied more than half of the entries of S can be dropped for the implementation of the smoother without affecting the numerical behaviour of M_{Mul}. This enables us to save a significant amount of floating-point operations involved in the smoother and consequently to reduce substantially its cost. Dropping 70% of the entries doubles the number of iterations while it divided by three the cost of applying the preconditioner. Depending on the target computer this latter situation might eventually lead to saving time.

$\mu_1 + \mu_2 = 2$											
Density	Dimension of the coarse space										
	-	1	2	3	4	5	6	7	8	9	10
29%	52	34	39	38	41	38	37	37	37	37	36
42%	52	30	27	25	22	20	19	18	18	19	18
49%	52	28	25	24	21	19	18	17	17	17	17
58%	52	28	25	24	21	19	18	17	17	17	17
100%	52	28	25	24	21	19	18	17	17	17	17

Table 7: Experiments with M_{Mul} on SPD systems arising in semiconductor device modelling when the density of $\tilde{S}$ is varied - $\omega = \frac{3}{2}\lambda_{\max}^{-1}(M_1 A)$.

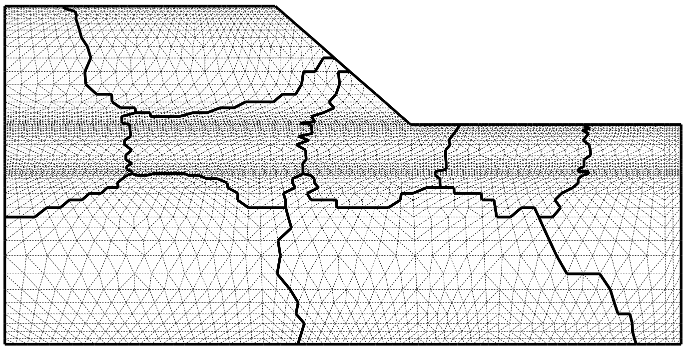


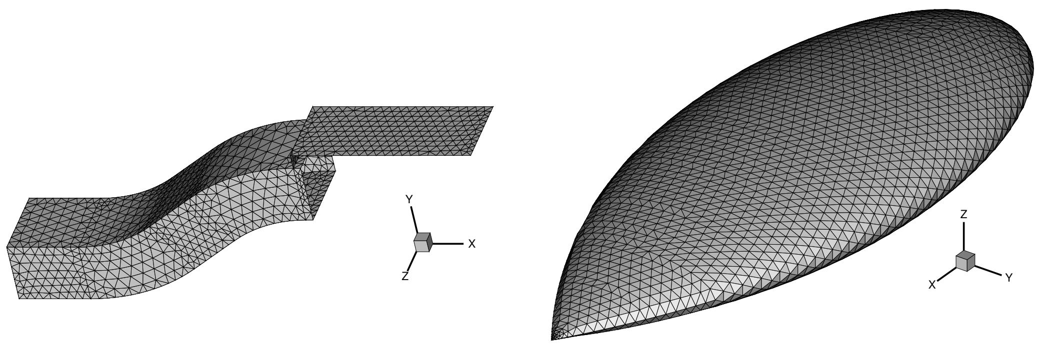
Figure 5: Mesh partitioning of the semiconductor discretization.

3.2.2 A case study in electromagnetism applications

Electromagnetic scattering problems give rise to linear systems that are challenging to solve by iterative methods. For 3D problems the boundary integral formulation of Maxwell's equations is often selected because of its nice approximation properties. This approach gives rise to dense complex systems. In recent years, the introduction of fast methods with reduced computational complexity and storage requirement has attracted an increasing interest in the use of preconditioned Krylov methods for the simulation of real-life electromagnetic problems. In this section, we consider the surface integral formulation of Maxwell's equations modelled via the electric-field integral equation (EFIE), that is the most general but the most difficult formulation to solve for iterative solvers. Thus preconditioning is crucial and approximate inverse methods based on Frobenius-norm minimization have proved to be amongst the most effective preconditioners for solving these systems efficiently. We refer the reader to [7] and the references therein for a more detailed presentation of this preconditioning technique on this class of problems.

The Frobenius-norm minimization preconditioner M_{FROB} is very effective in clustering most of the eigenvalues near the one, but tends to leave a few isolated eigenvalues close to zero. The presence of these very small eigenvalues can slow down the convergence of iterative solvers especially on large problems, where the memory constraints prevent the use of large restarts in GMRES that is the best suited solver for those problems. In this section, we apply the two-space spectral preconditioner on top of the Frobenius-norm minimization method. The preconditioner M_{FROB} is actually used to define a stationary iterative scheme implemented as a smoother. In the numerical experiments, the initial guess is the zero vector and we consider a right preconditioned GMRES method and the threshold for the stopping criterion is set to 10^{-3} on the normwise backward error $\frac{||r||}{||b||}$, where r denotes the residual and b the right-hand side of the linear system. This tolerance is accurate for engineering purposes, as it enables the correct reconstruction of the radar cross section of the objects. The parallel runs have been performed in single precision complex arithmetic on sixteen processors of a HP-Compaq Alpha server, a cluster of Symmetric Multi-Processors. We report on results on two industrial problems, namely a Cobra (Figure 6(a)) discretized with 60 695 dof and an Almond (Figure 6(b)) discretized with 104 793 dof. We mention that the eigenvectors are computed in forward mode by ARPACK in a preprocessing phase. The extra-cost associated with this pre-processing is quickly amortized as many right-hand sides have usually to be solved to compute the so call radar cross section.

For the matrix-vector products, we use the Fast Multipole Method (FMM) [13] that performs



(a) Cobra

(b) Almond

Figure 6: Mesh associated with test examples.

fast matrix-vector products in O($n$log$n$) arithmetic operations. More precisely, a highly accurate FMM is used for the standard matrix-vector product operation within GMRES, and a less accurate FMM is used for the extra matrix-vector products involved in the smoother of the preconditioning operation. The price to pay for the accuracy is mainly computational time, high accuracy also means more time consuming to compute.

In Table 8 we report on the number of iterations and the elapsed-time for different sizes of the coarse space and an increasing number of smoothing steps. For the two tests examples we show results with restarted and full GMRES. For these experiments we setup $\omega^{-1} = \frac{2}{3}\lambda_{max}(M_{FROB}A)$, that seems to be an overall good choice. On these problems M_{Add} and M_{Mul} give the same number of iterations. For this reason we only report on the M_{Add} preconditioner that is about 33 % faster than M_{Mul} as it requires less matrix-vector operations as indicated in Section 2.4. For instance on the Almond test problem using 3 smoothing steps and fifty eigenvectors both converge in 87 iterations with GMRES(10), M_{Mul} takes 10^m and M_{Add} only 6^m.

In this table we also display the number of iterations and the elapsed time of GMRES with only M_{FROB}. It can be seen that the use of M_{Add} is always beneficial both from a number iterations point of view but also from a computational time view point. The gain in time varies from fourteen to infinity with GMRES(10) and between two and three with GMRES(∞).

From a numerical point of view we observe on these examples the same behaviour as before. That are, the larger the coarse space, the better the preconditioner; the number of GMRES iterations decreases when the number of smoothing steps is increased. Furthermore, the gain is larger if restarted GMRES is considered than if full GMRES is used as solver. In particular on the Almond with GMRES(10) and less that fifty eigenvectors, the only way to get convergence is to perform a few steps of smoothing; with fifty eigenvectors the gain introduced by the smoothing iterations is still tremendous (i.e. larger than 21). On the Cobra problem, using fifteen eigenvectors the gain is far larger than two with GMRES(10), and close to two for full GMRES.

Not only the number of iterations is significantly reduced but also the solution time. On the Almond problem, using fifty eigenvectors and GMRES(10) we gain a factor of twelve in elapsed-time when we increase the number of smoothing steps from one to three. We mention that on large electromagnetic problems (of size larger than 0.5 million unknowns) the use of small restarts is recommended in order to save the heavy cost of reorthogonalization and reduce the final solution cost. The choice of a small restart is also dictated by memory constraints [7]. With full GMRES, the number of iterations is significantly decreased but the total solution cost is likely to only slightly decrease when the number of smoothing steps is large as the preconditioner is expensive to apply. The optimal selection of the size of the coarse space and of the number of smoothing steps remains an open question, and the choice mainly depends on the clustering properties of the initial preconditioner. In terms of computational cost the coarse grid correction and the smoothing mechanism are complementary components that have to be suitably combined. On the Cobra problem, for instance, using three smoothing steps and five eigenvectors we obtain better convergence rate but similar computational time than using one smoothing step but fifteen eigenvectors; on the Almond problem, the solution cost of GMRES(10) with three smoothing steps and thirty eigenvectors is similar to the time with with two smoothing steps but fifty eigenvectors. Enabling a reduction of the number of eigenvectors by using more smoothing steps is a desirable feature in the contexts where computing many eigenvalues can become very expensive which is for instance the case in this application for very large problems.

4 Concluding remarks

In this work, we exploit some ideas from the multigrid philosophy to derive new additive and multiplicative spectral two-level preconditioners for general linear systems. We propose a scheme that enables to improve a given preconditioner that leaves only few eigenvalues close to zero. We study the spectrum of the new preconditioned matrix associated with these schemes. The effectiveness and robustness of these preconditioners is mainly due to their ability to shift to one a selected set of eigenvalues and to cluster near one most of others. We illustrate the attractive numerical behaviour of these new preconditioners on a wide range set of matrices from Matrix Market as well as on two real-world applications. On electromagnetism industrial problems, we show that the preconditioner enables us to save a significant amount of time.

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Cobra problem			
With M_{FROB}	GMRES(10) 2719 iterations (1^h 10^m)		
	GMRES(∞) 378 iterations (18^m)		
$\mu_1 + \mu_2 = 1$			
	Dimension of the coarse space		
	5	10	15
GMRES(10)	1458 (42^m)	594 (12^m)	517 (11^m)
GMRES(∞)	262 (9^m)	216 (7^m)	188 (6^m)
$\mu_1 + \mu_2 = 2$			
	Dimension of the coarse space		
	5	10	15
GMRES(10)	471 (18^m)	209 (7^m)	201 (6^m)
GMRES(∞)	161 (8^m)	132 (6^m)	115 (5^m)
$\mu_1 + \mu_2 = 3$			
	Dimension of the coarse space		
	5	10	15
GMRES(10)	281 (12^m)	132 (6^m)	124 (5^m)
GMRES(∞)	120 (7^m)	98 (6^m)	85 (5^m)
Almond problem			
With M_{FROB}	GMRES(10) +3000 iterations		
	GMRES(∞) 242 iterations (14^m)		
$\mu_1 + \mu_2 = 1$			
	Dimension of the coarse space		
	10	30	50
GMRES(10)	+3000	+3000	1867 ($1^h 12^m$)
GMRES(∞)	229 (13^m)	157 (8^m)	132 (6^m)
$\mu_1 + \mu_2 = 2$			
	Dimension of the coarse space		
	10	30	50
GMRES(10)	552 (29^m)	245 (14^m)	176 (9^m)
GMRES(∞)	134 (9^m)	92 (6^m)	77 (6^m)
$\mu_1 + \mu_2 = 3$			
	Dimension of the coarse space		
	10	30	50
GMRES(10)	216 (16^m)	116 (9^m)	87 (6^m)
GMRES(∞)	97 (9^m)	66 (6^m)	56 (6^m)

Table 8: Experiments with M_{Add} on the electromagnetics problems.

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A Extensive numerical experiments

A.1 SPD matrices

$\mu_1 = 1,\ \mu_2 = 0$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	123	123	83	83	70	57	51	43	40	36	36	34
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	122	122	83	83	70	57	51	43	40	36	36	34
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	122	122	83	83	70	57	51	43	40	36	36	34
$\omega^{-1} = \lambda_{\max}(M_1A)$	122	122	84	83	70	56	51	43	40	36	36	34

$\mu_1 = 1,\ \mu_2 = 1$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	122	84	57	57	48	38	36	30	28	25	25	23
$\omega^{-1} = \lambda_{\max}(M_1A)$	122	99	66	66	56	45	42	34	32	29	29	27

$\mu_1 = 2,\ \mu_2 = 1$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	123	88	60	61	50	40	39	31	30	26	26	25
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	122	64	43	43	37	29	27	22	21	19	19	18
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	122	69	47	47	40	32	30	24	23	21	20	19
$\omega^{-1} = \lambda_{\max}(M_1A)$	122	83	56	56	47	38	35	29	28	25	25	23

Table 9: Number of iterations with M_{Mul} on 685BUS - $t = 4 \cdot 10^{-1}$ - $\lambda_{max}(M_1A) = 18.6$.

$\mu_1 = 1,\ \mu_2 = 0$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	295	295	212	162	130	112	101	98	96	83	78	76
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	295	295	224	172	134	111	107	101	101	86	79	78
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	299	299	221	170	136	112	107	97	102	87	83	79
$\omega^{-1} = \lambda_{\max}(M_1A)$	295	295	217	165	130	109	105	99	98	85	79	77

$\mu_1 = 1,\ \mu_2 = 1$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	299	250	186	140	113	99	85	81	81	74	73	69
$\omega^{-1} = \lambda_{\max}(M_1A)$	295	269	197	151	119	102	95	85	85	76	71	71

$\mu_1 = 2,\ \mu_2 = 1$	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	295	502	361	273	224	181	174	156	156	146	132	132
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	295	221	161	120	97	81	73	69	69	62	58	58
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	299	232	166	128	104	86	81	78	77	66	61	58
$\omega^{-1} = \lambda_{\max}(M_1A)$	295	251	185	137	111	91	83	79	77	74	69	66

Table 10: Number of iterations with M_{Mul} on 1138BUS - $t = 4 \cdot 10^{-1}$ - $\lambda_{max}(M_1A) = 3.4$.

$\mu_1 = 1,\ \mu_2 = 0$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	191	191	170	149	130	128	115	102	92	92	92	92
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	191	191	171	150	131	128	116	103	93	93	92	93
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	191	191	171	150	130	128	116	103	93	93	93	93
$\omega^{-1} = \lambda_{\max}(M_1A)$	191	191	170	150	131	128	115	103	93	93	92	93
$\mu_1 = 1,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	191	139	122	107	94	93	85	75	68	67	67	67
$\omega^{-1} = \lambda_{\max}(M_1A)$	191	159	141	123	108	106	96	85	77	76	76	76
$\mu_1 = 2,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	191	1011	896	810	702	688	621	563	505	503	503	501
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	191	110	98	85	74	73	65	59	53	53	53	53
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	191	119	105	91	80	79	71	63	56	56	56	56
$\omega^{-1} = \lambda_{\max}(M_1A)$	191	138	121	106	93	91	82	74	67	67	67	67

Table 11: Number of iterations with M_{Mul} on BCSSTK27 - $t = 1 \cdot 10^{-1}$ - $\lambda_{max}(M_1A) = 25.4$.

$\mu_1 = 1,\ \mu_2 = 0$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	114	114	91	80	76	70	64	56	54	53	53	48
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	116	116	94	82	79	74	66	58	56	55	53	50
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	116	116	92	83	80	72	66	56	56	55	54	50
$\omega^{-1} = \lambda_{\max}(M_1A)$	116	116	91	80	76	71	63	56	54	54	52	49
$\mu_1 = 1,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	116	94	77	68	64	60	55	46	46	45	44	40
$\omega^{-1} = \lambda_{\max}(M_1A)$	116	102	82	72	69	64	56	48	49	48	46	42
$\mu_1 = 2,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	114	163	134	117	109	104	93	83	82	82	80	73
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	116	81	66	58	54	51	46	39	39	38	37	34
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	116	85	70	61	57	53	47	42	40	40	39	36
$\omega^{-1} = \lambda_{\max}(M_1A)$	116	94	74	66	62	58	53	46	44	44	42	38

Table 12: Number of iterations with M_{Mul} on BCSSTK14 - $t = 1 \cdot 10^{-1}$ - $\lambda_{max}(M_1A) = 13.8$.

$\mu_1 = 1,\ \mu_2 = 0$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	150	150	115	97	83	73	70	70	68	63	65	65
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	150	150	123	102	86	78	73	73	72	67	68	68
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	151	151	125	103	88	77	74	74	73	69	69	69
$\omega^{-1} = \lambda_{\max}(M_1A)$	150	150	116	98	82	74	70	70	68	65	65	66
$\mu_1 = 1,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	151	132	107	89	75	67	65	64	62	58	58	58
$\omega^{-1} = \lambda_{\max}(M_1A)$	150	141	110	90	77	70	66	66	62	61	60	60
$\mu_1 = 2,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	150	254	196	167	142	134	125	124	120	116	116	118
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	150	122	95	79	67	61	58	57	56	52	54	53
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	151	125	99	83	69	63	61	59	58	54	56	56
$\omega^{-1} = \lambda_{\max}(M_1A)$	150	133	103	85	72	66	62	62	61	55	55	55

Table 13: Number of iterations with M_{Mul} on BCSSTK15 - $t = 1 \cdot 10^{-2}$ - $\lambda_{max}(M_1A) = 25.3$.

$\mu_1 = 1,\ \mu_2 = 0$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	49	49	45	42	38	35	34	32	30	28	27	26
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	49	49	45	41	38	35	34	32	30	28	27	26
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	49	49	45	41	38	35	34	32	30	28	27	26
$\omega^{-1} = \lambda_{\max}(M_1A)$	49	49	45	42	38	35	34	32	30	28	27	26
$\mu_1 = 1,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	49	29	26	24	22	20	20	18	17	16	16	15
$\omega^{-1} = \lambda_{\max}(M_1A)$	49	35	33	30	27	25	25	23	21	20	19	19
$\mu_1 = 2,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	49	26	24	22	21	19	19	17	16	15	15	14
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	49	27	25	23	22	20	19	18	17	16	16	15
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	49	24	23	21	19	18	17	16	15	14	13	13
$\omega^{-1} = \lambda_{\max}(M_1A)$	49	29	27	24	22	21	20	19	17	16	16	15

Table 14: Number of iterations with M_{Mul} on BCSSTK16 - $t = 5 \cdot 10^{-2}$ - $\lambda_{max}(M_1A) = 1.9$.

$\mu_1 = 1,\ \mu_2 = 0$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	204	204	133	114	104	90	89	89	83	82	78	77
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	204	204	133	114	104	90	89	89	83	82	78	77
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	204	204	133	114	104	90	89	89	83	82	78	77
$\omega^{-1} = \lambda_{\max}(M_1A)$	204	204	134	114	104	90	89	89	83	82	78	77
$\mu_1 = 1,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	204	123	80	68	62	54	53	53	50	50	47	46
$\omega^{-1} = \lambda_{\max}(M_1A)$	204	150	97	83	76	65	64	64	60	60	56	56
$\mu_1 = 2,\ \mu_2 = 1$												
	Dimension of the coarse space											
	-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$	204	273	179	154	141	122	120	120	111	111	108	104
$\omega^{-1} = \frac{1}{2}\lambda_{\max}(M_1A)$	204	114	72	60	56	50	49	49	46	46	44	43
$\omega^{-1} = \frac{2}{3}\lambda_{\max}(M_1A)$	204	105	69	58	53	46	45	45	43	43	40	39
$\omega^{-1} = \lambda_{\max}(M_1A)$	204	123	80	68	62	53	53	53	49	49	46	46

Table 15: Number of iterations with M_{Mul} on S1RMQ4M1 - $t = 1 \cdot 10^{-1}$ - $\lambda_{max}(M_1A) = 4.0$.

A.2 Unsymmetric matrices

		-	0	1	2	3	4	5	6	7	8	9	10
$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	286	286	148	121	92	64	54	48	43	39	36	35
	BiCGStab	112	112	73	64	50	41	34	33	26	24	24	20
M_{Mul}	GMRES(50)	286	286	147	121	92	65	55	48	43	39	36	35
	BiCGStab	112	112	72	68	52	41	34	35	27	25	25	20
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	286	147	121	92	64	54	48	43	39	36	35
	BiCGStab	117	117	78	63	52	42	36	35	29	25	25	21
M_{Mul}	GMRES(50)	286	286	147	121	92	65	55	48	43	39	36	35
	BiCGStab	117	117	72	67	52	42	34	34	26	25	25	20
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	286	147	121	92	64	55	48	43	39	36	35
	BiCGStab	109	109	77	66	48	41	37	34	30	26	25	21
M_{Mul}	GMRES(50)	286	286	147	121	92	65	55	48	43	39	36	35
	BiCGStab	109	109	81	69	48	41	34	35	24	25	25	21
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	286	–	–	388	–	348	197	244	195	177	186	109
	BiCGStab	112	198	171	171	157	136	107	102	92	103	99	92
M_{Mul}	GMRES(50)	286	–	–	390	–	348	198	244	149	188	191	125
	BiCGStab	112	198	164	169	143	133	122	125	98	86	89	77
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	101	60	48	43	37	31	28	25	23	21	20
	BiCGStab	117	64	52	39	29	26	23	20	16	15	13	11
M_{Mul}	GMRES(50)	286	101	60	48	43	37	31	28	25	23	21	20
	BiCGStab	117	64	46	38	29	25	21	21	18	15	15	11
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	153	95	66	53	45	38	34	31	28	26	25
	BiCGStab	109	77	55	47	34	30	24	26	19	18	16	14
M_{Mul}	GMRES(50)	286	153	95	65	52	45	38	34	31	28	26	25
	BiCGStab	109	77	53	44	35	29	26	24	17	15	15	14
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	286	106	80	52	45	40	34	30	27	25	23	22
	BiCGStab	112	73	48	44	33	28	23	23	16	14	14	14
M_{Mul}	GMRES(50)	286	106	79	52	45	40	34	30	27	25	23	22
	BiCGStab	112	73	50	43	33	29	23	23	15	14	14	14
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	55	47	42	37	33	28	25	22	20	18	17
	BiCGStab	117	57	37	31	26	21	16	15	13	11	11	9
M_{Mul}	GMRES(50)	286	55	47	42	37	32	27	24	22	20	18	18
	BiCGStab	117	57	34	31	27	21	17	15	13	11	11	9
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	286	101	60	49	43	37	31	28	25	23	21	20
	BiCGStab	109	60	44	36	30	25	20	18	16	13	13	11
M_{Mul}	GMRES(50)	286	101	60	48	43	37	31	28	25	23	21	20
	BiCGStab	109	60	42	38	29	25	19	18	16	13	13	12

Table 16: Number of iterations with M_{Mul} and M_{Add} on BFW398A - $t = 5 \cdot 10^{-1}$ - $\lvert\lambda_{max}(M_1A)\rvert = 2.17$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(20)	–	–	197	238	238	238	126	100	100	100	87	73
	BiCGStab	96	96	84	73	69	72	67	57	56	56	50	39
M_{Mul}	GMRES(20)	–	–	198	238	238	238	130	100	100	100	87	73
	BiCGStab	96	96	85	74	72	66	69	55	55	55	50	39
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	–	–	238	238	238	118	100	100	100	87	73
	BiCGStab	96	96	81	69	66	69	69	59	54	54	50	39
M_{Mul}	GMRES(20)	–	–	–	238	238	238	130	100	100	100	87	73
	BiCGStab	96	96	80	71	69	68	69	58	58	57	52	39
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	–	–	238	238	238	122	100	100	100	87	73
	BiCGStab	96	96	83	72	72	72	81	55	59	55	50	45
M_{Mul}	GMRES(20)	–	–	–	238	238	238	129	100	100	100	87	73
	BiCGStab	96	96	79	72	72	72	67	57	57	56	49	45
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(20)	–	151	115	118	118	118	116	97	97	97	99	115
	BiCGStab	96	96	53	54	54	54	44	44	44	44	40	40
M_{Mul}	GMRES(20)	–	151	115	118	118	118	96	97	97	97	99	115
	BiCGStab	96	96	61	54	54	54	44	44	44	44	40	42
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	206	198	97	97	97	66	58	58	58	46	40
	BiCGStab	96	55	45	43	42	42	39	32	32	32	27	26
M_{Mul}	GMRES(20)	–	206	191	97	97	97	71	58	58	58	47	40
	BiCGStab	96	55	46	43	42	43	38	32	32	32	29	26
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	284	287	112	112	112	90	73	73	73	60	51
	BiCGStab	96	68	55	58	55	53	46	41	41	41	37	30
M_{Mul}	GMRES(20)	–	284	304	112	112	112	82	73	73	73	61	51
	BiCGStab	96	68	57	55	54	51	46	41	41	41	39	30
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(20)	–	218	101	98	116	98	72	59	59	59	49	40
	BiCGStab	96	52	44	41	43	41	38	31	31	31	28	26
M_{Mul}	GMRES(20)	–	218	101	98	118	98	72	59	59	59	49	40
	BiCGStab	96	52	48	41	44	41	38	31	31	31	28	26
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	162	81	85	85	85	51	51	51	51	41	34
	BiCGStab	95	45	39	34	34	34	29	29	29	29	24	21
M_{Mul}	GMRES(20)	–	162	81	85	85	85	51	51	51	51	41	34
	BiCGStab	95	45	41	34	34	34	29	29	29	29	24	21
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(20)	–	203	99	97	97	97	66	58	58	58	48	40
	BiCGStab	96	53	45	40	40	40	38	33	33	33	28	28
M_{Mul}	GMRES(20)	–	203	99	97	97	97	71	58	58	58	47	40
	BiCGStab	96	53	49	40	40	40	36	33	33	33	29	28

Table 17: Number of iterations with M_{Mul} and M_{Add} on BWM200 - $t = 6 \cdot 10^{-1}$ - $|\lambda_{max}(M_1 A)| = 2.00$.

$\mu_1 = 0$, $\mu_2 = 1$												
		Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$												
M_{Add}	GMRES(20)	66	66	55	36	20	15	12	12	12	12	12	11
	BiCGStab	28	28	31	23	15	11	8	9	7	7	7	7
M_{Mul}	GMRES(20)	66	66	56	36	19	16	12	11	11	11	11	10
	BiCGStab	28	28	29	18	14	11	7	7	7	7	6	6
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	66	60	36	20	16	12	12	12	12	12	12
	BiCGStab	28	28	30	25	14	11	7	8	8	7	7	7
M_{Mul}	GMRES(20)	66	66	60	36	19	16	12	12	11	12	12	12
	BiCGStab	28	28	25	20	15	9	7	7	7	6	6	8
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	66	55	36	19	15	13	13	12	12	11	12
	BiCGStab	30	30	25	24	16	10	8	7	7	7	7	7
M_{Mul}	GMRES(20)	66	66	50	36	19	16	13	12	12	12	12	11
	BiCGStab	30	30	28	20	16	10	7	7	7	7	7	7
$\mu_1 = 1$, $\mu_2 = 1$												
		Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$												
M_{Add}	GMRES(20)	66	18	16	13	11	9	7	7	7	7	7	7
	BiCGStab	28	19	16	12	9	6	4	4	4	4	4	4
M_{Mul}	GMRES(20)	66	18	16	13	11	9	8	7	7	7	7	7
	BiCGStab	28	19	17	12	9	6	4	4	4	4	4	4
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	31	19	16	13	12	9	9	9	9	10	10
	BiCGStab	28	22	20	14	12	9	6	7	5	6	5	5
M_{Mul}	GMRES(20)	66	31	19	16	14	12	10	10	10	10	9	9
	BiCGStab	28	22	19	14	10	8	5	5	5	5	5	5
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	24	18	16	13	10	8	8	8	8	8	8
	BiCGStab	30	22	23	12	10	8	5	5	5	5	5	4
M_{Mul}	GMRES(20)	66	24	18	15	12	10	8	8	8	7	7	8
	BiCGStab	30	22	17	15	10	8	5	4	4	4	4	4
$\mu_1 = 2$, $\mu_2 = 1$												
		Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$												
M_{Add}	GMRES(20)	66	15	13	11	9	7	6	6	6	6	6	6
	BiCGStab	28	14	12	9	7	5	4	3	3	4	3	3
M_{Mul}	GMRES(20)	66	15	13	11	9	7	6	6	6	6	6	6
	BiCGStab	28	14	12	8	7	5	4	3	3	3	3	3
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	17	15	12	11	9	7	8	8	8	8	8
	BiCGStab	29	15	13	10	9	6	4	4	5	4	5	5
M_{Mul}	GMRES(20)	66	17	14	13	10	8	7	7	7	7	7	7
	BiCGStab	29	15	14	10	8	6	4	4	4	4	4	4
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$												
M_{Add}	GMRES(20)	66	16	14	12	9	8	7	6	6	6	6	6
	BiCGStab	30	16	12	10	7	5	4	4	4	4	3	3
M_{Mul}	GMRES(20)	66	16	14	12	9	8	6	6	5	5	6	5
	BiCGStab	30	16	13	11	8	5	3	4	3	3	3	3

Table 18: Number of iterations with M_{Mul} and M_{Add} on BWM2000 - $t = 3 \cdot 10^{-1}$ - $|\lambda_{max}(M_1 A)| = 1.3$.

		-	0	1	2	3	4	5	6	7	8	9	10
$\mu_1 = 0,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(100)	126	126	125	125	123	123	124	99	99	99	98	98
	BiCGStab	51	51	51	51	51	52	52	55	54	52	52	52
M_{Mul}	GMRES(100)	126	126	125	126	124	125	125	99	99	99	99	99
	BiCGStab	51	51	54	51	51	55	54	54	51	54	51	49
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	126	124	124	122	122	122	99	99	99	98	98
	BiCGStab	51	51	52	52	52	52	52	52	52	51	51	52
M_{Mul}	GMRES(100)	126	126	124	125	122	123	123	99	99	99	98	98
	BiCGStab	51	51	51	51	51	52	51	51	51	51	51	53
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	126	124	127	126	126	125	100	100	99	99	99
	BiCGStab	51	51	52	53	57	55	54	53	54	55	53	54
M_{Mul}	GMRES(100)	126	126	124	126	124	125	124	100	100	100	99	99
	BiCGStab	51	51	56	51	51	52	52	52	52	53	52	52
$\mu_1 = 1,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(100)	126	–	–	–	–	–	–	–	–	–	–	–
	BiCGStab	51	439	437	476	497	406	465	467	465	431	469	440
M_{Mul}	GMRES(100)	126	–	–	–	–	–	–	–	–	–	–	–
	BiCGStab	51	439	468	466	439	523	471	452	493	445	464	481
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	80	80	80	80	80	80	71	70	70	69	70
	BiCGStab	51	30	28	31	31	31	31	30	31	32	31	31
M_{Mul}	GMRES(100)	126	80	80	81	80	80	80	70	70	70	69	69
	BiCGStab	51	30	28	30	31	31	31	31	31	31	31	31
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	90	88	88	88	87	88	81	80	81	75	75
	BiCGStab	51	38	39	39	39	40	38	39	36	38	38	38
M_{Mul}	GMRES(100)	126	90	88	88	87	87	87	80	80	80	75	75
	BiCGStab	51	38	38	39	39	39	38	38	38	38	38	38
$\mu_1 = 2,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(100)	126	87	84	86	84	84	82	73	73	71	72	72
	BiCGStab	51	40	40	42	40	40	40	40	41	41	42	41
M_{Mul}	GMRES(100)	126	87	86	86	84	84	84	72	72	72	72	72
	BiCGStab	51	40	47	40	40	41	41	40	41	41	42	40
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	67	65	67	65	65	67	59	59	59	55	55
	BiCGStab	53	25	25	27	25	26	27	25	25	26	25	27
M_{Mul}	GMRES(100)	126	67	67	67	65	65	67	59	59	59	58	58
	BiCGStab	53	25	27	27	24	27	27	27	27	27	24	25
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(100)	126	78	77	77	77	76	76	71	71	71	67	67
	BiCGStab	51	31	32	29	28	32	32	32	32	32	29	32
M_{Mul}	GMRES(100)	126	78	77	77	77	77	77	71	71	71	70	69
	BiCGStab	51	31	31	31	31	31	31	31	31	31	31	32

Table 19: Number of iterations with M_{Mul} and M_{Add} on FS-541-4 - $t = 8 \cdot 10^{-1}$ - $|\lambda_{max}(M_1 A)| = 2.94$.

$\mu_1 = 0,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(40)	–	–	80	40	38	36	33	33	29	27	25	24
	BiCGStab	97	97	101	97	69	62	77	66	68	65	54	25
M_{Mul}	GMRES(40)	–	–	80	40	37	36	32	31	27	26	24	23
	BiCGStab	97	97	81	99	66	65	55	48	44	33	38	27
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	–	98	78	40	37	34	33	29	28	26	25
	BiCGStab	117	117	96	107	68	85	71	85	45	64	40	26
M_{Mul}	GMRES(40)	–	–	84	78	40	36	33	33	29	28	26	25
	BiCGStab	117	117	86	101	82	80	85	71	50	44	38	35
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	–	99	78	40	38	35	35	30	29	27	26
	BiCGStab	97	97	85	94	114	72	84	63	81	74	41	26
M_{Mul}	GMRES(40)	–	–	99	78	40	37	35	34	30	28	27	24
	BiCGStab	97	97	105	104	94	97	70	74	56	45	39	41

$\mu_1 = 1,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(40)	–	34	31	28	26	25	22	22	19	19	17	17
	BiCGStab	97	64	63	59	45	39	36	40	37	29	27	22
M_{Mul}	GMRES(40)	–	34	31	28	25	24	22	21	19	18	17	16
	BiCGStab	97	64	62	51	45	42	48	34	27	35	29	16
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	79	39	38	33	30	28	28	24	24	22	20
	BiCGStab	117	82	94	76	50	61	56	46	54	54	60	21
M_{Mul}	GMRES(40)	–	79	39	37	33	30	28	27	24	23	21	20
	BiCGStab	117	82	84	75	67	68	58	49	47	34	34	26
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	108	76	40	36	33	31	31	27	26	24	22
	BiCGStab	97	84	91	76	73	69	68	60	40	48	24	24
M_{Mul}	GMRES(40)	–	108	76	40	36	32	31	30	27	26	24	22
	BiCGStab	97	84	77	75	60	65	68	44	61	42	39	25

$\mu_1 = 2,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(40)	–	28	26	24	23	21	20	20	18	18	16	14
	BiCGStab	97	60	62	52	40	43	37	34	26	29	21	19
M_{Mul}	GMRES(40)	–	28	26	24	22	21	19	18	17	16	15	14
	BiCGStab	97	60	58	55	51	40	34	32	26	22	19	19
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	38	34	32	29	26	25	24	21	20	19	18
	BiCGStab	109	83	77	71	55	60	46	45	31	41	25	19
M_{Mul}	GMRES(40)	–	38	34	32	29	26	25	23	21	20	18	18
	BiCGStab	109	83	82	72	58	49	62	45	36	25	24	22
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(40)	–	77	38	35	33	29	28	27	24	23	21	20
	BiCGStab	111	70	86	63	59	70	66	50	38	40	38	21
M_{Mul}	GMRES(40)	–	77	38	35	33	29	27	26	24	22	20	19
	BiCGStab	111	70	82	69	57	60	48	48	39	30	36	21

Table 20: Number of iterations with M_{Mul} and M_{Add} on GRE1107 - $t = 1 \cdot 10^{-2}$ - $|\lambda_{max}(M_1A)| = 3.48$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	93	93	62	55	54	52	52	46	44	40	40	35
	BiCGStab	42	42	35	30	30	26	25	27	23	21	20	18
M_{Mul}	GMRES(30)	93	93	60	56	53	52	51	47	45	40	39	36
	BiCGStab	42	42	30	36	30	28	25	26	23	19	22	17
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	93	63	57	55	53	51	47	45	40	40	36
	BiCGStab	42	42	32	30	31	33	26	28	25	20	22	20
M_{Mul}	GMRES(30)	93	93	63	57	55	53	51	47	45	39	39	36
	BiCGStab	42	42	31	31	28	30	28	28	25	19	21	19
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	93	65	58	55	54	52	50	46	42	41	37
	BiCGStab	40	40	45	52	34	29	31	29	26	20	23	18
M_{Mul}	GMRES(30)	93	93	63	57	54	53	50	49	44	40	39	37
	BiCGStab	40	40	64	36	36	31	31	32	23	23	22	19
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	93	95	82	72	72	72	73	59	56	54	54	55
	BiCGStab	42	76	53	47	50	42	41	49	40	43	36	36
M_{Mul}	GMRES(30)	93	95	82	80	58	68	57	58	55	54	54	55
	BiCGStab	42	76	47	46	47	44	44	44	38	37	38	40
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	61	43	39	38	36	35	35	29	27	27	24
	BiCGStab	42	30	24	25	24	21	20	20	17	14	15	13
M_{Mul}	GMRES(30)	93	61	46	39	38	35	35	34	29	27	27	24
	BiCGStab	42	30	21	22	19	19	18	18	17	13	15	12
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	71	51	49	47	46	44	40	36	32	34	27
	BiCGStab	40	34	34	28	25	23	22	21	19	17	19	18
M_{Mul}	GMRES(30)	93	71	51	50	46	45	45	40	35	32	32	27
	BiCGStab	40	34	25	24	24	23	23	21	18	16	18	15
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	93	58	41	39	38	37	37	30	28	27	26	25
	BiCGStab	42	43	32	29	28	26	26	26	23	20	22	22
M_{Mul}	GMRES(30)	93	58	41	41	38	37	37	30	28	27	26	25
	BiCGStab	42	43	29	28	30	29	29	24	22	21	20	18
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	56	34	36	32	32	32	27	25	23	23	20
	BiCGStab	42	25	19	18	18	17	16	16	13	12	13	10
M_{Mul}	GMRES(30)	93	56	40	36	31	30	30	26	25	23	23	21
	BiCGStab	42	25	18	18	18	16	16	16	13	12	13	10
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	93	60	44	39	38	36	35	35	30	27	27	24
	BiCGStab	40	30	24	22	19	19	19	19	17	16	16	13
M_{Mul}	GMRES(30)	93	60	43	40	38	35	34	33	29	26	26	24
	BiCGStab	40	30	22	21	19	19	19	19	16	14	16	14

Table 21: Number of iterations with M_{Mul} and M_{Add} on HOR131 - $t = 2 \cdot 10^{-1}$ - $|\lambda_{max}(M_1A)| = 4.45$.

$\mu_1 = 0$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	160	160	141	131	122	122	120	115	106	105	104	103
	BiCGStab	441	441	139	144	93	86	176	152	123	100	85	91
M_{Mul}	GMRES(30)	160	160	140	130	121	120	117	115	105	101	103	101
	BiCGStab	441	441	267	120	85	88	84	94	84	88	79	70
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	160	141	131	122	122	120	115	106	105	104	103
	BiCGStab	126	126	150	94	105	90	111	112	102	143	90	82
M_{Mul}	GMRES(30)	160	160	140	130	120	120	117	115	105	101	103	101
	BiCGStab	126	126	182	87	92	106	86	96	81	89	89	60
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	160	140	129	123	122	117	114	106	102	102	101
	BiCGStab	184	184	267	126	82	106	105	77	97	80	80	63
M_{Mul}	GMRES(30)	160	160	140	131	121	122	118	115	105	102	101	100
	BiCGStab	184	184	110	193	113	93	106	170	120	93	114	63
$\mu_1 = 1$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	160	93	84	80	77	76	76	74	66	63	63	64
	BiCGStab	441	59	60	53	50	52	67	106	70	68	68	39
M_{Mul}	GMRES(30)	160	93	83	80	75	76	76	74	67	62	62	63
	BiCGStab	441	59	79	62	48	52	55	49	54	50	42	38
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	93	84	80	77	76	76	74	66	63	63	63
	BiCGStab	126	63	60	54	46	55	85	81	63	60	57	39
M_{Mul}	GMRES(30)	160	93	83	80	75	76	76	74	67	64	62	64
	BiCGStab	126	63	95	53	57	56	50	63	47	53	43	40
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	113	99	97	92	90	86	85	77	77	77	76
	BiCGStab	184	94	142	113	56	60	83	56	97	63	50	48
M_{Mul}	GMRES(30)	160	113	99	96	90	90	86	84	77	76	76	75
	BiCGStab	184	94	166	89	88	73	69	60	60	63	52	46
$\mu_1 = 2$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	160	83	74	73	69	69	65	63	60	54	54	54
	BiCGStab	441	58	55	50	45	47	53	43	49	37	44	37
M_{Mul}	GMRES(30)	160	83	74	74	69	68	66	63	57	54	53	54
	BiCGStab	441	58	69	47	41	42	51	38	42	43	35	35
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	84	74	73	69	70	67	63	59	54	54	54
	BiCGStab	135	60	50	51	41	43	51	43	54	41	35	35
M_{Mul}	GMRES(30)	160	84	74	74	69	68	66	63	57	54	53	54
	BiCGStab	135	60	60	43	43	44	53	38	46	42	54	34
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	160	89	82	79	75	75	75	71	65	60	59	60
	BiCGStab	143	127	65	56	57	55	67	80	52	60	54	40
M_{Mul}	GMRES(30)	160	89	82	79	75	75	74	71	65	59	59	60
	BiCGStab	143	127	74	63	52	49	60	50	41	43	40	37

Table 22: Number of iterations with M_{Mul} and M_{Add} on ORSIRR - $t = 3 \cdot 10^{-1}$ - $|\lambda_{max}(M_1 A)| = 1.50$.

$\mu_1 = 0$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(10)	266	266	66	44	39	28	25	22	19	17	16	16
	BiCGStab	40	40	32	25	19	16	14	12	10	10	8	8
M_{Mul}	GMRES(10)	266	266	59	42	34	28	24	20	19	16	15	15
	BiCGStab	40	40	31	22	19	16	14	13	10	10	8	8
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	266	56	44	37	27	24	20	19	17	16	16
	BiCGStab	40	40	28	23	19	16	14	12	10	10	9	8
M_{Mul}	GMRES(10)	266	266	56	44	37	28	24	20	19	17	16	15
	BiCGStab	40	40	31	25	20	14	14	11	10	10	8	8
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	266	60	44	39	28	24	20	18	16	16	15
	BiCGStab	40	40	31	25	19	16	14	11	10	9	8	8
M_{Mul}	GMRES(10)	266	266	60	42	38	28	24	20	19	16	15	15
	BiCGStab	40	40	31	25	19	15	14	11	10	9	8	8
$\mu_1 = 1$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(10)	266	127	48	46	35	50	47	48	48	49	49	49
	BiCGStab	40	28	19	23	20	20	20	20	20	20	20	20
M_{Mul}	GMRES(10)	266	127	48	46	37	49	49	50	50	50	50	50
	BiCGStab	40	28	19	24	20	20	20	20	20	20	20	20
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	100	30	24	19	16	14	12	10	9	9	8
	BiCGStab	40	24	19	12	11	9	9	7	6	5	4	4
M_{Mul}	GMRES(10)	266	100	30	25	18	15	14	12	10	9	8	8
	BiCGStab	40	24	17	11	11	10	8	7	6	5	4	4
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	172	43	28	25	20	17	15	13	12	12	12
	BiCGStab	40	29	23	15	14	11	9	8	7	7	7	5
M_{Mul}	GMRES(10)	266	172	43	29	25	20	17	15	13	12	10	10
	BiCGStab	40	29	20	14	13	11	10	8	8	7	6	5
$\mu_1 = 2$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(10)	266	120	34	24	19	17	15	13	10	10	9	9
	BiCGStab	40	24	18	14	10	8	8	7	6	6	6	6
M_{Mul}	GMRES(10)	266	120	33	24	19	16	14	13	10	10	9	9
	BiCGStab	40	24	16	14	11	9	7	6	6	6	6	4
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	100	28	19	17	14	12	10	9	8	7	7
	BiCGStab	40	21	14	13	9	7	8	6	5	4	4	4
M_{Mul}	GMRES(10)	266	100	28	19	17	13	11	10	9	8	7	7
	BiCGStab	40	21	14	10	10	7	7	6	5	4	4	4
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(10)	266	80	32	25	19	16	14	13	12	10	9	9
	BiCGStab	40	23	19	15	11	9	9	7	6	5	5	4
M_{Mul}	GMRES(10)	266	80	32	25	19	16	14	12	10	9	9	8
	BiCGStab	40	23	16	12	12	10	8	7	6	5	5	4

Table 23: Number of iterations with M_{Mul} and M_{Add} on PORES3 - $t = 1 \cdot 10^{-1}$ - $|\lambda_{max}(M_1A)| = 2.00$.

		-	0	1	2	3	4	5	6	7	8	9	10
$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
$\omega = 1.0$													
M_{Add}	GMRES(30)	–	–	–	–	–	–	–	–	268	177	99	101
	BiCGStab	360	360	–	397	–	350	605	194	216	214	104	126
M_{Mul}	GMRES(30)	–	–	–	–	–	–	–	414	270	178	112	89
	BiCGStab	360	360	282	405	322	310	266	177	509	157	100	142
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	–	–	–	–	–	–	–	354	184	140	104
	BiCGStab	349	349	394	383	364	256	284	259	211	192	383	119
M_{Mul}	GMRES(30)	–	–	–	–	–	–	–	–	326	199	120	105
	BiCGStab	349	349	612	396	259	360	477	221	147	159	130	114
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	–	–	–	–	–	–	–	324	210	138	108
	BiCGStab	413	413	379	392	616	962	452	237	197	–	141	92
M_{Mul}	GMRES(30)	–	–	–	–	–	–	–	–	348	210	141	108
	BiCGStab	413	413	307	–	401	379	236	233	196	–	190	216
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	–	266	236	206	149	140	126	63	57	54	46	42
	BiCGStab	360	143	163	131	158	628	108	85	113	69	80	53
M_{Mul}	GMRES(30)	–	266	236	207	148	135	118	59	56	54	42	43
	BiCGStab	360	143	129	155	141	167	92	104	85	86	74	57
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	–	–	–	–	505	–	177	120	89	81	59
	BiCGStab	349	315	582	205	181	–	157	122	119	144	75	83
M_{Mul}	GMRES(30)	–	–	–	–	–	–	500	177	120	89	82	59
	BiCGStab	349	315	264	210	231	191	282	154	98	153	73	66
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	–	–	–	–	–	–	238	178	114	87	77
	BiCGStab	413	291	221	640	187	312	178	141	139	113	98	93
M_{Mul}	GMRES(30)	–	–	–	–	–	–	–	240	146	113	86	77
	BiCGStab	413	291	258	232	245	268	157	526	135	125	106	76
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	–	110	87	81	76	60	52	30	29	28	26	25
	BiCGStab	360	109	103	118	94	75	86	62	67	57	43	36
M_{Mul}	GMRES(30)	–	110	86	81	69	58	49	30	29	28	26	25
	BiCGStab	360	109	126	107	88	72	87	56	77	67	45	34
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	327	268	258	179	170	179	90	85	59	49	48
	BiCGStab	317	157	202	195	143	135	135	102	93	74	55	60
M_{Mul}	GMRES(30)	–	327	268	265	180	152	172	90	81	57	48	48
	BiCGStab	317	157	301	190	174	171	204	100	79	107	61	50
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	–	507	–	–	417	419	299	131	115	84	56	57
	BiCGStab	448	299	248	240	165	–	168	145	102	102	71	85
M_{Mul}	GMRES(30)	–	507	–	–	437	360	319	131	111	83	57	57
	BiCGStab	448	299	258	178	216	178	138	147	126	92	92	67

Table 24: Number of iterations with M_{Mul} and M_{Add} on RDB2048 - $t = 4 \cdot 10^{-2}$ - $|\lambda_{max}(M_1A)| = 3.58$.

		Dimension of the coarse space											
$\mu_1 = 0$, $\mu_2 = 1$		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	–	–	–	–	201	118	86	72	63	56	49	44
	BiCGStab	137	137	145	107	75	47	34	28	29	25	23	22
M_{Mul}	GMRES(50)	–	–	–	–	219	106	88	73	68	55	49	44
	BiCGStab	137	137	96	85	61	41	32	29	30	29	22	22
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	–	–	–	209	100	86	72	63	59	49	43
	BiCGStab	150	150	135	100	71	45	37	35	33	28	26	26
M_{Mul}	GMRES(50)	–	–	–	–	250	117	88	78	68	55	49	44
	BiCGStab	150	150	103	81	64	40	34	33	29	31	26	27
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	–	–	–	214	109	85	74	55	55	46	44
	BiCGStab	132	132	123	106	76	48	40	27	34	25	26	24
M_{Mul}	GMRES(50)	–	–	–	456	242	113	87	76	59	60	49	44
	BiCGStab	132	132	111	81	61	40	33	27	27	27	22	22
$\mu_1 = 1$, $\mu_2 = 1$		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	–	–	284	100	49	43	36	33	33	30	26	23
	BiCGStab	137	74	76	54	39	25	20	19	17	16	14	15
M_{Mul}	GMRES(50)	–	–	300	100	49	42	36	35	33	30	26	23
	BiCGStab	137	74	57	48	37	25	21	17	15	17	13	13
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	–	388	142	56	47	43	36	36	32	29	26
	BiCGStab	150	77	64	54	43	27	21	19	18	16	14	14
M_{Mul}	GMRES(50)	–	–	441	141	56	47	40	39	36	33	29	26
	BiCGStab	150	77	63	49	40	28	22	19	17	19	13	14
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	–	–	295	113	67	48	44	39	39	36	32
	BiCGStab	132	96	117	84	53	35	26	21	24	20	17	18
M_{Mul}	GMRES(50)	–	–	–	290	106	70	48	47	44	39	36	32
	BiCGStab	132	96	80	61	45	31	26	22	20	20	19	17
$\mu_1 = 2$, $\mu_2 = 1$		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(50)	–	386	198	56	47	42	36	32	28	27	24	22
	BiCGStab	137	78	83	52	39	23	18	17	15	14	12	13
M_{Mul}	GMRES(50)	–	386	209	63	47	42	36	32	32	28	24	23
	BiCGStab	137	78	64	42	32	22	18	16	16	16	12	13
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	347	150	50	44	40	34	31	30	26	23	21
	BiCGStab	117	59	67	50	37	23	19	16	13	14	14	14
M_{Mul}	GMRES(50)	–	347	150	50	44	40	34	30	26	25	23	22
	BiCGStab	117	59	54	43	33	23	19	16	14	15	13	16
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(50)	–	–	347	139	55	47	39	36	36	33	29	26
	BiCGStab	116	89	93	58	42	30	21	20	23	22	17	15
M_{Mul}	GMRES(50)	–	–	350	139	55	47	40	39	36	33	29	26
	BiCGStab	116	89	71	52	39	28	21	21	21	19	13	13

Table 25: Number of iterations with M_{Mul} and M_{Add} on SAYLR1 - $t = 4 \cdot 10^{-1}$ - $|\lambda_{max}(M_1A)| = 1.88$.

$\mu_1 = 0$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	38	38	28	17	15	15	15	14	14	12	12	12
M_{Mul}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	38	38	28	17	15	15	15	14	14	12	12	12
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	39	39	25	19	16	16	16	14	14	12	12	12
M_{Mul}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	39	39	25	19	16	16	16	14	14	12	12	12
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	37	37	26	20	16	16	16	14	13	11	12	11
M_{Mul}	GMRES(30)	89	89	51	35	29	29	27	25	25	23	21	21
	BiCGStab	37	37	26	20	16	16	16	14	13	11	12	11

$\mu_1 = 1$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	89	53	47	49	43	30	30	30	30	30	30	30
	BiCGStab	38	34	27	20	20	20	20	20	20	20	19	19
M_{Mul}	GMRES(30)	89	53	47	49	42	30	30	30	30	30	30	29
	BiCGStab	38	34	27	20	20	20	20	20	20	20	19	19
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	33	25	21	17	16	16	15	15	14	13	13
	BiCGStab	39	22	15	11	8	8	8	7	7	7	7	7
M_{Mul}	GMRES(30)	89	33	25	21	17	17	16	15	15	14	13	12
	BiCGStab	39	22	15	11	8	8	8	7	7	7	7	7
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	49	31	25	20	20	20	18	18	16	15	15
	BiCGStab	37	26	18	14	11	10	10	9	8	8	9	8
M_{Mul}	GMRES(30)	89	49	31	25	20	20	20	18	18	16	16	15
	BiCGStab	37	26	18	14	11	10	10	9	8	8	8	8

$\mu_1 = 2$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(30)	89	33	24	19	17	16	15	14	14	13	12	12
	BiCGStab	38	21	15	11	10	9	9	7	7	7	8	7
M_{Mul}	GMRES(30)	89	33	24	19	17	16	15	14	14	13	12	12
	BiCGStab	38	21	15	11	10	9	9	7	7	7	8	7
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	29	21	17	14	14	13	12	12	11	10	10
	BiCGStab	37	19	13	9	8	7	7	6	6	6	6	5
M_{Mul}	GMRES(30)	89	29	21	17	14	14	13	12	12	11	10	10
	BiCGStab	37	19	13	9	8	7	7	6	6	6	6	5
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(30)	89	34	25	20	18	17	15	14	14	14	13	13
	BiCGStab	37	22	15	11	8	8	7	7	7	6	7	6
M_{Mul}	GMRES(30)	89	34	25	20	18	17	15	15	14	14	13	13
	BiCGStab	37	22	15	11	8	8	7	7	7	6	7	6

Table 26: Number of iterations with M_{Mul} and M_{Add} on SAYLR4 - $t = 3 \cdot 10^{-1}$ - $|\lambda_{max}(M_1A)| = 2.00$.

A.3 Unsymmetric matrices full-GMRES

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	106	92	82	73	63	53	48	43	39	36	35
M_{Mul}	GMRES(∞)	106	106	92	82	73	63	53	48	43	39	36	35
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	106	93	83	73	63	53	48	43	39	36	35
M_{Mul}	GMRES(∞)	106	106	92	82	73	63	53	48	43	39	36	35
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	106	92	82	73	63	53	48	43	39	36	35
M_{Mul}	GMRES(∞)	106	106	92	82	73	63	53	48	43	39	36	35
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	110	98	98	93	87	83	80	77	76	74	74
M_{Mul}	GMRES(∞)	106	110	98	98	93	87	83	80	77	76	74	74
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	77	66	59	52	45	38	34	31	28	26	25
M_{Mul}	GMRES(∞)	106	77	66	59	52	45	38	34	31	28	26	25
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	63	54	48	43	37	31	28	25	23	21	20
M_{Mul}	GMRES(∞)	106	63	54	48	43	37	31	28	25	23	21	20
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	66	57	51	45	40	34	30	27	25	23	22
M_{Mul}	GMRES(∞)	106	66	57	51	45	40	34	30	27	25	23	22
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	63	55	49	43	37	31	28	25	23	21	20
M_{Mul}	GMRES(∞)	106	63	55	48	43	37	31	28	25	23	21	20
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	106	54	47	42	37	33	28	25	22	20	18	17
M_{Mul}	GMRES(∞)	106	54	47	42	37	32	27	24	22	20	18	18

Table 27: Number of iterations with M_{Mul} and M_{Add} on BFW398A - $t = 5 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	100	100	99	94	95	94	86	77	77	77	69	61
M_{Mul}	GMRES(∞)	100	100	99	94	95	94	86	77	77	77	69	61
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	100	100	94	94	94	86	77	77	77	69	61
M_{Mul}	GMRES(∞)	100	100	100	94	94	94	86	77	77	77	69	61
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	100	99	94	94	94	77	77	77	77	69	61
M_{Mul}	GMRES(∞)	100	100	99	94	94	94	77	77	77	77	69	61
$\mu_1 = 1$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	100	93	83	79	83	79	75	69	69	69	64	62
M_{Mul}	GMRES(∞)	100	93	83	79	82	79	75	69	69	69	64	62
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	76	72	67	67	67	61	54	54	54	49	43
M_{Mul}	GMRES(∞)	100	76	72	67	67	67	61	54	54	54	49	43
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	61	58	55	55	55	44	44	44	44	40	35
M_{Mul}	GMRES(∞)	100	61	58	55	55	55	44	44	44	44	40	35
$\mu_1 = 2$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	100	65	60	56	56	56	50	45	45	45	40	36
M_{Mul}	GMRES(∞)	100	65	60	56	59	56	50	45	45	45	40	36
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	63	59	55	55	55	50	44	44	44	40	35
M_{Mul}	GMRES(∞)	100	63	59	55	55	55	50	44	44	44	40	35
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	100	55	52	48	48	48	39	39	39	39	35	31
M_{Mul}	GMRES(∞)	100	55	52	48	48	48	39	39	39	39	35	31

Table 28: Number of iterations with M_{Mul} and M_{Add} on BWM200 - $t = 6 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	29	29	26	23	20	15	12	12	12	12	12	11
M_{Mul}	GMRES(∞)	29	29	26	22	19	16	12	11	11	11	11	10
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	29	26	23	19	15	13	13	12	12	11	12
M_{Mul}	GMRES(∞)	29	29	26	22	19	16	13	12	12	12	12	11
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	29	26	22	20	16	12	12	12	12	12	12
M_{Mul}	GMRES(∞)	29	29	26	22	19	16	12	12	11	12	12	12

$\mu_1 = 1$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	29	18	16	13	11	9	7	7	7	8	7	7
M_{Mul}	GMRES(∞)	29	18	16	13	11	9	8	7	7	7	7	7
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	21	18	16	13	10	8	8	8	8	8	8
M_{Mul}	GMRES(∞)	29	21	18	15	12	10	8	8	8	7	7	8
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	22	19	16	13	12	9	9	9	9	10	10
M_{Mul}	GMRES(∞)	29	22	19	16	14	12	10	10	10	10	9	9

$\mu_1 = 2$, $\mu_2 = 1$													
			Dimension of the coarse space										
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	29	15	13	11	9	7	6	6	6	6	6	6
M_{Mul}	GMRES(∞)	29	15	13	11	9	7	6	6	6	6	6	6
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	16	14	12	9	8	7	6	6	6	6	6
M_{Mul}	GMRES(∞)	29	16	14	12	9	8	6	6	5	5	6	5
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	29	17	15	12	11	9	7	8	8	8	8	8
M_{Mul}	GMRES(∞)	29	17	14	13	10	8	7	7	7	7	7	7

Table 29: Number of iterations with M_{Mul} and M_{Add} on BWM2000 - $t = 3 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	106	103	106	102	103	103	99	99	99	98	98
M_{Mul}	GMRES(∞)	106	106	103	106	102	103	103	99	99	99	99	99
$\omega^{-1} = \|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	106	103	104	104	104	104	100	100	100	99	99
M_{Mul}	GMRES(∞)	106	106	103	104	104	104	104	99	100	100	99	99
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	106	102	103	102	103	103	99	99	99	98	98
M_{Mul}	GMRES(∞)	106	106	103	103	102	103	103	99	99	99	98	98

$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	168	163	168	167	167	167	172	172	175	174	169
M_{Mul}	GMRES(∞)	106	168	163	163	163	163	163	167	167	175	167	167
$\omega^{-1} = \|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	90	88	88	88	87	88	81	80	81	75	75
M_{Mul}	GMRES(∞)	106	90	88	88	87	87	87	80	80	80	75	75
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	80	80	80	80	80	80	71	70	71	69	70
M_{Mul}	GMRES(∞)	106	80	80	81	80	80	80	70	70	70	69	69

$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	106	87	84	86	84	84	82	73	73	71	72	72
M_{Mul}	GMRES(∞)	106	87	86	86	84	84	84	72	72	72	72	72
$\omega^{-1} = \|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	78	77	77	77	76	76	71	71	71	67	67
M_{Mul}	GMRES(∞)	106	78	77	77	77	77	77	71	71	71	70	69
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1A)\|$													
M_{Add}	GMRES(∞)	106	67	65	67	65	65	67	59	59	59	55	55
M_{Mul}	GMRES(∞)	106	67	67	67	65	65	67	59	59	59	58	58

Table 30: Number of iterations with M_{Mul} and M_{Add} on FS-541-4 - $t = 8 \cdot 10^{-1}$.

$\mu_1 = 0,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	49	49	44	40	38	35	33	33	29	27	25	24
M_{Mul}	GMRES(∞)	49	49	44	40	37	34	32	31	27	26	24	23
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	49	45	42	40	38	35	34	30	28	27	26
M_{Mul}	GMRES(∞)	49	49	45	42	40	37	35	33	30	28	27	26
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	49	45	42	40	36	33	32	29	28	26	25
M_{Mul}	GMRES(∞)	49	49	45	42	40	36	34	32	29	27	26	25
$\mu_1 = 1,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	49	34	31	28	26	24	22	22	19	19	17	17
M_{Mul}	GMRES(∞)	49	34	31	28	25	23	22	21	19	18	17	16
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	45	42	39	36	33	31	29	27	26	24	22
M_{Mul}	GMRES(∞)	49	45	42	39	36	32	30	29	27	25	24	22
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	43	39	36	33	30	28	27	24	24	22	20
M_{Mul}	GMRES(∞)	49	43	39	36	34	30	28	26	24	23	21	20
$\mu_1 = 2,\ \mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	49	28	26	24	23	21	20	20	18	18	16	14
M_{Mul}	GMRES(∞)	49	28	26	24	22	21	19	18	17	16	15	14
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	42	38	35	33	29	28	27	24	23	21	20
M_{Mul}	GMRES(∞)	49	42	38	35	33	29	27	26	24	22	20	19
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	49	38	34	32	29	26	25	24	21	20	19	18
M_{Mul}	GMRES(∞)	49	38	34	32	29	26	25	23	21	20	18	18

Table 31: Number of iterations with M_{Mul} and M_{Add} on GRE1107 - $t = 1 \cdot 10^{-2}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	66	66	54	50	49	47	47	43	40	36	37	32
M_{Mul}	GMRES(∞)	66	66	54	51	49	47	47	43	40	36	37	32
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	66	55	52	51	49	49	44	42	37	38	33
M_{Mul}	GMRES(∞)	66	66	55	52	51	49	48	43	41	37	38	33
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	66	55	51	50	48	48	43	41	36	37	33
M_{Mul}	GMRES(∞)	66	66	55	52	50	48	48	43	41	36	37	33
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	66	52	45	44	44	41	41	41	40	37	38	37
M_{Mul}	GMRES(∞)	66	52	45	44	44	41	41	41	40	37	37	37
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	59	49	44	43	41	41	37	34	31	31	27
M_{Mul}	GMRES(∞)	66	59	49	44	42	42	41	36	34	31	31	27
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	53	44	40	38	36	36	32	29	27	27	24
M_{Mul}	GMRES(∞)	66	53	44	40	38	37	36	32	29	27	27	24
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	66	42	36	33	33	32	32	30	28	27	26	25
M_{Mul}	GMRES(∞)	66	42	36	33	32	32	32	30	28	27	26	25
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	51	43	37	36	36	35	32	30	27	27	24
M_{Mul}	GMRES(∞)	66	51	43	39	39	37	37	31	29	26	26	24
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	66	44	35	34	31	31	31	27	25	23	23	20
M_{Mul}	GMRES(∞)	66	44	35	33	31	30	30	26	25	23	23	21

Table 32: Number of iterations with M_{Mul} and M_{Add} on HOR131 - $t = 2 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	121	121	117	112	110	110	101	98	90	89	88	84
M_{Mul}	GMRES(∞)	121	121	117	112	110	110	101	98	90	89	88	84
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	121	118	113	111	111	103	99	90	90	89	84
M_{Mul}	GMRES(∞)	121	121	118	113	112	110	102	99	90	90	88	84
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	121	117	112	110	110	101	98	90	89	88	84
M_{Mul}	GMRES(∞)	121	121	117	112	110	110	101	98	91	89	88	84
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	121	84	80	76	74	74	68	66	61	60	59	56
M_{Mul}	GMRES(∞)	121	84	80	76	74	74	68	66	61	60	59	56
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	97	92	88	86	86	80	76	70	68	68	64
M_{Mul}	GMRES(∞)	121	97	92	88	86	86	79	75	70	69	68	64
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	84	80	76	74	74	68	66	61	60	58	56
M_{Mul}	GMRES(∞)	121	84	80	76	74	74	68	66	61	60	59	56
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	121	73	71	67	66	66	61	60	55	53	52	49
M_{Mul}	GMRES(∞)	121	73	71	67	66	66	59	58	54	53	52	49
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	83	77	74	73	73	68	65	61	58	58	54
M_{Mul}	GMRES(∞)	121	83	77	74	73	73	66	65	59	58	58	54
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	121	73	71	67	66	66	60	59	55	55	52	49
M_{Mul}	GMRES(∞)	121	73	70	67	66	66	59	58	54	53	52	50

Table 33: Number of iterations with M_{Mul} and M_{Add} on ORSIRR - $t = 3 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
M_{Mul}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
M_{Mul}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
M_{Mul}	GMRES(∞)	47	47	42	36	30	26	22	20	18	16	15	14
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	47	43	35	32	27	27	27	26	25	25	25	24
M_{Mul}	GMRES(∞)	47	43	35	32	27	27	27	26	25	24	24	24
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	35	31	26	22	18	16	14	13	12	11	11
M_{Mul}	GMRES(∞)	47	35	31	26	22	18	16	14	13	11	10	10
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	28	25	21	18	15	13	11	10	9	9	8
M_{Mul}	GMRES(∞)	47	28	25	21	17	15	13	11	10	9	8	8
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	47	30	26	22	18	15	13	12	10	10	9	9
M_{Mul}	GMRES(∞)	47	30	26	21	18	15	13	12	10	10	9	9
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	29	25	21	18	15	13	12	11	10	9	9
M_{Mul}	GMRES(∞)	47	29	25	21	18	15	13	12	10	9	9	8
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	47	26	22	18	15	13	11	10	9	8	7	7
M_{Mul}	GMRES(∞)	47	26	22	18	15	13	11	10	9	8	7	7

Table 34: Number of iterations with M_{Mul} and M_{Add} on PORES3 - $t = 1 \cdot 10^{-1}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	83	83	83	78	74	69	64	60	57	54	52	49
M_{Mul}	GMRES(∞)	83	83	83	78	74	69	64	60	57	54	52	49
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	83	84	80	76	72	67	62	60	57	54	51
M_{Mul}	GMRES(∞)	83	83	84	80	76	72	67	62	59	57	54	51
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	83	84	79	76	71	65	61	59	56	52	50
M_{Mul}	GMRES(∞)	83	83	84	79	76	71	65	61	58	56	53	50
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	83	53	52	49	47	43	41	38	37	35	33	31
M_{Mul}	GMRES(∞)	83	53	52	50	47	43	41	38	36	35	33	31
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	74	74	70	66	62	57	54	52	49	44	44
M_{Mul}	GMRES(∞)	83	74	74	70	66	62	57	54	51	49	46	44
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	68	69	65	61	56	53	49	47	45	41	40
M_{Mul}	GMRES(∞)	83	68	69	65	61	56	53	49	47	45	42	40
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	83	42	42	40	37	35	33	30	29	28	26	25
M_{Mul}	GMRES(∞)	83	42	42	40	37	35	33	30	29	28	26	25
$\omega^{-1} = \|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	66	66	62	59	54	51	47	46	43	39	39
M_{Mul}	GMRES(∞)	83	66	66	62	58	54	51	47	45	43	40	39
$\omega^{-1} = \frac{2}{3}\|\lambda_{\max}(M_1 A)\|$													
M_{Add}	GMRES(∞)	83	59	59	55	52	48	45	42	41	38	36	34
M_{Mul}	GMRES(∞)	83	59	59	55	52	48	45	42	40	38	35	34

Table 35: Number of iterations with M_{Mul} and M_{Add} on RDB2048 - $t = 4 \cdot 10^{-2}$.

$\mu_1 = 0$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	116	116	107	96	86	78	67	61	53	55	49	44
M_{Mul}	GMRES(∞)	116	116	107	96	86	77	67	65	61	55	49	44
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	116	108	97	88	78	67	61	54	54	46	44
M_{Mul}	GMRES(∞)	116	116	108	96	87	78	67	61	61	55	49	44
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	116	107	96	85	77	67	61	54	55	49	43
M_{Mul}	GMRES(∞)	116	116	107	96	85	77	67	65	61	55	49	44

$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	116	72	63	56	49	43	36	33	33	30	26	23
M_{Mul}	GMRES(∞)	116	72	63	56	49	42	36	35	33	30	26	23
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	89	81	73	63	57	48	44	39	39	36	32
M_{Mul}	GMRES(∞)	116	89	81	72	63	57	48	47	44	39	36	32
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	72	67	59	52	47	43	36	36	32	29	26
M_{Mul}	GMRES(∞)	116	72	67	59	52	47	40	39	36	33	29	26

$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	116	64	58	53	47	42	36	32	28	27	24	22
M_{Mul}	GMRES(∞)	116	64	58	53	47	42	36	32	32	28	24	23
$\omega^{-1} = \lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	73	66	59	52	47	39	36	36	33	29	26
M_{Mul}	GMRES(∞)	116	73	66	59	52	47	40	39	36	33	29	26
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1A)\rvert$													
M_{Add}	GMRES(∞)	116	62	56	50	44	40	34	31	30	26	23	21
M_{Mul}	GMRES(∞)	116	62	56	50	44	40	34	30	26	25	23	22

Table 36: Number of iterations with M_{Mul} and M_{Add} on SAYLR1 - $t = 4 \cdot 10^{-1}$.

		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\mu_1 = 0$, $\mu_2 = 1$													
$\omega = 1.0$													
M_{Add}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
M_{Mul}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
M_{Mul}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
M_{Mul}	GMRES(∞)	58	58	42	34	29	29	27	25	25	23	21	21
$\mu_1 = 1$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	58	45	38	34	31	30	30	30	30	30	30	30
M_{Mul}	GMRES(∞)	58	45	38	34	31	30	30	30	30	30	30	29
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	41	32	25	20	20	20	18	18	16	15	15
M_{Mul}	GMRES(∞)	58	41	32	25	20	20	20	18	18	16	16	15
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	34	25	21	17	16	16	15	15	14	13	13
M_{Mul}	GMRES(∞)	58	34	25	21	17	17	16	15	15	14	13	12
$\mu_1 = 2$, $\mu_2 = 1$													
		Dimension of the coarse space											
		-	0	1	2	3	4	5	6	7	8	9	10
$\omega = 1.0$													
M_{Add}	GMRES(∞)	58	32	24	19	17	16	15	14	14	13	12	12
M_{Mul}	GMRES(∞)	58	32	24	19	17	16	15	14	14	13	12	12
$\omega^{-1} = \lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	34	25	20	18	17	15	14	14	14	13	13
M_{Mul}	GMRES(∞)	58	34	25	20	18	17	15	15	14	14	13	13
$\omega^{-1} = \frac{2}{3}\lvert\lambda_{\max}(M_1 A)\rvert$													
M_{Add}	GMRES(∞)	58	29	21	17	14	14	13	12	12	11	10	10
M_{Mul}	GMRES(∞)	58	29	21	17	14	14	13	12	12	11	10	10

Table 37: Number of iterations with M_{Mul} and M_{Add} on SAYLR4 - $t = 3 \cdot 10^{-1}$.