Boundary preconditioners for mixed finite-element discretizations of fourth-order elliptic problems

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CERFACS Technical Report TR/PA/04/43

Abstract

We extend the preconditioning approach of Glowinski and Pironneau, and of Peisker to the case of mixed finite element general fourth-order elliptic problems. We show that $H^{-1/2}$-preconditioning on the boundary leads to mesh-independent performance of iterative solvers of Krylov subspace type. In particular, we show that the field of values of the boundary Schur complement preconditioned by a discrete $H^{-1/2}$ boundary norm is bounded independently of the discretization.

Keywords: $H^{-1/2}$-preconditioning, Schur complements, biharmonic and fourth-order elliptic problems, field of values.

1 Problem description

Let $\Omega \in \mathbb{R}^2$ denote an open set with boundary Γ. Given a function f, we are interested in the solution of the general fourth-order elliptic problem

$$\left\{ \begin{array}{rlr} \Delta^2 u - \nabla \cdot (\mathbf{a}\nabla u) + (\vec{b}\cdot\nabla)u + cu & = f & \text{in } \Omega, \\ u = \partial u/\partial\nu & = 0 & \text{on } \Gamma. \end{array} \right. \tag{1}$$

where, for the moment, we assume that the coefficients $\vec{b}, c$, as well as the entries in the symmetric positive definite 2×2 matrix $\mathbf{a}$ are $C^\infty(\Omega)$ functions.
Setting $v = -\Delta u$ we obtain the following equivalent system of PDE in variables u, v

$$\left\{ \begin{array}{rlr} -\Delta v - \nabla \cdot (\mathbf{a}\nabla u) + (\vec{b}\cdot\nabla)u + cu & = f & \text{in } \Omega, \\ v + \Delta u & = 0 & \text{in } \Omega, \\ u = \partial u/\partial\nu & = 0 & \text{on } \Gamma. \end{array} \right. \tag{2}$$

This system of equations arises for example as the streamfunction-vorticity formulation of time-dependent linearized problems in oceanography [20], [22], [15]. More common, however, is the case when $\mathbf{a}, \vec{b}, c$ are all zero, which corresponds to the standard biharmonic problem

$$\begin{cases} -\Delta v &= f \qquad \text{in } \Omega, \\ v + \Delta u &= 0 \qquad \text{in } \Omega, \\ u = \partial u/\partial \nu &= 0 \qquad \text{on } \Gamma. \end{cases} \tag{3}$$

This latter problem has been extensively considered in the literature from an approximation viewpoint [14], [9], [11], [4], [5], [8] etc. Moreover, the related issue of efficient solution algorithms for the discretization of (3) has also received considerable attention [16], [3], [19]. However, we single out the now classic approach to solving (3) provided by Glowinski and Pironneau [9], combined with the optimal preconditioning technique of Peisker [16]. The method is essentially a boundary Schur complement technique for solving the linear system arising from the standard mixed finite element formulation of (3). It is shown in [9] that the Schur complement is a symmetric and positive operator, which is $H^{-1/2}(\Gamma)$-elliptic. Moreover, if the Schur complement is available, system (3) decouples into two smaller Poisson problems. For this reason, the authors of [9] construct and factorize the boundary Schur complement. To circumvent this rather inefficient approach, an optimal Schur complement preconditioner was subsequently constructed by Peisker, who showed spectral equivalence with a discrete $H^{-1/2}(\Gamma)$-norm. This discrete norm is a direct sum of small but full matrices (one-dimensional square-root-laplacians), corresponding to each segment of the boundary of the computational domain.

In this paper we provide the generalization of the method in [9],[16] to (2). Our approach also requires the solution of a linear system involving a boundary Schur complement operator. For this operator we show that the same discrete $H^{-1/2}(\Gamma)$ norm is a useful preconditioner. In particular, we show that the field of values of the preconditioned Schur complement is in the right-half plane, bounded independently of the size of the problem. Standard Krylov theory implies therefore convergence in a constant number of iterations. On the other hand, unlike the case of the biharmonic problem, the general system (2) does not decouple into smaller problems with Dirichlet boundary conditions (interior problems). For this reason, we will also consider the issue of 'interior' preconditioners.

The outline of the paper is as follows. In the next section we review the approach of Glowinski and Pironneau together with the boundary preconditioner introduced by Peisker [16]. In section 3 we present the generalization of these results to mixed formulations of fourth-order elliptic problems. In particular, we show that in the general case also, the 'vorticity' boundary operator for (2) is continuous and $H^{-1/2}(\Gamma)$-elliptic. Hence, discrete $H^{-1/2}$-norms are ideal candidates as preconditioners for the boundary Schur complement.

In section 4 we introduce and prove optimality of an 'interior' preconditioner, that allows the decoupling of our problem into smaller easy-to-invert sub-problems. Finally, section 5 validates the analysis on numerical experiments drawn from oceanographic applications .

2 Preliminaries: the biharmonic problem

In this section we review a solution method first introduced by Glowinski and Pironneau [9]. We also show that the method can be interpreted as a Schur complement method for the linear system arising from standard mixed finite element discretizations of the streamfunction-vorticity formulation of the biharmonic problem. Finally, we describe the optimal boundary preconditioner introduced by Peisker [16].

Throughout the paper we will use the following notation and standard results. Ω is an open simply-connected bounded domain in $\mathbb{R}^2$ with boundary Γ and exterior unit normal vector $\vec{\nu} = \vec{\nu}(x, y)$. We denote by $L^p(\Omega)$ the usual Lebesgue spaces of p-integrable functions and by $H^m(\Omega)$ the usual Sobolev space of order m equipped with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$, with $H^0(\Omega) = L^2(\Omega)$. We denote by $\langle\cdot,\cdot\rangle$ the $L^2(\Omega)$ inner-product. Given a positive weight ω we also define the norms and semi-norms

$$\|v\|_{\omega,m,\Omega} = \left(\sum_{|\alpha|\leq m} \langle \omega D^\alpha v, D^\alpha v\rangle\right)^{1/2}, \quad |v|_{\omega,m,\Omega} = \left(\sum_{|\alpha|=m} \langle \omega D^\alpha v, D^\alpha v\rangle\right)^{1/2};$$

by default, we write $\|\cdot\|_{1,m,\Omega} = \|\cdot\|_{m,\Omega}$. We also define trace operators $\gamma_0, \gamma_1 : H^1(\Omega) \to L^2(\Gamma)$ via

$$\gamma_0 v = v|_\Gamma, \quad \gamma_1 u = \gamma_0 \partial v/\partial\nu = \vec{\nu}\cdot\nabla v.$$

It is known that if $v \in H^2(\Omega)$, $\gamma_1 : H^2(\Omega) \to H^{1/2}(\Gamma)$ is bounded [1]

$$\|\gamma_1 v\|_{1/2,\Gamma} \leq c(\gamma_1)\|v\|_{2,\Omega}. \tag{4}$$

We write $H^1_0(\Omega)$ for the subspace of $H^1(\Omega)$ of functions u for which $\gamma_0 u = 0$.

2.1 The method of Glowinski and Pironneau

Consider the standard biharmonic problem

$$\begin{cases} \Delta^2 u &= f \qquad \text{in } \Omega, \\ u = \partial u/\partial\nu &= 0 \qquad \text{on } \Gamma, \end{cases} \tag{5}$$

and let $v = -\Delta u$. If $\lambda = v|_\Gamma$ is known, we can compute the solution of (5) as the solution of two Poisson problems

$$\left\{ \begin{array}{rll} -\Delta v & = f & \text{in } \Omega, \\ v & = \lambda & \text{on } \Gamma, \end{array} \right. \qquad \left\{ \begin{array}{rll} -\Delta u & = v & \text{in } \Omega, \\ u & = 0 & \text{on } \Gamma, \end{array} \right. \tag{6}$$

Thus, for any λ one can define a linear operator $\lambda \mapsto -\partial u/\partial \nu|_\Gamma$, where u is obtained solving (6). We denote this operator by $\mathcal{S}$. It is shown in [9] that $\mathcal{S}$ is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ and that it induces a symmetric, continuous, $H^{-1/2}(\Gamma)$-elliptic bilinear form via

$$s(\lambda, \mu) = \langle \mathcal{S}\lambda, \mu \rangle = -\int_\Gamma \frac{\partial u}{\partial \nu} \mu \, \mathrm{d}\Gamma \tag{7}$$

for all $\mu \in H^{-1/2}(\Gamma)$. Moreover, if we denote by (u_λ, v_λ) the solution of (6) then

$$s(\lambda, \mu) = -\int_\Gamma \frac{\partial u_\lambda}{\partial \nu} \mu \, \mathrm{d}\Gamma = -\int_\Omega \Delta u_\lambda \tilde{\mu} \, \mathrm{d}\Omega - \int_\Omega \nabla u_\lambda \cdot \nabla \tilde{\mu} \, \mathrm{d}\Omega = \int_\Omega v_\lambda \tilde{\mu} \, \mathrm{d}\Omega - \int_\Omega \nabla u_\lambda \cdot \nabla \tilde{\mu} \, \mathrm{d}\Omega$$

where $\tilde{\mu}$ is the extension to Ω of μ. Then solving for u, v is equivalent to solving the following problems (see [9])

$$\left\{ \begin{array}{rll} -\Delta v_0 & = f & \text{in } \Omega, \\ v_0 & = 0 & \text{on } \Gamma, \end{array} \right. \qquad \left\{ \begin{array}{rll} -\Delta u_0 & = v_0 & \text{in } \Omega, \\ u_0 & = 0 & \text{on } \Gamma, \end{array} \right. \tag{8}$$

$$\mathcal{S}\lambda \quad = \quad \partial u_0/\partial \nu \qquad \text{on } \Gamma, \tag{9}$$

$$\left\{ \begin{array}{rll} -\Delta v_1 & = 0 & \text{in } \Omega, \\ v_1 & = \lambda & \text{on } \Gamma, \end{array} \right. \qquad \left\{ \begin{array}{rll} -\Delta u_1 & = v_1 & \text{in } \Omega, \\ u_1 & = 0 & \text{on } \Gamma, \end{array} \right. \tag{10}$$

the final solution being $(u, v) = (u_0 + u_1, v_0 + v_1)$. Note that equation (9) essentially enforces the second boundary condition in (5) since, given λ in (10) and the definition of $\mathcal{S}$, equation (9) is equivalent to

$$-\partial u_1/\partial \nu = \partial u_0/\partial \nu.$$

The advantage of this approach is obvious – except for one equation, it only requires solutions of Poisson problems. The apparent drawback is in the discretization and solution of (9). However, as discussed below, this issue has been successfully investigated in the literature [16].

Consider now the following standard mixed finite element method for (3). Let $S^h \subset H^1(\Omega)$ be a finite-dimensional space of functions defined on some triangulation $\mathcal{T}_h$ of Ω into simplices T of maximum diameter h. In particular, we will be interested in the choice

$$S^h = S^{h,k} = \left\{ w \in C^0(\Omega) : w|_T \in P_k \;\; \forall T \in \mathcal{T}_h \right\},$$

where P_k is the space of degree k polynomials in two variables. Let $S_I^h, S_B^h \subset S^h$ satisfy $S_I^h \oplus S_B^H \equiv S^h$ where $S_I^h = \{w \in S^h : w|_\Gamma = 0\}$. The discrete weak formulation is then

Find $(u_h, v_h) \in S_I^h \times S^h$ such that $\forall (w_h, z_h) \in S_I^h \times S^h$

$$l(v_h, w_h) = \langle f, w_h \rangle \tag{11a}$$

$$l(u_h, z_h) - m(v_h, z_h) = 0 \tag{11b}$$

where

$$l(z, w) := \langle \nabla z, \nabla w \rangle, \quad m(z, w) := \langle z, w \rangle .$$

As described in [9], (11) is equivalent to the discrete versions of (8–10) given by the following three weak formulations

I. Find $(u_{0h}, v_{0h}) \in S_I^h \times S_I^h$ such that $\forall (w_h, z_h) \in S_I^h \times S_I^h$

$$l(v_{0h}, w_h) = \langle f, w_h \rangle \tag{12a}$$

$$l(u_{0h}, z_h) - m(v_{0h}, z_h) = 0 \tag{12b}$$

II. Find $\lambda_h \in S_B^h$ such that $\forall \mu_h \in S_B^h$

$$s(\lambda_h, \mu_h) = -s(\lambda_{0h}, \mu_h) \tag{13a}$$

III. Find $(u_{1h}, v_{1h}) \in S_I^h \times S^h$, $v_{1h} - \lambda_h \in S_I^h$, such that $\forall (w_h, z_h) \in S_I^h \times S^h$

$$l(v_{1h}, w_h) = 0 \tag{14a}$$

$$l(u_{1h}, z_h) - m(v_{1h}, z_h) = 0 \tag{14b}$$

Let now $\text{span}\{\Psi_i\}_{1 \le i \le n} = S^h$ so that $w_h \in S^h, z_h \in S_I^h$ can be written

$$w_h = \sum_{i=1}^{n} \mathbf{w}_i \Psi_i, \quad z_h = \sum_{i=1}^{n_I} \mathbf{z}_i \Psi_i$$

where $n = |S^h|, n_I = |S_I^h|$. Problem (11) is then equivalent to the following linear system of equations

$$\begin{pmatrix} 0 & L_{II} & L_{IB} \\ L_{II} & -M_{II} & -M_{IB} \\ L_{IB}^T & -M_{IB}^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

where

$$(L_{II})_{ij} = l(\Psi_j, \Psi_i), \quad (L_{IB})_{ik} = l(\Psi_k, \Psi_i),$$

and

$$(M_{II})_{ij} = m(\Psi_j, \Psi_i), \quad (M_{IB})_{ik} = m(\Psi_k, \Psi_i), \quad (M_{BB})_{kl} = m(\Psi_l, \Psi_k)$$

for $1 \leq i, j, \leq m,\ 1 \leq k, l \leq n - n_I$. We also write (15) in the more compact form

$$\begin{pmatrix} L & Z \\ Z^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ 0 \end{pmatrix} \tag{16}$$

where

$$L = \begin{pmatrix} 0 & L_{II} \\ L_{II} & -M_{II} \end{pmatrix}, \quad Z = \begin{pmatrix} L_{IB} \\ -M_{IB} \end{pmatrix}, \ \ \mathbf{x} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \end{pmatrix}.$$

The Schur complement associated with L in the matrix of (16) is then

$$S = -M_{BB} - Z^T L^{-1} Z$$

and the corresponding Schur complement approach reads
(i) Solve $L\mathbf{x}^0 = \mathbf{g}$
(ii) Solve $S\mathbf{v}_B = -Z\mathbf{x}^0$
(iii) Solve $L\mathbf{x}^1 = -Z^T\mathbf{v}_B$,
with final solution $(\mathbf{x}, \mathbf{v}_B) = (\mathbf{x}^0 + \mathbf{x}^1, \mathbf{v}_B)$. We note here that (ii) can be interpreted as
(ii') 'Solve' for $\mathbf{v}_B$

$$Z^T\mathbf{x}^1 - M_{BB}\mathbf{v}_B = -Z^T\mathbf{x}^0$$

where $\mathbf{x}^1$ is the solution of (3).

It is now somewhat evident that the method of Glowinski-Pironneau is, in matrix terms, a boundary Schur complement approach, as the following result shows.

Lemma 2.1 *The Schur complement approach* (i–iii) *is equivalent to solving the mixed finite element problems* (I–III).

Proof Step (i) above requires the solution of

$$\begin{aligned} L_{II}\mathbf{v}_I^0 &= \mathbf{f} \\ L_{II}\mathbf{u}_I^0 &= M_{II}\mathbf{v}_I^0 \end{aligned}$$

which is the matrix representation of I (12). Similarly, step (iii) is equivalent to solving III (14)

$$\begin{aligned} L_{II}\mathbf{v}_I^1 &= -L_{IB}\mathbf{v}_B \\ L_{II}\mathbf{u}_I^1 &= M_{II}\mathbf{v}_I^1 + M_{IB}\mathbf{v}_B. \end{aligned}$$

Finally,

$$s(\lambda_h, \mu_h) = -s(\lambda_{0h}, \mu_h)$$

can be written as (cf. (7))

$$m(v_{1h}, \tilde{\mu}_h) - l(u_{1h}, \tilde{\mu}_h) = l(u_{0h}, \tilde{\mu}_h) - m(v_{0h}, \tilde{\mu}_h)$$

for $v_{1h}, \tilde{\mu}_h \in S^h, u_{0h}, u_{1h}, v_{0h} \in S_I^h$. Choosing $\tilde{\mu}_h = \Psi_k, n_I + 1 \leq k \leq n$, we obtain the following $(n - n_I) \times (n - n_I)$ system of equations for $\mathbf{v}_B$

$$M_{IB}^T \mathbf{v}_I^1 + M_{BB}\mathbf{v}_B - L_{IB}^T \mathbf{u}_I^1 = L_{IB}^T \mathbf{u}_I^0 - M_{IB}^T \mathbf{v}_I^0$$

which is (ii'). □

2.2 A boundary preconditioner

The $H^{-1/2}(\Gamma)$-ellipticity of the bilinear form $s(\cdot, \cdot)$ introduced above was used to advantage by Peisker [16] to construct a boundary preconditioner for the biharmonic problem. In particular, for uniform meshes of size h, he showed that the ellipticity holds also in a discrete sense, i.e.,

$$c_1 \|\gamma_0 \lambda_h\|^2_{-1/2,\Gamma} \leq s(\lambda_h, \lambda_h) \leq c_2 \|\gamma_0 \lambda_h\|^2_{-1/2,\Gamma}$$

where $\lambda_h \in S_B^h$. Moreover, he provided the following discrete interpretation of the $H^{-1/2}(\Gamma)$ for the case where piecewise linear basis functions Ψ_i are employed in the mixed formulation (11). Let $\Gamma = \bigcup_{1 \leq k \leq r} \Gamma_k$ consist of a finite number of straight lines. Let also $\Phi_i = \Psi_i|_\Gamma, n_I + 1 \leq i \leq n$, denote the piecewise linear functions obtained by restricting the basis functions Ψ_i to the boundary Γ. We note here that $\{\Phi_i\}_{n_I+1 \leq i \leq n}$ span a subspace V^h of $H^1(\Gamma)$ of dimension $n - n_I$. Let now M_k, L_k denote the mass and Laplacian matrices assembled on the interior nodes of each boundary segment Γ_k

$$(M_k)_{ij} = \langle\langle \Phi_j, \Phi_i \rangle\rangle_{\Gamma_k}, \quad (L_k)_{ij} = h \left\langle\left\langle \Phi_j', \Phi_i' \right\rangle\right\rangle_{\Gamma_k}.$$

Define

$$H_k = M_k L_k^{-1/2} M_k$$

and let

$$H = \begin{pmatrix} H_1 & & & & & & \\ & h^2 & & & & & \\ & & H_2 & & & & \\ & & & h^2 & & & \\ & & & & \ddots & & \\ & & & & & H_r & \\ & & & & & & h^2 \end{pmatrix} \tag{17}$$

Theorem 2.2 ([16]) *There exist constants* c_3, c_4 *such that*

$$c_3\|\gamma_0\lambda_h\|^2_{-1/2,\Gamma} \leq \|\boldsymbol{\lambda}\|_H \leq c_4\|\gamma_0\lambda_h\|^2_{-1/2,\Gamma} \tag{18}$$

where $\boldsymbol{\lambda}$ *is the vector of coefficients of any nonzero* $\lambda_h|_\Gamma$ *expanded in the basis* $\{\Phi_i\}$

$$\lambda_h|_\Gamma = \sum_{i=n_I+1}^{n} \boldsymbol{\lambda}_i\Phi_i.$$

The immediate consequence is that

$$\frac{c_1}{c_4} \leq \frac{\boldsymbol{\lambda}^T S\boldsymbol{\lambda}}{\boldsymbol{\lambda}^T H\boldsymbol{\lambda}} \leq \frac{c_2}{c_3}.$$

Thus, the eigenvalues of the Schur complement preconditioned by H are real, positive and bounded independently of h.

In the following section we consider the generalization of these ideas. In particular, we define the Schur complement in a similar fashion and show that the resulting bilinear form is also continuous and $H^{-1/2}(\Gamma)$-elliptic. Since in the general case the Schur complement is not symmetric, the resulting preconditioned system will in general have complex eigenvalues. In this case, a useful alternative for convergence analysis may be the field of values of the preconditioned matrix. In our case, we will make use of the H-field of values of a matrix M, defined via [10]

$$\mathcal{W}_H(M) = \left\{\mathbf{x} \in \mathbb{C} \setminus \{0\} : \frac{\mathbf{x}^* HM\mathbf{x}}{\mathbf{x}^* H\mathbf{x}}\right\}; \tag{19}$$

clearly, the set $\mathcal{W}_H(M)$ contains the eigenvalues of M. We also define the following parameters

$$\theta = \min_{z\in\mathcal{W}_H(M)} \text{re} z = \min_{\mathbf{x}\in\mathbb{R}^n\{\mathbf{0}\}} \frac{\mathbf{x}^t HM\mathbf{x}}{\mathbf{x}^t H\mathbf{x}}, \quad \Theta = \max_{z\in\mathcal{W}_H(M)} |z|\,.$$

Then the GMRES residuals satisfy [7], [18, Thm 6.7]

$$\|\mathbf{r}^k\|_H/\|\mathbf{r}^0\|_H \leq \left(1 - \theta^2/\Theta^2\right)^{k/2}. \tag{20}$$

In the next section we show that the H-field of values, and therefore the eigenvalues of S preconditioned by H are in the right half-plane in a region bounded independently of the meshsize. Therefore, the above estimates for convergence of GMRES apply with constant parameters θ, Θ and guarantee convergence in a number of iterations bounded by a constant.

We end this section with some remarks concerning the solution of (16). First, we are not interested in solving the Schur complement problem. The main reason for this is the fact that the Schur complement is expensive to construct. A suitable approach is to precondition the Schur complement implicitly; this is achieved, for instance, by using block triangular left or right preconditioners of the form

$$P_L = \begin{pmatrix} L & Z \\ 0 & H \end{pmatrix}, \quad P_R = \begin{pmatrix} L & 0 \\ Z^T & H \end{pmatrix}.$$

The resulting preconditioned systems are

$$P_L^{-1}K = \begin{pmatrix} I & L^{-1}Z \\ 0 & H^{-1}S \end{pmatrix}, \quad KP_R^{-1} = \begin{pmatrix} I & 0 \\ Z^T L^{-1} & SH^{-1} \end{pmatrix} \tag{21}$$

and the convergence will depend entirely on the distribution of eigenvalues of SH^{-1}. We remark here that this approach requires the solution of a system involving P_L, P_R, and thus, the solution of systems with L and H. While H is small, the solution of the 2×2 block matrix L may be expensive. In the case of the biharmonic problem, this can be achieved by inverting two discrete Laplacians. In general, we may have to provide an efficient algorithm to achieve this goal. This issue is addressed again in section 4.

3 Boundary preconditioners for fourth order elliptic problems

In this section we present the generalization of the preconditioning technique described in the previous section for the biharmonic problem.

Consider again our general problem

$$\begin{cases} -\Delta v - \nabla \cdot (\mathbf{a}\nabla u) + (\vec{b} \cdot \nabla)u + cu &= f \qquad \text{in } \Omega, \\ v + \Delta u &= 0 \qquad \text{in } \Omega, \\ u = \partial u/\partial \nu &= 0 \qquad \text{on } \Gamma. \end{cases} \tag{22}$$

The generalization of the Glowinski-Pironneau method is straightforward. As before, we define an operator S via

$$S\lambda = -\partial u/\partial \nu|_\Gamma$$

where u is the solution of (22). Then solving for u, v is equivalent to solving the following three problems

$$\begin{cases} -\Delta v_0 - \nabla \cdot (\mathbf{a}\nabla u_0) + (\vec{b} \cdot \nabla)u_0 + cu_0 &= f \qquad \text{in } \Omega, \\ v_0 + \Delta u_0 &= 0 \qquad \text{in } \Omega, \\ u_0 &= 0 \qquad \text{on } \Gamma, \\ v_0 &= 0 \qquad \text{on } \Gamma, \end{cases} \tag{23}$$

$$\mathcal{S}\lambda \;\; = \partial u_0/\partial\nu \qquad \text{on } \Gamma, \tag{24}$$

$$\begin{cases} -\Delta v_1 - \nabla\cdot(\mathbf{a}\nabla u_1) + (\vec{b}\cdot\nabla)u_1 + cu_1 & = 0 \qquad \text{in } \Omega, \\ v_1 + \Delta u_1 & = 0 \qquad \text{in } \Omega, \\ u_1 & = 0 \qquad \text{on } \Gamma, \\ v_1 & = \lambda \qquad \text{on } \Gamma, \end{cases} \tag{25}$$

the final solution being $(u, v) = (u_0 + u_1, v_0 + v_1)$. We remark here that unlike in the case of the biharmonic problem, systems (23), (25) do not decouple into two scalar problems for u and v.

As before, we define the bilinear form induced by the operator S via

$$s(\lambda, \mu) = \langle \mathcal{S}\lambda, \mu\rangle = -\int_\Gamma \frac{\partial u_1}{\partial \nu}\mu \,\mathrm{d}\Gamma \tag{26}$$

for $\lambda, \mu \in H^{-1/2}(\Gamma)$.

Before we prove our first result for $s(\cdot,\cdot)$, we first note that problem (25) represents an elliptic system of order 2 in the sense of Agmon, Douglis and Nirenberg [2]. As a consequence the following two results hold for $u_1, v_1 \in H^2(\Omega)$

$$\|\lambda\|_{r-1/2,\Gamma} \le C_1(\|u_1\|_{r+2,\Omega} + \|v_1\|_{r+2,\Omega}), \tag{27}$$

$$\|u_1\|_{r+2,\Omega} + \|v_1\|_{r+2,\Omega} \le C_2\|\lambda\|_{r-1/2,\Gamma}. \tag{28}$$

for $r \ge 0$ and for some constants C_1, C_2 which depend on the coefficients $\mathbf{a}, \vec{b}, c$ and the domain Ω. In particular, (27) is a consequence of the continuity of the matrix operator in (25) and (28) is obtained as an application of the general regularity result contained in [2]. Moreover, if (u_1, v_1) is the solution of (25) then the following norm equivalence holds.

Lemma 3.1 *Let $\lambda \in H^{-1/2}(\Gamma)$ and let (u_1, v_1) be the solution of (25). Then there exist constants C_3, C_4 such that*

$$C_3(\|u_1\|^2_{1,\Omega} + \|v_1\|^2_{0,\Omega}) \le \|u_1\|^2_{2,\Omega} + \|v_1\|^2_{2,\Omega} \le C_4(\|u_1\|^2_{1,\Omega} + \|v_1\|^2_{0,\Omega}) \tag{29}$$

Proof The lower bound is a simple consequence of the definitions of Sobolev norms (with $C_3 = 1$). For the upper bound, we first note that for bounded, convex $\Omega \in \mathbb{R}^2$, $\|v_1\|_{2,\Omega} \le \|\Delta v_1\|_{0,\Omega}$ [6]. Similarly, $\|u_1\|_{2,\Omega} \le \|\Delta u_1\|_{0,\Omega} = \|v_1\|_{0,\Omega}$ (cf. second equation of (25)). Then using the first equation of (25) we get

$$\begin{aligned} \|v_1\|^2_{2,\Omega} &\le \|\Delta v_1\|^2_{0,\Omega} \\ &\le \|\nabla\cdot(\mathbf{a}\nabla u_1)\|^2_{0,\Omega} + \|\vec{b}\cdot\nabla u_1\|^2_{0,\Omega} + \|cu_1\|^2_{0,\Omega} \\ &\le \|\mathbf{a}\|^2_{\infty,\Omega}\|\Delta u_1\|^2_{0,\Omega} + \|\vec{b}\|^2_{0,\Omega}|u_1|^2_{1,\Omega} + \|c\|^2_{0,\Omega}\|u_1\|^2_{0,\Omega} \\ &= \|\mathbf{a}\|^2_{\infty,\Omega}\|v_1\|^2_{0,\Omega} + \|\vec{b}\|^2_{0,\Omega}|u_1|^2_{1,\Omega} + \|c\|^2_{0,\Omega}\|u_1\|^2_{0,\Omega}. \end{aligned}$$

Hence

$$\begin{aligned}
\|u_1\|_{2,\Omega}^2 + \|v_1\|_{2,\Omega}^2 &\leq \|v_1\|_{0,\Omega}^2 + \|v_1\|_{2,\Omega}^2 \\
&\leq (1 + \|\mathbf{a}\|_{\infty,\Omega}^2)\|v_1\|_{0,\Omega}^2 + \|\vec{b}\|_{0,\Omega}^2 |u_1|_{1,\Omega}^2 + \|c\|_{0,\Omega}^2 \|u_1\|_{0,\Omega}^2 \\
&\leq C_4(\|u_1\|_{1,\Omega}^2 + \|v_1\|_{0,\Omega}^2)
\end{aligned}$$

for $C_4 = \max\left\{1 + \|\mathbf{a}\|_{\infty,\Omega}^2, \|\vec{b}\|_{0,\Omega}^2, \|c\|_{0,\Omega}^2\right\}$. □

We are now able to prove the following

Theorem 3.2 *The bilinear form $s(\cdot,\cdot)$ is coercive and continuous on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, i.e., there exist constants c, C such that for all $\lambda, \mu \in H^{-1/2}(\Gamma)$*

$$s(\lambda,\mu) \leq C\|\lambda\|_{H^{-1/2}(\Gamma)}\|\mu\|_{H^{-1/2}(\Gamma)} \tag{30}$$

$$s(\lambda,\lambda) \geq c\|\lambda\|_{H^{-1/2}(\Gamma)}^2 \tag{31}$$

Proof The continuity of $s(\cdot,\cdot)$ follows directly from (28) with $r = 0$, and the trace inequality (4)

$$s(\lambda,\mu) \leq \|\gamma_1 u\|_{1/2,\Gamma}\|\mu\|_{-1/2,\Gamma} \leq c(\gamma_1)\|u\|_{2,\Omega}\|\mu\|_{-1/2,\Gamma} \leq c(\gamma_1)C_2\|\lambda\|_{-1/2,\Gamma}\|\mu\|_{-1/2,\Gamma}.$$

In order to derive the lower bound we note that

$$s(\lambda,\lambda) = -\int_\Gamma \frac{\partial u_1}{\partial \nu}\lambda \mathrm{d}\Gamma.$$

Now, multiplying the second equation of (25) by v_1 we get

$$-\int_\Gamma \frac{\partial u_1}{\partial \nu}\lambda \mathrm{d}\Gamma = -\langle \nabla u_1, \nabla v_1\rangle + \|v_1\|_{0,\Omega}^2$$

and multiplying the first equation of (25) by u_1 we get

$$-\langle \nabla u_1, \nabla v_1\rangle = \langle c u_1, u_1\rangle + \langle \mathbf{a}\nabla u_1, \nabla u_1\rangle$$

so that

$$s(\lambda,\lambda) = \|v_1\|_{0,\Omega}^2 + |u_1|_{c,0,\Omega}^2 + |u_1|_{\mathbf{a},1,\Omega}^2.$$

Hence, using Lemma (3.1) and (27) with $r = 0$ we get

$$s(\lambda,\lambda) \geq c_5\left(\|v_1\|_{0,\Omega}^2 + \|u_1\|_{1,\Omega}^2\right) \geq \frac{c_5}{C_4}\left(\|v_1\|_{2,\Omega}^2 + \|u_1\|_{2,\Omega}^2\right) \geq \frac{c_5}{C_1 C_4}\|\lambda\|_{-1/2,\Gamma}^2,$$

where $c_5 = \min\{1, \min_{x\in\Omega}\sigma_{min}(\mathbf{a}(x))\}$, where $\sigma(\mathbf{a})$ denotes an eigenvalue of $\mathbf{a}$. □

The above result holds also in a discrete sense. Assuming the notation introduced in the previous section, the weak formulation corresponding to (22) reads

Find $(u_h, v_h) \in S_I^h \times S^h$ such that $\forall (w_h, z_h) \in S_I^h \times S^h$

$$\begin{aligned} a(u_h, w_h) + l(v_h, w_h) &= \langle f, w_h \rangle && (32a) \\ l(u_h, z_h) - m(v_h, z_h) &= 0 && (32b) \end{aligned}$$

where $a(\cdot, \cdot) : S_I^h \times S_I^h$ is defined as

$$a(u_h, w_h) := \langle \mathbf{a} \nabla u_h, \nabla w_h \rangle + \left\langle \vec{b} \cdot \nabla u_h + c u_h, w_h \right\rangle .$$

The Schur complement approach is described by the following three problems:

I. Find $(u_{0h}, v_{0h}) \in S_I^h \times S_I^h$ such that $\forall (w_h, z_h) \in S_I^h \times S_I^h$

$$\begin{aligned} a(u_{0h}, w_h) + l(v_{0h}, w_h) &= \langle f, w_h \rangle && (33a) \\ l(u_{0h}, z_h) - m(v_{0h}, z_h) &= 0 && (33b) \end{aligned}$$

II. Find $\lambda_h \in S_B^h$ such that $\forall \mu_h \in S_B^h$

$$s(\lambda_h, \mu_h) = -s(\lambda_{0h}, \mu_h) \qquad (34a)$$

III. Find $(u_{1h}, v_{1h}) \in S_I^h \times S^h$, $v_{1h} - \lambda_h \in S_I^h$, such that $\forall (w_h, z_h) \in S_I^h \times S^h$

$$\begin{aligned} a(u_{1h}, w_h) + l(v_{1h}, w_h) &= 0 && (35a) \\ l(u_{1h}, z_h) - m(v_{1h}, z_h) &= 0 && (35b) \end{aligned}$$

As in the case of the biharmonic problem, the bilinear form $s(\cdot, \cdot)$ can also be expressed in terms of integrals over the domain Ω; given an extension $\tilde{\mu}_h$ to Ω of μ_h, we have

$$s(\lambda_h, \mu_h) = m(v_{1h}, \tilde{\mu}_h) - l(u_{1h}, \tilde{\mu}_h);$$

in particular, we note that if $(u_{\lambda h}, v_{\lambda h}), (u_{\mu h}, v_{\mu h})$ are solutions of (35) for respective boundary data λ_h, μ_h, then

$$\begin{aligned} s(\lambda_h, \mu_h) &= m(v_{\lambda h}, v_{\mu h}) - l(u_{\lambda h}, v_{\mu h}) \\ &= m(v_{\lambda h}, v_{\mu h}) + a(u_{\mu h}, u_{\lambda h}) \end{aligned}$$

and thus

$$s(\lambda_h, \lambda_h) = \|v_{\lambda h}\|_{0,\Omega}^2 + |u_{\lambda h}|_{c,0,\Omega}^2 + |u_{\lambda h}|_{a,1,\Omega}^2 .$$

Moreover, it is clear that the symmetry of $s(\cdot,\cdot)$ is decided by that of $a(\cdot,\cdot)$.

In the following, we will also need a standard a priori estimate as derived for the biharmonic problem in [11],[5] etc. Let u, v be the solution of (2) with $-\Delta u = v \in H^{k+2}(\Omega)$, and let $(u_h, v_h) \subset S_I^h \times S^h$, $S^h = S^{h,k}$, be the corresponding finite element solution using the mixed formulation (32). We assume that the following (sub-optimal) estimate holds

$$\|u - u_h\|_{1,\Omega} + \|v - v_h\|_{0,\Omega} \le C_5 h^k \|v\|_{k+2,\Omega} \quad \forall k \ge 1. \tag{36}$$

Remark 3.1 *Optimal a priori estimates were provided by Li* [12] *for the case of regular rectangular meshes and for $\vec{b} = 0$. In fact, the proof in* [12] *can be easily modified to hold for nonzero $\vec{b}$, provided*
(i) $c(x, y) - \nabla \cdot \vec{b}(x, y) \ge c_0$ for some $c_0 > 0$ and for all $(x, y) \in \Omega$
(ii) if u is prescribed only on $\Gamma_D \subset \Gamma$ we require $\vec{n} \cdot \vec{b}|_{\Gamma\setminus\Gamma_D} \ge 0$.
The resulting improved estimate is

$$\|u - u_h\|_{1,\Omega} + \|v - v_h\|_{0,\Omega} \le C_5' h^{k+1} \|v\|_{k+2,\Omega} \quad \forall k \ge 1. \tag{37}$$

We shall also assume that, given a quasi-uniform subdivision of Ω, the following two inverse inequalities hold [1, p. 38]

$$\|\gamma_0 v_h\|_{0,\Gamma} \le C_6 h^{-1/2} \|v_h\|_{0,\Omega} \tag{38}$$

for all $v_h \in S^h$ and for all $\lambda_h \in S_B^h$ [17, p. 48]

$$\|\gamma_0 \lambda_h\|_{1/2,\Gamma} \le C_7 h^{-1/2} \|\gamma_0 \lambda_h\|_{0,\Gamma}. \tag{39}$$

Theorem 3.3 *Let $\lambda_h, \mu_h \in S_B^h$. Then there exist constants c', C' such that for all λ_h, μ_h*

$$c' \|\gamma_0 \lambda_h\|^2_{-1/2,\Gamma} \le s(\lambda_h, \lambda_h)$$

and

$$s(\lambda_h, \mu_h) \le C' \|\gamma_0 \lambda_h\|_{-1/2,\Gamma} \|\gamma_0 \mu_h\|_{-1/2,\Gamma}.$$

Proof Let $\lambda_h \in S_B^h$ and denote by $(u_{\lambda h}, v_{\lambda h})$ the corresponding solution of (35). Let also $\lambda = \gamma_0 \lambda_h \in H^{-1/2}(\Gamma)$, and let u_λ, v_λ denote the corresponding solutions of (25). Since

$$s(\lambda_h, \lambda_h) = \|v_{\lambda h}\|^2_{0,\Omega} + |u_{\lambda h}|^2_{c,0,\Omega} + |u_{\lambda h}|^2_{\mathbf{a},1,\Omega} \ge c_5 \left(\|v_{\lambda h}\|^2_{0,\Omega} + \|u_{\lambda h}\|^2_{1,\Omega} \right)$$

we find using Lemma (3.1), (27) with $r = 0$ and (36) with $k = 1$

$$\begin{aligned} s(\lambda_h, \lambda_h) &\ge c_5 \left(\|v_\lambda\|^2_{0,\Omega} + \|u_\lambda\|^2_{1,\Omega} \right) - c_5 \left(\|v_\lambda - v_{\lambda h}\|^2_{0,\Omega} + \|u_\lambda - u_{\lambda h}\|^2_{1,\Omega} \right) \\ &\ge \frac{c_5}{C_1 C_4} \|\lambda\|^2_{-1/2,\Gamma} - c_5 C_5^2 h^2 \|v_\lambda\|^2_{3,\Omega} \end{aligned}$$

Now using (28) with $r = 1$ we find $\|v_\lambda\|_{3,\Omega} \le C_2\|\lambda\|_{1/2,\Gamma}$. Thus, using (39),

$$s(\lambda_h, \lambda_h) \ge \frac{c_5}{C_1C_4}\|\lambda\|^2_{-1/2,\Gamma} - c_5C_2C_5^2h^2\|\lambda\|^2_{1/2,\Gamma} \ge \frac{c_5}{C_1C_4}\|\lambda\|^2_{-1/2,\Gamma} - c_5C_2C_5^2C_7^2h\|\lambda\|^2_{0,\Gamma}.$$

Since we also have by using (38)

$$s(\lambda_h, \lambda_h) \ge c_5\|v_{\lambda h}\|^2_{0,\Omega} \ge \frac{c_5}{C_6^2}h\|\lambda\|^2_{0,\Gamma}$$

we finally get

$$(1+\tilde{c})s(\lambda_h, \lambda_h) \ge \frac{c_5}{C_1C_4}\|\lambda\|^2_{-1/2,\Gamma},$$

where $\tilde{c} = C_2(C_5C_6C_7)^2$. For the upper bound we proceed in a similar way. Let $\mu = \gamma_0\mu_h \in H^{-1/2}(\Gamma)$, where $\mu_h \in S_B^h$ and let $(u_{\mu h}, v_{\mu h})$ be the corresponding solution of (35). We have

$$\begin{aligned} s(\lambda_h, \mu_h) &= m(v_{\lambda h}, v_{\mu h}) + a(u_{\mu h}, u_{\lambda h}) \\ &\le \|v_{\lambda h}\|_{0,\Omega}\|v_{\mu h}\|_{0,\Omega} + \|\vec{b}\|_{0,\Omega}|u_{\mu h}|_{1,\Omega}\|u_{\lambda_h}\|_{0,\Omega} \\ &\quad + \|u_{\mu h}\|_{\mathbf{a},1,\Omega}\|u_{\lambda h}\|_{\mathbf{a},1,\Omega} + \|u_{\mu h}\|_{c,0,\Omega}\|u_{\lambda h}\|_{c,0,\Omega} \\ &\le \widehat{c}(\|v_{\lambda h}\|_{0,\Omega} + \|u_{\lambda h}\|_{1,\Omega})(\|v_{\mu h}\|_{0,\Omega} + \|u_{\mu h}\|_{1,\Omega}) \end{aligned}$$

and the result follows from the norm-equivalence (29) and the regularity result (28). □

It follows immediately that we can use the preconditioning strategy suggested in [16]. Let S denote the representation of $s(\cdot,\cdot)$ in the basis $\{\Phi_i\}_{n_I+1\le i\le n}$ (cf. Thm 2.2) and let H denote the matrix representation (17) of the $H^{-1/2}(\Gamma)$-norm.

Theorem 3.4 *The H-field of values (and thus spectrum) of $H^{-1}S$ is bounded independently of h.*

Proof We include here the proof for completeness. For general results see [21], [13]. Using Thms 2.2, 3.3 we get for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^{n-n_I}$

$$\tilde{c} \le \frac{\boldsymbol{\lambda}^T S\boldsymbol{\lambda}}{\boldsymbol{\lambda}^T H\boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}^T H(H^{-1}S)\boldsymbol{\lambda}}{\boldsymbol{\lambda}^T H\boldsymbol{\lambda}}, \qquad \frac{\boldsymbol{\lambda}^T S\boldsymbol{\mu}}{\|\boldsymbol{\lambda}\|_H\|\boldsymbol{\mu}\|_H} \le \tilde{C}.$$

Thus, a lower bound on the H-field of values is $\tilde{c}$. Since the H-norm of a matrix is an upper bound on its H-field of values [10], the upper bound we seek is $\tilde{C}$

$$\|H^{-1}S\|_H = \max_{\boldsymbol{\lambda}\neq 0}\frac{\|H^{-1}S\boldsymbol{\lambda}\|_H}{\|\boldsymbol{\lambda}\|_H} = \max_{\boldsymbol{\lambda},\boldsymbol{\mu}\neq 0}\frac{\boldsymbol{\lambda}^T S\boldsymbol{\mu}}{\|\boldsymbol{\lambda}\|_H\|\boldsymbol{\mu}\|_H} \le \tilde{\kappa}_2.$$

□

4 Block preconditioners for the interior

While the results in the previous section ensure that we have an optimal preconditioner on the boundary, the discussion at the end of §2 highlighted the fact that a block-preconditioning approach of type (21) requires also an efficient algorithm for the solution of the problem involving the operator acting in the interior of the domain (the 1-1 block in P_L, P_R). This we aim to provide here.

Let us write the discretization of (32) as

$$\begin{pmatrix} A_{II} & L_{II} & L_{IB} \\ L_{II} & -M_{II} & -M_{IB} \\ L_{IB}^T & -M_{IB}^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \\ 0 \end{pmatrix} \tag{40}$$

where, in the notation of §2,

$$(A_{II})_{ij} = a(\Psi_j, \Psi_i),$$

with $1 \leq i, j \leq m$. We recall here the definition of $a(\cdot, \cdot)$

$$a(u_h, w_h) := \langle \mathbf{a}\nabla u_h, \nabla w_h \rangle + \left\langle \vec{b} \cdot \nabla u_h + c u_h, w_h \right\rangle .$$

We are interested in preconditioning

$$A = \begin{pmatrix} A_{II} & L_{II} \\ L_{II} & -M_{II} \end{pmatrix},$$

and we are going to investigate the suitability of the following candidate

$$P = \begin{pmatrix} L_{II} & L_{II} \\ L_{II} & -M_{II} \end{pmatrix}.$$

The advantage of using such a preconditioner is that its inverse can be written as

$$P^{-1} = \begin{pmatrix} L_{II}^{-1} M_{II} Q^{-1} & Q^{-1} \\ Q^{-1} & -Q^{-1} \end{pmatrix}$$

where $Q = L_{II} + M_{II}$. Thus an application of the preconditioner requires solutions of problems with either L_{II} or $L_{II} + M_{II}$ as coefficient matrices, for which optimal solvers are available. Moreover,

$$AP^{-1} = \begin{pmatrix} S_A S_L^{-1} & (A_{II} - L_{II}) Q^{-1} \\ 0 & I \end{pmatrix}$$

where

$$S_A = A_{II} + L_{II} M_{II}^{-1} L_{II}, \quad S_L = L_{II} + L_{II} M_{II}^{-1} L_{II}$$

are the Schur complements of $-M_{II}$ in A and P, respectively. Thus, the eigenvalue distribution of interest for convergence is that of $S_A S_L^{-1}$ – we will see that the coercivity and continuity of $a(\cdot,\cdot)$ are sufficient for these eigenvalues and hence the eigenvalues of AP^{-1} to be bounded independently of the meshsize. We now derive some properties for the matrices involved in our problem.

First, it is straightforward to show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive with respect to the norm $|\cdot|_{1,\Omega}$, i.e., for all $v_h, w_h \in S^h$

$$c_5 \|v_h\|_{1,\Omega}^2 \leq a(v_h, v_h), \quad a(v_h, w_h) \leq c_6 |v_h|_{1,\Omega} |w_h|_{1,\Omega}$$

with $c_6 = 3\max\left\{\|\mathbf{a}\|_{\infty,\Omega}, C(\Omega)\|\vec{b}\|_{\infty,\Omega}, \|c\|_{\infty,\Omega} C^2(\Omega)\right\}$, where $C(\Omega)$ is Poincaré's constant. Moreover,

$$\frac{1}{2}\left(a(v_h, w_h) - a(w_h, v_h)\right) \leq c_7 |w_h|_{1,\Omega} |v_h|_{1,\Omega}, \tag{41}$$

where $c_7 = \|\vec{b}\|_{\infty,\Omega}$. It follows that the following discrete relations hold for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n_I} \setminus \{0\}$

$$c_1 \leq \frac{\mathbf{v}^T A_{II} \mathbf{v}}{\mathbf{v}^T L_{II} \mathbf{v}}, \quad \frac{\mathbf{w}^T A_{II} \mathbf{v}}{\|\mathbf{w}\|_{L_{II}} \|\mathbf{v}\|_{L_{II}}} \leq c_2.$$

Let H_{II}, N_{II} denote the symmetric and skew-symmetric parts of A_{II}. Then (41) yields the bound

$$\frac{\mathbf{w}^T N_{II} \mathbf{v}}{\|\mathbf{w}\|_{L_{II}} \|\mathbf{v}\|_{L_{II}}} \leq c_7. \tag{42}$$

Define now $S_H = H_{II} + L_{II} M_{II}^{-1} L_{II}$. Then both S_H, S_L are symmetric and positive-definite matrices, which induce equivalent norms $\|\cdot\|_{S_H}, \|\cdot\|_{S_L}$, with the following constants of equivalence

$$\min\{1, 1/c_6\} \leq \frac{\mathbf{v}^T S_L \mathbf{v}}{\mathbf{v}^T S_H \mathbf{v}} \leq \max\{1, 1/c_5\}. \tag{43}$$

We are now ready to prove the following result.

Theorem 4.1 *The S_L-field-of-values (and thus, spectrum) of $S_L^{-1} S_A$ is bounded independently of h.*

Proof The lower bound follows from the above norm-equivalence

$$\frac{\mathbf{v}^T S_L (S_L^{-1} S_A) \mathbf{v}}{\mathbf{v}^T S_L \mathbf{v}} = \frac{\mathbf{v}^T S_H \mathbf{v}}{\mathbf{v}^T S_L \mathbf{v}} \geq \min\{1, c_5\}.$$

For the upper bound we proceed as before and consider the S_L-norm of $S_L^{-1}S_A$. We have, using (43), (42)

$$\begin{aligned}
\|S_L^{-1}S_A\|_{S_L} &= \max_{\mathbf{v},\mathbf{w}\neq 0} \frac{\mathbf{w}^T S_A \mathbf{v}}{\|\mathbf{w}\|_{S_L}\|\mathbf{v}\|_{S_L}} \\
&\leq \max_{\mathbf{v},\mathbf{w}\neq 0} \frac{\mathbf{w}^T S_A \mathbf{v}}{\|\mathbf{w}\|_{S_H}\|\mathbf{v}\|_{S_H}} \max\left\{1, c_6^2\right\} \\
&\leq \left\{1, c_6^2\right\}\left(1 + \max_{\mathbf{v},\mathbf{w}\neq 0} \frac{\mathbf{w}^T N_{II} \mathbf{v}}{\|\mathbf{w}\|_{S_H}\|\mathbf{v}\|_{S_H}}\right) \\
&= \max\left\{1, c_6^2\right\}\left(1 + \max_{\mathbf{v},\mathbf{w}\neq 0} \frac{\mathbf{w}^T N_{II} \mathbf{v}}{\|\mathbf{w}\|_{L_{II}}\|\mathbf{v}\|_{L_{II}}} \frac{\|\mathbf{w}\|_{L_{II}}\|\mathbf{v}\|_{L_{II}}}{\|\mathbf{w}\|_{S_L}\|\mathbf{v}\|_{S_L}} \frac{\|\mathbf{w}\|_{S_L}\|\mathbf{v}\|_{S_L}}{\|\mathbf{w}\|_{S_H}\|\mathbf{v}\|_{S_H}}\right) \\
&\leq \max\left\{1, c_6^2\right\}\left(1 + \max_{\mathbf{v},\mathbf{w}\neq 0} \frac{\mathbf{w}^T N_{II} \mathbf{v}}{\|\mathbf{w}\|_{L_{II}}\|\mathbf{v}\|_{L_{II}}} \max\left\{1, c_5^{-2}\right\}\right) \\
&\leq \max\left\{1, c_6^2\right\}\left(1 + c_7 \max\left\{1, c_5^{-2}\right\}\right)
\end{aligned}$$

where we used the fact that $\|\mathbf{v}\|_{L_{II}} \leq \|\mathbf{v}\|_{S_L}$, for all $\mathbf{v} \in \mathbb{R}^{n_I}$. □

The bounds on the field of values derived separately for the boundary and interior preconditioned matrices do not lead to bounds on the field of values of the globally preconditioned matrix. They do however provide bounds on its eigenvalues – this motivates our use of these preconditioners in the next section.

5 Experiments

In this section we present the results obtained for our general elliptic problem

$$\left\{ \begin{aligned} -\Delta v - \nabla\cdot(\mathbf{a}\nabla u) + (\vec{b}\cdot\nabla)u + cu &= f && \text{in } \Omega, \\ v + \Delta u &= 0 && \text{in } \Omega, \\ u = \partial u/\partial \nu &= 0 && \text{on } \Gamma. \end{aligned} \right. \tag{44}$$

where we chose $\mathbf{a} = \nu I_2$, $c = 0$ and where ν took a range of values. The choice of $\vec{b}$ was motivated by applications from oceanography [20]; we first tested the symmetric case ($\vec{b} = (0,0)$) and the nonsymmetric case with $\vec{b} = (1,0)$. The weak formulation of (44) was discretized on uniform triangular meshes. For the solution of the linear system we used full GMRES with stopping tolerance 10^{-10} and preconditioner

$$P_L = \begin{pmatrix} P_A & Z \\ 0 & P_S \end{pmatrix}.$$

where Z is as defined in section 2. With first employed the choice $P_A = A$, in order to test the parameter independence of the $H^{1/2}(\Gamma)$-norm preconditioner ($P_S = H$). We compared this choice with the case where we used no boundary preconditioner ($P_S = I$). The results are presented for the symmetric and nonsymmetric cases in Tables 1 and 2, respectively. We see indeed that mesh-independence is achieved in for both examples, while the expected mesh-dependent asymptotic behaviour for the unpreconditioned case is rather noticeable (experimentally of order $O(\log h)$).

Remark 5.1 *The implementation of H given in (17) was simplified in the following manner. We recall that H is the direct sum of $H_k = M_k L_k^{-1/2} M_k$, where M_k are one-dimensional mass matrices which on a uniform mesh scale like hI. Thus, the application of H_k^{-1} can be taken to be simply a multiplication by $L_k^{1/2} h^{-2}$, which is spectrally equivalent to the original choice.*

The same behaviour is noticed when we replace $P_A = A$ with $P_A = \nu L$, for which the theoretical result of Thm 4.1 holds. Combined with $P_S = H$, the resulting preconditioner is still mesh-independent – in fact, Table 3 reveals that the performance is only slightly affected, compared to the case where $P_S = I$.

We tested our preconditioner for the case of quasi-uniform meshes and also non-constant $\vec{b}$, but found the convergence to be very similar; we chose not to include the results here. Instead we include a numerical study of the dependence on $\vec{b}$. Given that the motivation of

$\nu =$	10^2	1	10^{-2}	10^2	1	10^{-2}
$n = 8,712$	14	13	14	38	40	41
$33,800$	14	14	14	51	50	53
$133,128$	15	14	14	63	59	62

Table 1: *GMRES iterations for $P_A = A$, with $P_S = H$ and $P_S = I$: symmetric case* $\vec{b} = (0, 0)$.

$\nu =$	10^2	1	10^{-2}	10^2	1	10^{-2}
$n = 8,712$	18	18	18	45	46	47
$33,800$	19	19	19	57	56	57
$133,128$	18	16	16	70	64	68

Table 2: *GMRES iterations for $P_A = A$, with $P_S = H$ and $P_S = I$: nonsymmetric case* $\vec{b} = (1, 0)$.

$\nu =$	10^2	1	10^{-2}	10^2	1	10^{-2}
$n = 8,712$	20	20	20	68	68	68
$33,800$	21	20	20	85	84	84
$133,128$	21	18	17	104	99	99

Table 3: *GMRES iterations for $P_A = \nu L$, with $P_S = H$ and $P_S = I$: nonsymmetric case* $\vec{b} = (1,0)$.

this work came from oceanographic problems, where large values of $\vec{b}$ are frequent, we tested our preconditioner also on a range of values of constant, horizontal $\vec{b}$, of the form $\vec{b} = (\beta, 0)$. The results are displayed in Table 4. We find as expected the independence of the size of the problem, though there is a dependence on β which is relatively mild (a difference of about 17 iterations for a variation of β of five orders of magnitude, cf. Table 2).

Finally, we include the results for the more practical choice $P_A = \nu L$, which requires only the inversion of either a laplacian or a reaction-diffusion discrete operator. While this choice is practical from an implementation point of view, preconditioning a problem with a large skew-symmetric part with its symmetric part may not lead to an optimal preconditioner. This can be seen in Table 5, where the number of iterations grows considerably with β. On the other hand, as expected given our theoretical results, the mesh-independence is preserved, maintaining the usefulness of the choice $P_S = H$ given our preconditioning context.

$\nu = 1, \beta =$	10^3	10^4	10^5	10^3	10^4	10^5
$n = 8,712$	22	27	34	50	49	42
$33,800$	22	27	34	61	63	53
$133,128$	22	27	33	73	79	70

Table 4: *GMRES iterations for $P_A = A$, with $P_S = H$ and $P_S = I$.*

$\nu = 1, \beta =$	10^3	10^4	10^5	10^3	10^4	10^5
$n = 8,712$	33	56	130	113	143	206
$33,800$	31	56	132	136	163	207
$133,128$	30	55	132	145	164	223

Table 5: *GMRES iterations for $P_A = \nu L$, with $P_S = H$ and $P_S = I$.*

6 Summary

We extended the preconditioning approach in [9], [16] to the case of general fourth-order elliptic problems. The resulting preconditioners require the solution of a discrete boundary operator, which needs to be assembled in a piecewise fashion on the boundary segments. Both theory and results were presented for the case of quasi-uniform meshes. Our analysis showed that the field of values of the preconditioned boundary Schur complement is bounded independently of the size of the problem. Theoretical bounds on the convergence of GMRES guarantee mesh-independent convergence, which we verified numerically. We also provided and analyzed suitable candidates for preconditioning in the interior of the domain. The resulting method is an optimal block algorithm which requires only the solution of smaller problems, involving the discretization of either laplacian or reaction-diffusion operators. We expect future work to consider the generalization of these ideas to the case of multiply-connected domains which arise in some large-scale applications such as oceanography.

Acknowledgments I thank Martin van Gijzen for suggesting this topic, for the subsequent useful discussions and for his comments on the manuscript.

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