On Lidskii's algorithm to quantify the first order terms in the asymptotics of a defective eigenvalue. Part II

Françoise Chaitin-Chatelin[1]

CERFACS Technical Report TR/PA/05/04

[1]Université Toulouse 1 and CERFACS, 42 Ave. G. Coriolis, 31057 Toulouse Cedex 1, France. E-mail: chatelin@cerfacs.fr

Abstract

This report is a follow-up for [3]. Part II addresses a question left open in Part I: it consists in the analysis of the computation which can be realised at step $j+1$ of Lidskii's algorithm when step j is non generic. This question is examined in the context of an appropriate modification of the existing Homotopic Deviation theory [1, 2]. Connections are made with the theory of square matrix pencils [9, 10].

Key words: Simple and Augmented Homotopic Deviation, Schur complement, Sherman-Morrison formula, polynomial, rational fraction, regular matrix pencil

1. Introduction

1.1. Presentation

This report aims at answering the question left open in [3], whose notation is kept. $\deg P$ denotes the degree of the polynomial $z \mapsto P(z)$.

We present below the algebraic problem which underlies the question: what happens at step $j+1$ of Lidskii's algorithm when ϕ_j is singular, hence the Schur complement Ω_{j+1} does not exist, yet $j < q$?

The analysis that we propose here is related to the theory of Homotopic Deviation. We introduce a new viewpoint deriving from the existence of at least one rational form for $\det \phi_{j+1}(z)$ when f_j and $r_{j+1} \geq 2$, and possibly two. These forms arise from computing $\det \phi_{j+1}(z)$ by partitioning.

1.2. The basic algebraic problem

Let ϕ_j be singular, $j < q$. At step $j+1$, we face the following problem: find $z \in \mathbb{C}$ such that $\det \phi_{j+1}(z) = 0$ with

$$\phi_{j+1}(z) \quad = \quad \begin{pmatrix} \phi_j & R_{j+1} \\ L_{j+1} & \Delta_{j+1} - zI_{r_{j+1}} \end{pmatrix},$$

where the 2×2 block representation corresponds to the splitting

$$f_{j+1} \quad = \quad f_j \ + \ r_{j+1}.$$

We drop the subscripts j and $j+1$ below, and write generically

$$\phi(z) \quad = \quad \begin{pmatrix} \Gamma & R \\ L & \Delta - zI_r \end{pmatrix},$$

where the Γ represents ϕ_j. $Z(\det \phi)$ is the set of zeros of $\det \phi$.

1. When Γ is invertible, the determination of $Z(\det \phi)$ is classical [3, 6]: $Z(\det \phi) = \sigma(\Omega)$ where

   $$\Omega - zI_r \quad = \quad \Delta \ - \ L\Gamma^{-1}R \ - \ zI_r$$

   represents the Schur complement of Γ in $\phi(z)$. Therefore

   $$\det \phi(z) \quad = \quad \det(\Gamma)\ \det(\Omega - zI_r).$$

2. When Γ is singular, a direct application of Laplace's theorem [5] to compute $\det \phi(z)$ shows that the coefficient of z^r is $\det \Gamma$. This is done by considering the partitioning $f+r$. Therefore $\det \phi(z)$ is a polynomial in z of degree $< r$ when $\det \Gamma = 0$: it has at most $r-1$ roots. At least one root is missing to obtain the r roots required to apply the Lidskii algorithm, when Γ is singular.

   What light is shed on this difficulty by an application of the alternative Schur formula? Let us assume that $z \notin \sigma(\Delta)$. Then

$$\det \phi(z) \quad = \quad \det(\Delta - zI_r) \;\; \det\big(\Gamma - R(\Delta - zI_r)^{-1}L\big)\,.$$

   The zeros of $\det \phi(z) = 0$ not in $\sigma(\Delta)$ are given by the second factor. We assume that L and R are nonzero. If either R or L is 0, the second factor reduces to $\det \Gamma = 0$, hence $\det \phi(z) \equiv 0$ for $z \in \mathbb{C}$.

We set

$$W_z \;\; = \;\; R(zI - \Delta)^{-1}L :$$

this matrix of order f defined for $z \notin \sigma(\Delta)$ plays for $(\Delta,\ LR)$ the role played by

$$M_z \;\; = \;\; V^H(zI - A)^{-1}U$$

for $(A,\ E = UV^H)$. However, the equation $\det(\Gamma + W_z) = 0$ differs from $\det(I + M_z) = 0$ previously encountered in Homotopic Deviation: the **identity** matrix I is replaced by a possibly **rank-deficient** matrix Γ, which may even be 0.

In Section 2, we extend the *simple* Homotopic Deviation (HD) theory which deals with an identity matrix to take into account the case of a possibly rank-deficient matrix. This variant of HD is called *augmented* HD because we are interested in the roots of $\det \phi(z)$ where $\phi(z)$ is an *augmented* matrix.

2. A modification of Homotopic Deviation by augmentation

2.1. Simple HD and Schur complement

We recall that the classical Homotopic Deviation theory for (A, E) is related to the notion of Schur complement in an augmented matrix $\hat{A}(z,t)$ of order

$N = n + r, \ 1 \le r < n$ such that

$$\hat{A}(z,t) \;=\; \left(\begin{array}{c|c} zI_n - A & tU \\ \hline V^H & I_r \end{array}\right), \qquad E \;=\; UV^H,$$

and rank E = rank U = rank $V = r$. $\hat{A}(z,t)$ has the two equivalent forms given below which are both block-diagonal, for any $z \in \mathrm{re}(A)$:

$$\left(\begin{array}{c|c} zI_n - A & 0 \\ \hline 0 & I_r - tM_z \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} zI_n - A - tE & 0 \\ \hline 0 & I_r \end{array}\right).$$

The matrix $M_z = V^H(zI - A)^{-1}U$ plays an essential role in the analyticity of

$$t \;\longmapsto\; R(t,z) \;=\; (A + tE - zI)^{-1}$$

derived from the Sherman-Morrison formula. It is the *coupling matrix* of order $r \le n$ which expresses algebraically the intimate connection between A and E realised by the coupling $A + tE$. It plays a role similar to that of the transfer matrix in Linear Systems Theory [4, 5]. It is equal to the W_z matrix of Section 1 when A, $E = UV^H$ are replaced by Δ, LR respectively.

The equation $\det(I_r - tM_z) = 0$ plays also an important role for getting the eigenvalues in $\sigma(A(t))$: z in $\mathrm{re}(A)$ is an eigenvalue of $A + tE$ iff $t\mu_z = 1$ for $\mu_z \in \sigma(M_z) \setminus \{0\}$. See [1, 2] for more. In the Appendix, the set

$$\{t = 1/\mu_z, \ \mu_z \in \sigma(M_z) \setminus \{0\}\}$$

is interpreted as the set of *finite* eigenvalues for the regular pencil $A - zI + tE$, z given in $\mathrm{re}(A)$.

2.2. Augmented HD

In this variant of Homotopic Deviation, I_r is replaced by a matrix Γ of order r ($0 \le \mathrm{rank}\,\Gamma \le r$) and n and r are not necessarily constrained by $r \le n$. In particular, $n = 1$ together with $r > 1$ is possible since this has a meaning in the context of Lidskii's algorithm (corresponding to $r = 1$ and $f > 1$) [3, 6]. We assume that U, V are nonzero in $\mathbb{C}^{n \times r}$, but U, V have a rank between 1 and $\min(n, r)$. We set $r_0 = \min(\mathrm{rank}\,U, \mathrm{rank}\,V) \ge 1$. It is important to keep in mind that r now denotes a size (and not a rank, as it did in the case of simple HD).

$E = UV^H$ and $G = V^H U$ of respective order n and r have a rank $\leq \min(n,r)$. They share the same spectrum, plus $\{0\}$ as the case may be: 0 lies in $\sigma(E)$ (resp. $\sigma(G)$) for $r < n$ (resp. $r > n$). Observe that $G = 0$ implies $E^2 = U(V^H U)V^H = 0$: if $E \neq 0$, it is nilpotent so that $E^2 = 0$. Similarly, $E = 0$ implies $G^2 = 0$.

The assumption that $U, V \in \mathbb{C}^{n\times r}$ are nonzero entails that when $r = 1$ (resp. $n = 1$) $E = UV^H$ (resp. $G = V^H U$) has rank 1 and is $\neq 0$. This need not be true for r and $n \geq 2$. The condition $\det G = \det V^H U \neq 0$ may hold only if $r_0 = r \leq n$. $\det G \neq 0$ is then equivalent to the condition (Σ) in simple HD ($0 \in \sigma(E)$ is semi-simple).

Another important consequence of $\Gamma \neq I$ for the augmented matrix

$$\hat{A}(z,t) \quad = \quad \left(\begin{array}{c|c} zI - A & tU \\ \hline V^H & \Gamma \end{array}\right)$$

is that the *second* equivalent block-diagonal form, which is now

$$\left(\begin{array}{c|c} zI - A - tU\Gamma^{-1}V^H & 0 \\ \hline 0 & \Gamma \end{array}\right), \quad \text{for} \quad z \in \mathrm{re}(A),$$

exists iff $\det \Gamma \neq 0$. Accordingly, the *first* equivalent block-diagonal form becomes

$$\left(\begin{array}{c|c} zI - A & 0 \\ \hline 0 & \Gamma - tM_z \end{array}\right) \quad \text{for} \quad z \in \mathrm{re}(A).$$

Proposition 2.1. *A necessary condition for M_z to be invertible for some z in* $\mathrm{re}(A)$ *is that $r_0 = r \leq n$.*

Proof. We write $M_z = \frac{1}{\pi(z)} Q(z)$, with $\pi(z) = \det(zI - A)$, and $Q(z) = V^H \mathrm{adj}(zI - A)\, U \in \mathbb{C}^{r\times r}$. When $r > n$, rank $Q(z) \leq n < r$. When $r = n$, $\det Q(z) = \det U \det V^H \det \mathrm{adj}(zI - A)$. Therefore when $r > n$ or $r_0 < r \leq n$, $\det Q(z) \equiv 0$. Conversely $\det Q(z) \not\equiv 0$ implies $r_0 = r \leq n$. □

When $\det Q(z) \not\equiv 0$, the set of zeros $Z(\det Q)$ for $\det Q$ is discrete. In this case, we define $F(A,E) = Z(\det Q) \cap \mathrm{re}(A)$ to be the set of frontier points [1], which are the points z in $\mathrm{re}(A)$ where M_z is not invertible. In order to maintain the possibility that F is discrete, we assume below that $r_0 = r \leq n$. We observe that the condition $r_0 = r \leq n$ (which holds in simple

HD) does not guarantee that F is discrete. See Example 3.1 in [4] where $r_0 = r = 2 < 3 = n$, and $F(A, E) = \mathrm{re}(A)$. However the additional condition $\det G \neq 0$ implies that $Z(\det Q)$, hence F, are discrete [1]. In particular, $r_0 = r = n$ implies that $F(A, E) = \emptyset$. More generally, F is discrete iff there exists z_0 in $\mathrm{re}(A)$ such that M_z is invertible [1]. When M_z is invertible ($z \in \mathrm{re}(A) \setminus F(A, E)$ [1, 2, 3]) then

$$\Gamma - tM_z \quad = \quad t\left(s\Gamma M_z^{-1} - I_r\right) M_z, \qquad s \quad = \quad \frac{1}{t},$$

and

$$(\Gamma - tM_z)^{-1} \quad = \quad -sM_z^{-1} \sum_{k=0}^{\infty} \left(s\Gamma M_z^{-1}\right)^k$$

converges for

$$|s| \quad < \quad \frac{1}{\rho(\Gamma M_z^{-1})},$$

that is

$$|t| > \rho(\Gamma M_z^{-1}).$$

Analyticity of the resolvent $R(t, z)$ is preserved around $|t| = \infty$ ($|s| \sim 0$). But there is *no* analyticity for $|t|$ small when Γ is singular, which is the case which motivates this report. The rate of convergence of the analytic representation at ∞ is not given by the frontier portrait $z \mapsto \rho(M_z^{-1})$, but rather, by the *resultant* portrait $z \mapsto \rho(\Gamma M_z^{-1})$. We observe also that the notion of criticality vanishes: $R(t, z)$ cannot be represented as a polynomial in t for certain critical points z in $\mathrm{re}(A)$.

2.3. The equation $\det \hat{A}(z, t) = 0$ for n and $r \geq 2$ resolved by using the Schur formulae

We set $\Delta(z, t) = \det \hat{A}(z, t)$: this is a polynomial in z of degree at most n, as well as a polynomial in t of degree at most r. The coefficient of z^n is $\det \Gamma$: it is nonzero iff $\det \Gamma \neq 0$.

We want to compare the direct evaluation $\Delta(z, t)$ of $\det \hat{A}(z, t)$ with the computation by means of the partitioning $N = n + r$ realised by the Schur formulae when at least one of the diagonal blocks $zI_n - A$ or Γ is invertible.

- First, when $\det \Gamma \neq 0$,

(S1) $\quad \Delta(z,t) \;=\; (\det\Gamma)\det(zI - A - tU\Gamma^{-1}V^H)$

for any z, t in $\mathbb{C}$.

- Second, for $z \in \mathrm{re}(A)$, $\pi(z) = \det(zI - A) \neq 0$, then

(S2) $\quad \Delta(z,t) \;=\; \pi(z)\det(\Gamma - tM_z)$

is valid for $z \notin \sigma(A)$ and $t \neq 0$.

We remark that $\Delta(z, 0) = (\det\Gamma)\pi(z)$ for any z in $\mathbb{C}$. Let *Var* be the algebraic manifold in $\mathbb{C} \times \mathbb{C}$ defined by

$$Var \;=\; \{(z,t),\, \Delta(z,t) = 0\}.$$

What is the description of *Var* provided by **(S1)** or **(S2)**? We analyse this question by distinguishing whether $\det\Gamma$ is 0 or not.

2.4. $\det\Gamma \neq 0$

We set $E_1 = U\Gamma^{-1}V^H$, and $A_1(t) = A + tE_1$. Then it is plain that $\Delta(z,t) = 0$ iff $\det(zI - A - tE_1) = 0$. *Var* can be fully described by the n spectral rays:

$$t \;\longmapsto\; \lambda_i(t) \in \sigma(A_1(t)), \quad i = 1, \ldots, n,$$

when one uses **(S1)** with $\det\Gamma \neq 0$. This represents the map

$$\textbf{(a):} \quad t \in \mathbb{C} \;\longmapsto\; \sigma(A_1(t)) \in \mathbb{C}^n.$$

We turn to **(S2):** $\det(\Gamma - tM_z) = \det\Gamma\det(I_r - tM_{1\,z})$ with $M_{1\,z} = V^H(zI - A)U\Gamma^{-1} = M_z\Gamma^{-1}$, for any $z \notin \sigma(A)$. Observe that $F(A, E) = F(A, E_1)$.

Any point in *Var* such that $z \notin \sigma(A)$ can be obtained by solving $\det(I_r - tM_{1\,z}) = 0$. This defines the vector map

$$\textbf{(b):} \quad z \in \mathrm{re}(A) \;\longmapsto\; t(z),$$

where $\frac{1}{t} \in \sigma(M_{1\,z}) \setminus \{0\}$.

The values $t(z)$ are the finite eigenvalues of the regular pencil $A - zI + tE_1$, for z given in $\mathrm{re}(A)$. There are at most r such t, and at least 1, unless M_z is nilpotent [1]. For $z \in F(A, E) = F(A, E_1)$, let p_z, $0 < p_z \leq r$, be the algebraic multiplicity of $0 \in \sigma(M_{1\,z})$. There are exactly $q_z = r - p_z$ nonzero eigenvalues with $q_z > 0 \iff p_z < r$ and $q_z = 0 \iff p_z = r \iff M_{1\,z}$ is nilpotent. The latter happens for $z \in C(A, E_1)$, the set of critical points for (A, E_1) [1]. Any z in $C(A, E_1)$ has no image by **(b)**: the pencil $A - zI + tE_1$ has no finite eigenvalue.

How is the information given by **(b)** related to the spectral information $\sigma(A_1(t))$ given by **(a)**? We write

$$zI_n - A - tE_1 \;=\; \left(I_n - tE_1(zI_n - A)^{-1}\right)(zI_n - A)$$

for $z \notin \sigma(A)$. z in $\mathrm{re}(A)$ is an eigenvalue of $A_1(t)$ iff t is such that

$$\frac{1}{t} \;\in\; \sigma(E_1(zI_n - A)^{-1}) \setminus \{0\}.$$

And the nonzero elements of the spectra of $U\Gamma^{-1}V^H(zI - A)^{-1}$ and of $M_{1\,z} = V^H(zI_n - A)^{-1}U\Gamma^{-1}$ are identical.

We readily conclude that the map **(b):** $z \in \mathrm{re}(A) \;\longmapsto\; t(z)$ represents the backward (vector) distance to singularity of z in $\mathrm{re}(A)$: $A + t(z)E_1 - zI_n$ is singular, iff $M_{1\,z}$ is not nilpotent. When $M_{1\,z}$ is nilpotent, the map **(b)** is not defined at z, and $(A + tE_1 - zI_n)^{-1}$ is a polynomial in t for this z: it exists for any t in $\mathbb{C}$.

The formulae **(S1)** and **(S2)** give *two* different and complementary descriptions of *Var*: one description is *forward*, the second is *backward*.

A simplification may occur when $G = \Gamma$.

2.5. $G = \Gamma$ is invertible

When $G = \Gamma$, one can factor $G - tM_z$ for $z \in \mathrm{re}(A)$:

$$G - tM_z \;=\; V^H\,(zI_n - A)^{-1}\,[(z - t)I_n - A]\,U.$$

We assume that $\det\Gamma = \det G \neq 0$: this requires that $r_0 = r \leq n$, hence $\mathrm{rank}\ E = \mathrm{rank}\ E_1 = r \leq n$.

Lemma 2.1. *When $G = \Gamma$ is invertible, E_1 is a projection. Moreover, $I_n - E_1$ is the eigenprojection for $E = UV^H$ associated with 0 when $r < n$, or $E_1 = I_n$ for $r = n$.*

Proof. $E_1 = UG^{-1}V^H$ satisfies $E_1^2 = E_1$. By assumption, $0 \in \sigma(E)$ has algebraic and geometric multiplicity $g = n - r$ for $n > r$. □

Lemma 2.2. *When $G = \Gamma$ is invertible, if there exists $\lambda \in \sigma(A)$ with geometric multiplicity g_λ such that $g_\lambda > g$, then, for any $z \in \mathrm{re}(A)$, $t = z - \lambda \neq 0$ satisfies* $\det(G - tM_z) = 0$.

Proof. This follows from the factorization for $G - tM_z$: for $r_0 = r \leq n$,

$$\mathrm{rank}(G - tM_z) \quad \leq \quad \mathrm{rank}((z-t)I_n - A) \quad < \quad r$$

is equivalent to

$$\dim \mathrm{Ker}((z-t)I - A) \quad > \quad n - r.$$

This is, for $z - t = \lambda \in \sigma(A)$, the condition $g_\lambda > g$. □

Proposition 2.3. *When $r_0 = r = n$ and $G = \Gamma$, the two maps* **(a)** *and* **(b)**: $\mathbb{C} \mapsto \mathbb{C}^n$ *reduce to the linear translation $z = \lambda + t$, $\lambda \in \sigma(A)$.*

Proof. When $r = n$, and G is invertible, $E_1 = I_n$ and $A(t) = A + tI$ has the eigenvalues $z(t) = \lambda + t$ for $t \in \mathbb{C}$. □

We introduce the condition on A:

- **(H)**: there exists at least one eigenvalue $\lambda \in \sigma(A)$ such that

$$g_\lambda > g = n - r \geq 1 \quad \text{for} \quad n > r.$$

When **(H)** holds, $g_\lambda \geq 2$ and there exists a component of **(b)** which takes the linear form: $z \in \mathrm{re}(A) \quad \longmapsto \quad t = z - \lambda \neq 0$. The analogue does not necessarily hold for **(a)** when $r < n$. Indeed, $A + (t - z)I$ singular for $t - z = -\lambda$ does not necessarily imply $A + tE_1 - zI$ singular, when $E_1 \neq I$.
The following is immediate:

Lemma 2.4. **(H)** *cannot be satisfied by a nonderogatory matrix.*

The proof is obvious since $g_\lambda = 1$ for a non derogatory A. When $r = 1 < n$, **(H)** can hold only if $g_\lambda = n$ hence $A = \lambda I_n$. More precisely, one has the

Lemma 2.5. *When $A = \lambda I_n$, the matrix $A_1(t) - zI$ (resp. $G - tM_z$) is singular (resp. zero) for $t = z - \lambda \neq 0$.*

Proof. From

$$I - t(zI - A)^{-1} \;=\; I - \frac{t}{z-\lambda} I \;=\; 0$$

for $t = z - \lambda \neq 0$, we conclude that $G - tM_z = 0$, whether $\det G \neq 0$ or not. When $\det G \neq 0$, $A_1(t) - zI$ is singular (resp. 0) for $z = t + \lambda \neq \lambda$ (resp. $z = \lambda$) by **(S1)**. □

2.6. $\det \Gamma = 0$ and $\Gamma \neq 0$

When $\Gamma \neq 0$ is not invertible, E_1 does not exist and computation by **(S1)** is impossible. However computation is still possible by means of **(S2)**, that is, by solving $\det(\Gamma - tM_z) = 0$ for any z in $\mathrm{re}(A)$, when the pencil $\Gamma - tM_z$ is not singular [10], see Appendix. $\det \Gamma = 0$ and $\det M_z = 0$ are necessary conditions under which $\det(\Gamma - tM_z) \equiv 0$ for z given in F, that is, the pencil is singular.

When F is discrete, then for $z \in \mathrm{re}(A) \setminus F$, **(S2)** leads to

$$\textbf{(c)}: \quad z \;\longmapsto\; t \in \sigma(\Gamma M_z^{-1}) \in \mathbb{C}^r.$$

(c) would be equivalent to **(b)** if Γ were invertible. But now that Γ is not invertible, there is no obvious interpretation for **(c)** in terms of A and E. Observe that $0 \in \sigma(\Gamma M_z^{-1})$.

What are the modifications provided by the particular case $G = \Gamma$ singular?

1. Firstly, the pencil $G - tM_z$ is singular when $1 \leq r_0 < r$ or $1 \leq n < r$. This requires $r \geq 2$.

2. Second, for $1 \leq r_0 = r < n$, **(H)** implies the linearisation of at least two (identical) components of **(c)**: $z \in \mathrm{re}(A) \;\longmapsto\; t = z - \lambda$, with $g_\lambda \geq 2$ since $n > r$.

Example 2.1. For $r = 1 < n$, $g_\lambda = n$ implies $A = \lambda I_n$. Therefore, when $\gamma = v^H u$,

$$\det \hat{A}(z,t) \;=\; (z-\lambda)^{n-1} v^H u (z - \lambda - t) \;=\; 0$$

yields either $z = \lambda$, t arbitrary, or $z = \lambda + t$ if $v^H u \neq 0$, and z, t arbitrary in $\mathbb{C}$ if $v^H u = 0$. △

This analysis shows that the complex dimension of the manifold *Var* in $\mathbb{C}^2$ takes the generic value 1. The value 2 is possible non generically, as illustrated by Example 2.1.

2.7. $\Gamma = 0$

When $\Gamma = 0$,

$$\det(-tM_z) \;=\; (-t)^r \det M_z.$$

We set

$$\hat{\pi}(z) \;=\; \Delta_0(z,-1) \;=\; \det \hat{A}_0(z,-1)$$

for $\Gamma = 0$, that is

$$\hat{A}_0(z,-1) \;=\; \left(\begin{array}{c|c} zI - A & -U \\ \hline V^H & 0 \end{array}\right) \;=\; \hat{A}(z).$$

For $z \in \mathrm{re}(A)$,

$$\hat{\pi}(z) \;=\; \pi(z) \det M_z,$$

by **(S2)** for $t = -1$, and

$$\Delta(z,t) \;=\; (-t)^r \,\hat{\pi}(z)$$

for any z in $\mathbb{C}$.

Corollary 2.6.

$$\det Q(z) \;=\; (\pi(z))^{r-1}\,\hat{\pi}(z)$$

with

$$\hat{\pi}(z) \;=\; \det \hat{A}(z),$$

and

$$\hat{A}(z) \;=\; \begin{pmatrix} zI - A & -U \\ V^H & 0 \end{pmatrix}$$

of order $n + r$.

Proof. For $y \in \mathrm{re}(A)$, one has the relations:

$$\begin{aligned} \det Q(z) &= (\pi(z))^r \det M_z, \\ \hat{\pi}(z) &= \pi(z) \det M_z. \end{aligned}$$

Therefore

$$\det Q(z) \quad = \quad (\pi(z))^{r-1}\,\hat{\pi}(z)$$

holds for $z \in \text{re}(A)$. The relation extends to $z \in \mathbb{C}$ by continuity of polynomials. □

It is remarkable that the influence of U and V on det $Q(z)$ is limited, for $r \geq 2$, to the factor $\hat{\pi}(z)$.

We recall that $G = V^H U$.

Corollary 2.7. *The following equivalences hold:*

i) $\det Q \not\equiv 0 \iff \hat{\pi}(z) \not\equiv 0,$

ii) $\deg\det Q(z) = (n-1)r \iff \deg\det\hat{\pi}(z) = n - r \iff \det G \neq 0.$

Proof.

i) Clear by Corollary 2.6.

ii)

$$Q(z) \quad = \quad V^H \text{adj}(zI - A)\,U \quad = \quad z^{n-1}\,V^H U + \ldots$$

Hence

$$\det Q(z) \quad = \quad z^{(n-1)r}\det V^H U + \ldots$$

And

$$(n-1)r \quad = \quad n(r-1) + n - r.$$

□

2.8. Summary

Of the two Schur computing formulas, the second one **(S2)** has a larger domain of validity, but gives less information.

1. When $\det\Gamma \neq 0$, **(S2)** gives only the finite eigenvalues of the regular pencil $A - zI + tE_1$, for z in re(A). Whereas **(S1)** gives the complete spectrum of the matrix $A + tE_1$, for $t \in \mathbb{C}$.

2. When $\det \Gamma = 0$ with $\Gamma \neq 0$, **(S1)** cannot be used, but **(S2)** gives the finite eigenvalues of the pencil $\Gamma - tM_z$, which is certainly regular for z in $\text{re}(A) \setminus F(A, E)$.

   We leave for a future paper the analysis of the case when $\Gamma - tM_z$ is singular, which may happen for $z \in F(A, E)$.

3. When $\Gamma = 0$, $\Delta(z, t) = (-1)^r \hat{\pi}(z)$ for $z, t \in \mathbb{C}$.

We continue our investigation of the role of the second formula **(S2)** in the next Section.

3. The role of (S2) in the computation of the zeros of $\Delta(z, t)$

3.1. $\det \Gamma \neq 0$

The manifold V is described directly by **(a)** which gives the n spectral rays contained in $\sigma(A_1(t))$, $t \in \mathbb{C}$. For $z \in \text{re}(A) \setminus F$, **(b)** provides a restriction of **(a)** to the values of z not in $\sigma(A)$ such that $\det M_z \neq 0$ and $t \neq 0$ in $\mathbb{C}$.

When $z \in F$, the pencil $\Gamma - tM_z$ has at least one infinite eigenvalue that is not accessible, since **(S2)** gives only the finite eigenvalues. If moreover $z \in C(A, E_1) \neq \emptyset$ is a critical point, $\Gamma - tM_z$ has *no* finite eigenvalue. When card $C(A, E_1) \geq n$, a direct approach shows that $\sigma(A_1(t)) = \sigma(A)$ for any t: the n spectral rays reduce to the n invariant eigenvalues in $\sigma(A)$. Such information cannot be delivered by **(b)**.

This discussion shows that, even in the most favorable case where $\det \Gamma \neq 0$, the backward information given by **(b)** is *incomplete,* when compared with the forward formulation given by **(a)**.

3.2. $\det \Gamma = 0$

To assess the information given by **(S2)**, one should compare it directly to the zeros of $\Delta(z, t)$ since computation **(S1)** is now impossible.

If one has no access to this direct information, one *cannot* assess the validity of the information delivered by **(S2)**, unless $\Gamma = 0$.

Let us suppose that $(z_0, t_0) \in \mathrm{re}(A) \times \mathbb{C}$ satisfies $\det \hat{A}(z_0, t_0) = 0$. If $z_0 \notin F(A, E)$, then $t_0 \in \sigma(\Gamma M_{z_0}^{-1})$, and there exists $u \neq 0$ in $\mathbb{C}^r$ such that $\Gamma u = t_0 M_{z_0} u \neq 0$. If $z_0 \in F(A, E)$, then $\Gamma u = t M_{z_0} u = 0$ for any $t \in \mathbb{C}$ is possible iff $u \in \operatorname{Ker} \Gamma \cap \operatorname{Ker} M_{z_0}$. Then the pencil $\Gamma - t M_{z_0}$ is singular: $\det(\Gamma - t M_{z_0}) \equiv 0$ for $t \in \mathbb{C}$ and (z_0, t) satisfies **(S2)** for any t. However, as we know, $(z_0, 0)$ cannot be a root of $\det \hat{A}(z, t)$ when $z_0 \notin \sigma(A)$.

This shows that the information delivered by **(S2)** may also not be reliable when $\det \Gamma = 0$, if, for example, the pencil $\Gamma - t M_z$ is singular. The case $\Gamma = 0$ is an exception: a factorization occurs, and $\Delta(z, t) = (-t)^r \hat{\pi}(z)$ (see Proposition 4.3).

4. The rational forms associated with $\det(\Gamma - tM_z)$

We consider the following two ways to associate a rational form with

$$\det(\Gamma - tM_z) \;=\; \Theta(z, t)$$

when $\Theta \not\equiv 0$. We assume that $0 \leq \operatorname{rank} \Gamma \leq r$.

1. For $z \in \mathrm{re}(A)$,

   $$\det(\Gamma - tM_z) \;=\; \frac{1}{\pi^r} \det(\pi\Gamma - tQ(z)).$$

   Then **(S2)** yields for $z \in \mathrm{re}(A)$:

   $$\Delta \;=\; \Delta(z, t) \;=\; \pi \det(\Gamma - tM_z) \;=\; \frac{1}{\pi^{r-1}} \det(\pi\Gamma - tQ).$$

   We write

   $$\mathcal{R}_1 \;=\; \frac{\mathcal{N}_1}{\mathcal{D}_1}$$

   with $\mathcal{N}_1 = \det(\pi\Gamma - tQ)$ and $\mathcal{D}_1 = \pi^{r-1}$.

2. We suppose that $r_0 = r \leq n$ and $F(A, E)$ is discrete, therefore $\det Q \not\equiv 0$. For $z \in \mathrm{re}(A) \setminus F(A, E)$

   $$\det(\Gamma - tM_z) \;=\; \det M_z \, \det(\Gamma M_z^{-1} - tI_r)$$

and

$$\Delta \;=\; \pi \det(\Gamma - tM_z) \;=\; \frac{1}{(\pi \det Q)^{r-1}} \det(\pi\Gamma \operatorname{adj} Q - t \det Q\, I_r).$$

We set $\mathcal{N}_2 = \det P$ with

$$P(z,t) \;=\; \pi(z)\,\Gamma \operatorname{adj} Q(z) \;-\; t \det Q(z)\, I_r,$$

$\mathcal{D}_2 = (\pi \det Q)^{r-1}$ and $\mathcal{R}_2 = \mathcal{N}_2/\mathcal{D}_2$.

The equalities $\Delta = \mathcal{R}_1$ or $\mathcal{R}_2$ are conditional to $z \in \operatorname{re}(A)$, or $z \in \operatorname{re}(A) \setminus F$. It is not evident that they should hold for all z in $\mathbb{C}$.

We look at this question by considering first the particular cases $r \geq n = 1$ or $n \geq r = 1$, which can be treated by a direct approach.

4.1. The special cases $r \geq n = 1$, or $n \geq r = 1$

The first case ($n = 1$) corresponds to A being the scalar a in $\mathbb{C}$, then $(zI - A)^{-1}$ becomes $\frac{1}{z-a}$ for $z \neq a$, and

$$M_z \;=\; \frac{1}{z-a} V^H U$$

is a rank 1 matrix of order r, with $U \in \mathbb{C}^{1\times r}$ and $V^H \in \mathbb{C}^r = \mathbb{C}^{r\times 1}$ being $\neq 0$.

$Q(z) \equiv V^H U$ is a *constant* polynomial of order r and rank 1. There are only two possibilities, depending on r, for $Z(\det Q)$:

1. either the scalar $E = UV^H = 0$ and $Z(\det Q) = \mathbb{C}$ for $r \geq 2$,
2. or $E = UV^H \neq 0$ and $Z(\det Q) = \mathbb{C}$ (resp. $\emptyset$) for $r \geq 2$ (resp. $r = 1$).

When $n = 1 \leq r$,

$$\det \hat{A}(z,t) \;=\; \det \begin{pmatrix} z-a & tU \\ V^H & \Gamma \end{pmatrix}$$

can be computed directly for any $z \in \mathbb{C}$ as $(z-a)\det\Gamma \;-\; tU \operatorname{adj}\Gamma\, V^H$ [5, p. 65]. The term $(a-z)\det\Gamma$ cancels itself when $\det\Gamma = 0$.

1. For $r \geq 2$ and $\det \Gamma \neq 0$, $z = a + tU\Gamma^{-1}V^H$. When $\det \Gamma = 0$,

$$\det \hat{A}(z,t) \;\; = \;\; t\, U\, (\mathrm{adj}\, \Gamma)\, V^H$$

   is a scalar independent of z. The solutions are either $\mathbb{C}$ (for $U(\mathrm{adj}\,\Gamma)V^H = 0$) or $\emptyset$ (for $U(\mathrm{adj}\,\Gamma)V^H \neq 0$), when $t \neq 0$. When $t = 0$, then

$$\det \hat{A}(z,0) \;\; \equiv \;\; 0.$$

2. For $r = 1$,

$$\hat{A}(z,t) \;\; = \;\; \begin{pmatrix} z-a & tu \\ v & \gamma \end{pmatrix}$$

   is 2×2, where U, V are now the scalars $u, v \in \mathbb{C}$ respectively. By assumption u and v are nonzero. Therefore

$$\det \hat{A}(z,t) \;\; = \;\; (z-a)\gamma \, - \, tuv \;\; = \;\; 0$$

   yields $t = \frac{\gamma}{uv}(z-a)$. When $\gamma = uv \neq 0,\; t = z - a$.

The second case ($r = 1$) is treated similarly: for $r = 1 \leq n$ we get

$$\hat{A}(z,t) \;\; = \;\; \begin{pmatrix} zI_n - A & tU \\ V^H & \gamma \end{pmatrix}$$

where $U, V \in \mathbb{C}^{n\times 1} = \mathbb{C}^n$, and $V^H U$ is a scalar. Therefore

$$\det \hat{A}(z,t) \;\; = \;\; \gamma\pi(z) \, - \, tQ(z).$$

For $\gamma = 0$,

$$\det \hat{A}(z,t) \;\; = \;\; -t\, V^H \,\mathrm{adj}\,(zI - A)\, U \;\; = \;\; -t\, Q(z)$$

where $Q(z)$ is a scalar polynomial ($r = 1$) of degree at most $n-1$: it has at most $n-1$ roots for $n \geq 2$ and none for $n = 1$, assuming that U and V are nonzero scalars. When $\gamma = V^H U = 0$, $Q(z)$ of degree $\leq n-2$ has no root in z if $n = 2$ and $V^H \mathrm{adj} A\, U \neq 0$. This gives $Var = \mathbb{C} \times \{0\}$ of dimension 1. Moreover, if $A = \lambda I_n$, we get $Var = \mathbb{C}^2$ of dimension 2. See Example 2.1.

4.2. An application to Krylov methods

The case $r = 1$ is computationally significant since rank 1 deviation matrices of the type $E = UV^H$, $U, V \in \mathbb{C}^{n\times 1} = \mathbb{C}^n$ are ubiquitous in numerical algorithms.

As an example we reexamine now Krylov methods, as they are presented in [2, 7]. The reader is referred to [7] for a presentation of the incomplete Arnoldi method in the framework of simple HD. We keep the notation of [7]. The deviation matrix E has rank $r = 1$, it is nilpotent so that $E^2 = 0$. It has a unique Jordan block of size 2, the others being trivial Jordan blocks.

Proposition 5.2 [7] specifies $\mathrm{Lim} = \sigma(\Omega)$ under the Lidskii assumption that $a_{n-1\,n}$ (which plays the role of Γ of order 1) is $\neq 0$. Ω is the Schur complement of $a_{n-1\,n}$ of order $n-2$, and $l_* = \mathrm{card\ Lim} = n-2$. We left open, in [7], the characterisation of Lim when $a_{n-1\,n} = 0$. We assume that $n \geq 3$.

The sizes $r = 1, n$ should be replaced respectively by $f = 1$, $g' = n-2$ [7]. The generic case $a_{n-1\,n} \neq 0$ yields $l_* = g'$ as expected: two eigenvalues escape to ∞. When $a_{n-1\,n} = 0$, Lim consists of the zeros of the scalar polynomial $v^T \operatorname{adj}(zI_{g'} - A_{n-2})\, u$ of degree at most $g' - 1$ (notation of [7], p. 11). We get $l_* = g' - 1 = n - 3$ (resp. $l_* \leq g' - 2 = n - 4$) when $v^T u \neq 0$ (resp. $= 0$), with $v^T u = a_{n-1\,n-2}\; a_{n-2\,n}$ when A is in upper Hessenberg form, hence $A \neq \lambda I$. At least three eigenvalues escape to ∞ (with orders < 1 for $\nu(s)$).

Numerical experiments are performed by M. Ahmadnasab to examine how the transition from 2 to ≥ 3 eigenvalues escaping to ∞ occurs in finite precision as $a_{n-1\,n} \to 0$.

This is easily recast in the context of the Lidskii process with 2 steps $j = 1, 2$. The first step $j = 1$ is non generic with $\det \phi_1 = a_{n-1\,n} = 0$.

1. At least one of the $g' = n-2$ eigenvalues $\nu(s)$, which generically should converge with order 1, actually does converge with order < 1.

2. No eigenvalue converges with order 1 ($l_* = 0$) when $n = 3$ (resp. $n \leq 4$) and $v^T u \neq 0$ (resp. $= 0$).

4.3. The rational forms $\mathcal{R}_i$ when $r, n \geq 2$

4.3.1. $\mathcal{R}_1 = \mathcal{N}_1/\mathcal{D}_1$

We recall that

$$\mathcal{N}_1 \quad = \quad \det(\pi\Gamma - tQ) \quad = \quad (-t)^r \det Q + \cdots + \pi^r \det \Gamma,$$

and $\mathcal{D}_1 = \pi^{r-1}$. $\deg \det Q \leq (n-1)r$. In what follows, deg denotes the degree in z for polynomials in z and t.

Lemma 4.1. *When* $\det \Gamma \neq 0$, $\deg \mathcal{N}_1 = nr$ *and* $\deg \mathcal{R}_1 = \deg \mathcal{N}_1 - \deg \mathcal{D}_1 = n$.

Proposition 4.2. *When* $\det \Gamma \neq 0$*, the rational form* $\mathcal{R}_1$ *reduces to the polynomial* $\Delta(z,t) = \Delta$.

Proof. For $z \in \mathrm{re}(A)$, $t \neq 0$, $\mathcal{R}_1(z,t) = \Delta(z,t)$. For $t = 0$, $\mathcal{R}_1(z,0) = \pi \det \Gamma = \Delta(z,0)$ for any $z \in \mathbb{C}$. The conclusion follows by continuity. $\square$

The situation is more complex when $\det \Gamma = 0$. We know from Section 3 that the reduction of $\mathcal{R}_1$ to Δ need not take place. We defer to a future report a more complete analysis of the case $\Gamma \neq 0$.

The case $\Gamma = 0$ is easily treated by the

Proposition 4.3. *When* $\Gamma = 0$*, the rational form* $\mathcal{R}_1$ *reduces to the factored polynomial* $\Delta(z,t) = (-t)^r \hat{\pi}(z)$.

Proof. For $\Gamma = 0$, $\mathcal{N}_1 = (-t)^r \det Q$. Hence

$$\frac{\mathcal{N}_1}{\mathcal{D}_1} \quad = \quad \mathcal{R}_1 \quad = \quad (-t)^r \hat{\pi}(z) \quad = \quad \Delta(z,t).$$

$\square$

We observe that the two situations where the polynomial reduction is easy to establish are the two extreme cases: either $\det \Gamma \neq 0$, or $\Gamma = 0$.

4.3.2. $\mathcal{R}_2 = \mathcal{N}_2/\mathcal{D}_2$

$\mathcal{D}_2 = (\pi \det Q)^{r-1} = (\pi^r \hat{\pi})^{r-1}$ has a degree which depends not only on A, but on U and V as well.

Lemma 4.4. *The leading coefficient in* $Q(z)$ *(resp.* $\det Q(z)$*) is* $V^H U$ *(resp.* $\det V^H U$*).*

Proof. Simple calculation (see Corollary 2.7).

$Q(z)$ is a matrix polynomial of degree $n-1$ iff $V^H U \neq 0$, that is $E^2 \neq 0$, $E = UV^H$. $\det Q(z)$ is a scalar polynomial of degree $r(n-1)$ iff $\det V^H U \neq 0$. This is condition (Σ): $0 \in \sigma(E)$ is semi-simple. □

Lemma 4.5. *The leading coefficient in* $P(z, 1)$ *is*

$$C \quad = \quad \Gamma \operatorname{adj}\left(V^H U\right)$$

iff $V^H U \neq 0$.

Proof. $\operatorname{adj}\left(V^H U\right) \neq 0 \iff V^H U \neq 0$.

The leading coefficient for $\pi(z)\,\Gamma \operatorname{adj} Q(z)$ is $\Gamma \operatorname{adj}\left(V^H U\right)$ when C is nonzero. When $V^H U$ and Γ are $\neq 0$,

$$\deg P \quad = \quad n + (n-1)(r-1) \quad = \quad r(n-1) + 1 \quad \geq \quad \deg \det Q + 1.$$

When $V^H U = 0$, $\Gamma \neq 0$,

$$\deg P \quad \leq \quad n \;+\; (n-2)(r-1).$$

□

Lemma 4.6. $\det P(z,1)$ *is a scalar polynomial of degree* $r^2(n-1) + r$ *when* $\det \Gamma \det G \neq 0$.

Proof. By Lemma 4.5, the coefficient of $z^{r^2(n-1)+r}$ in $\det P(z, 1)$ is

$$\det\left(\Gamma \operatorname{adj}\left(V^H U\right)\right) \quad = \quad \det \Gamma (\det G)^{r-1}.$$

The conclusion follows. □

One writes

$$\begin{aligned} \mathcal{N}_2(z,t) \quad &= \quad \det(\pi \Gamma \operatorname{adj} Q - t \det Q\, I) \\ &= \quad (-t)^r\, (\det Q)^r + \cdots + \pi^r \det \Gamma\, (\det Q)^{r-1} \end{aligned}$$

as a polynomial in t with coefficients depending on z.

Lemma 4.7. *$z_0 \in Z(\det Q)$ is a necessary condition for $\mathcal{N}_2(z_0, t)$ to be identically 0 in t.*

Proof. Clear from the expansion above. □

Lemmas 4.6 and 4.7 show that an analogue of Proposition 4.2 need not hold for $\mathcal{R}_2$ when $\det \Gamma \neq 0$ alone.

Proposition 4.8. *If $\det \Gamma \det G \neq 0$, the rational form $\mathcal{R}_2$ reduces to Δ.*

Proof. When $\det G \neq 0$, $Z(\det Q)$ is finite. When $\det \Gamma \neq 0$, for $z \in \mathrm{re}(A) \backslash F$, $t \neq 0$, $\mathcal{R}_2(z,t) = \Delta(z,t)$. For $t = 0$, $\mathcal{N}_2(z,0) = \pi^r \det \Gamma \, (\det Q)^{r-1}$ hence $\mathcal{R}_2(z,0) = \pi \det \Gamma = \Delta(z,0)$ for any $z \in \mathbb{C}$. The conclusion follows by continuity. □

A more complete analysis is beyond reach at the moment when $\Gamma \neq 0$.

Proposition 4.9. *When $\Gamma = 0$, the rational form $\mathcal{R}_2$ reduces to $\Delta = (-t)^r \hat{\pi}(z)$.*

Proof. Easy calculation. □

5. Application to Lidskii's algorithm

We go back to the question asked in Section 1: How can we compute the coefficients ξ of the eigenvalues converging with order $1/n_{j+1}$, when $\det \phi_j = 0$, hence Ω_{j+1} does not exist?

The reasoning of Lidskii is based on the existence of the polynomial form for

$$\det \phi_{j+1}(z) \quad = \quad \det \phi_j \, \cdot \, \det \left(\Omega_{j+1} - z I_{r_{j+1}}\right)$$

with

$$\Omega_{j+1} \quad = \quad \Delta_{j+1} \; - \; L_{j+1} \, \phi_j^{-1} \, R_{j+1},$$

and

$$z \quad = \quad \xi^{n_{j+1}}, \qquad \xi \, \in \, \sigma(\Omega_{j+1}).$$

When $\det \phi_j = 0$, the reasoning breaks down. The difficulty can be analysed, but not resolved, by augmented HD. There are fewer solutions z for $\det \phi_{j+1}(z) = 0$ than the generic value r_{j+1}, which holds when $\det \phi_j \neq 0$.

We drop the subscript $j+1$ and use the results of Sections 2 and 3, together with the change of notation:

$$\Downarrow \quad \begin{array}{ccccccc} r & n & A & U & V^H & \Gamma & \det \hat{A}(z,1) \\ \hline f & r & \Delta & L & R & \Gamma & \det \phi(z). \end{array}$$

We first review the special cases $f = 1 \leq r$, or $r = 1 \leq f$ at step $j+1$, when step j is non generic ($\det \phi_j = 0$).

5.1. The special cases at step $j+1$ when step j is non generic

$f = 1$ is possible for $j = 1$ only: this happens if there is a unique Jordan block of maximal size n_1 (see Section 3.4). There are always fewer solutions than r, the generic value.

$r = 1$ is satisfied at step $j+1$ whenever there is a *unique* block of order n_{j+1}. For $f \geq 2$, the number of zeros for $\det \phi(z) = 0$ is ∞ or 0, depending on whether the scalar product $L(\operatorname{adj} \Gamma)R$ is zero or not.

5.2. r and $f \geq 2$ at step $j+1$

$r \geq 2$ requires that there be *several* Jordan blocks of order n_{j+1}. At least n_{j+1} eigenvalues (out of $n_{j+1}\, r_{j+1}$) remain unclassified at step $j+1$.

5.3. Consequence for the Lidskii algorithm

We have seen in Part I [3] that, when the Lidskii process hits the first singular step j for $1 \leq j < q$, then only $n_j(r_j - \omega_j)$ eigenvalues are classified at step j, with ω_j = the algebraic multiplicity of $0 \in \sigma(\Omega_j)$. Classification at step $j+1$ is also difficult for the Lidskii process because there are *too few* solutions. We have access only to at most $r_{j+1} - 1$ solutions $z = \xi^{n_{j+1}}$ (instead of r_{j+1}, the generic value). This means that $n_{j+1}\delta_{j+1}$ eigenvalues converge with unknown orders smaller than $1/n_j$.

In summary, when step j is singular we are unable to classify $n_j\, \omega_j$ eigenvalues at step j and $n_{j+1}\delta_{j+1}$ eigenvalues at step $j+1$. The two question marks displayed on Figure 2 of [3] for the Lidskii algorithm cannot be removed by a direct computation of $\det \phi_{j+1}(z)$.

When the step j is generic, then $\det \phi_{j+1}(z)$ has a polynomial form in z, and the classification is straightforward. But when step j is *non generic,* $\det \phi_{j+1}(z)$ evaluated by the only possible Schur formula can be written as a *rational fraction.* This leads to an incomplete classification. This may also lead to computational difficulties when *finite precision arithmetic* is used.

6. Conclusion

Our analysis of the algorithmic consequences of the event $\det \phi_j = 0$ points to a new difficulty arising from the rational form for $\det \phi_{j+1}(z)$. The use of the dual Schur complement of order f_j yields either too few (exact arithmetic) or too many (inexact arithmetic) solutions for $\det \phi_{j+1}(z) = 0$, instead of the generic value r_{j+1}. One way to progress in this uncharted territory has been to explore numerically the case in which Γ is invertible. In this situation, we have the two Schur complements of order r_{j+1} and f_j at our disposal simultaneously. It is therefore possible to compare the solutions for $\det \phi_{j+1}(z) \neq 0$ given by each Schur complement, all the more that there are *two* rational forms (one based on ΓW_z^{-1}, and the other on $\Gamma^{-1} W_z$ associated with the dual Schur complement of order f_j) [8].

Appendix

A1. Simple Homotopic Deviation [1, 4]

Let $E \in \mathbb{C}^{n\times n}$ be given of rank $r < n$. By SVD, it can always be written as $E = UV^H$, where $U, V \in \mathbb{C}^{n\times r}$ have rank r. 0 is an eigenvalue of E with algebraic (resp. geometric) multiplicity m (resp. $g = n - r \leq m$). (Σ) is the condition that $0 \in \sigma(E)$ is semi-simple, that is, $m = g = n - r$.

We consider the pencil $A - zI + tE$ for z given in $\text{re}(A)$: it is a *regular* pencil, its determinant is nonzero for $t = 0$.

What is the structure of the pencil as z varies in $\text{re}(A)$?

Lemma A1. *For $z \in \mathrm{re}(A) \setminus F(A,E)$, the pencil $A - zI + tE$ is strictly equivalent to the 2×2 block matrix*

$$\left(\begin{array}{ccc|ccc} 1 & t\varepsilon & 0 & & & \\ & \ddots & \ddots & & & \\ & & 1 & & & \\ \hline & & & \frac{1}{\mu_{1\,z}} & \varepsilon & 0 \\ & & & & \ddots & \ddots \\ & & & & & \frac{1}{\mu_{r\,z}} \end{array}\right)$$

corresponding to the partition $n = g + r$, where ε represents 0 or 1 and $\mu_{i\,z} \in \sigma(M_z)$, $i = 1, \ldots, r$, $t \in \mathbb{C}$.

Proof. For any z such that M_z is invertible, $\det(A - zI + tE) = 0$ iff $t \in \sigma(M_z^{-1})$. □

The structure of the pencil is given by the partition $n = g + r$ for any $z \notin F(A,E)$: it depends only on $r = \mathrm{rank}\ E$ and *not* on A. This is the generic situation when F is a discrete set. When z is close to being a frontier point, then some of the finite eigenvalues of the pencil are large.

When $z \in F(A,E)$ there is an abrupt change. Let p_z, $1 \leq p_z \leq r$ be the algebraic multiplicity of $0 \in \sigma(M_z)$, for $z \in F(A,E)$.

Lemma A2. *For $z \in F(A,E)$, the structure of the pencil $A - zI + tE$ is determined by the partition $n = (g + p_z) + (r - p_z)$ which depends on z in $F(A,E)$.*

For example, if the critical set $C(A,E)$ is not empty and $z \in C(A,E)$, then $p_z = r$ and the pencil has *no* finite eigenvalues.

When the frontier set is discrete, there are at most $(n-1)r$ frontier points z in $\mathrm{re}(A)$ where the *structure* of the pencil depends on A. In particular, the number of finite eigenvalues $r - p_z$ for the regular pencil is always smaller then the generic value r, the rank of E

These frontier points signal a tight algebraic coupling between A and E such that $R(\infty, z)$ does not exist (there is no analyticity at ∞).

A2. Limit and frontier points in re(A)

As remarked in [4], not all the frontier points are realized as $\lim_{|t|\to\infty}\lambda(t)$. In general, only the inclusion Lim $\subset Z(\det Q)$ is valid in $\mathbb{C}$. Equality holds for $r = 1$ and $g = n-1 = m$. That is, under (Σ) with $r = 1$, Lim $= \sigma(\Pi) = Z(\det Q)$ [1]. For $n > r \geq 2$, the strict inclusion holds under (Σ), with

$$\frac{1}{r^2} \quad \leq \quad \frac{\text{card Lim}}{\text{card } Z(\det Q)} \quad = \quad \frac{1}{r}\,\frac{n-r}{n-1} \quad < \quad \frac{1}{r}.$$

$\xi \in$ Lim iff $\nu = 0$ and $\xi = \nu'(0)$ is finite. This direct analytic interpretation: $\nu(s) \longrightarrow 0$ with order ≥ 1 for $\nu(s) \in E(s) = E + sA$ is not possible for the frontier points z which are not in Lim. They are characterized by the algebraic condition $0 \in \sigma(M_z)$ for $z \in \text{re}(A)$.

The factorization

$$\det Q(z) \quad = \quad (\pi(z))^{r-1}\,\hat{\pi}(z),$$

proved in Corollary 2.6, shows that the only zeros of $\det Q$ which may not be eigenvalues of A are among the roots of $\hat{\pi}(z) = 0$. For $\hat{\pi} \not\equiv 0$ we set $\hat{Z} = Z(\hat{\pi}(z))$. Hence $Z = Z(\det Q) = (\sigma(A))^{r-1} \cup \hat{Z}$ and $F(A, E) = \hat{Z} \cap \text{re}(A)$, with card $F \leq n - r$. And card $\hat{Z} = n - r \iff (\Sigma)$ (Corollary 2.7).

The matrix

$$\tilde{\Pi}(z) \quad = \quad \begin{pmatrix} \Gamma & R \\ L & \Pi' - zI_{g'} \end{pmatrix}$$

of order $g = f + g'$ has been defined in [1, 3]. By Lidskii's theory, $\tilde{Z} = Z(\det \tilde{\Pi}(z)) \subset$ Lim. When $\det \Gamma \neq 0$, $\tilde{Z} = \sigma(\Omega)$ where $\Omega = \Pi' - L\Gamma^{-1}R$ is the Schur complement of Γ in $\tilde{\Pi}(0)$.

Moreover, Lim $= \sigma(\Omega)$ under the condition that certain quantities are nonzero: $\det \Gamma$, and also certain specific principal minors of Γ, deriving from the Jordan structure of the defective eigenvalue $0 \in \sigma(E)$ [3].

Proposition A3.

i) *When $\hat{\pi} \not\equiv 0$, the following inclusions hold*

$$\tilde{Z} \quad \subset \quad \text{Lim} \quad \subset \quad (\sigma(A))^{r-1} \cup \hat{Z}.$$

ii) *When $\hat{\pi} \equiv 0$, then $\hat{Z} = \mathbb{C}$.*

Proof. See [3]. □

We observe readily that $\hat{\pi}(z)$ is easily defined, whereas $\tilde{\Pi}(z)$ and Ω entangle E and A nonlinearly [1].

Proposition A4. *For* $z \in re(A) \setminus F(A, E)$

$$M_z^{-1} \quad = \quad \frac{1}{(\pi(z))^{r-2}\,\hat{\pi}(z)}\,\mathrm{adj}\,Q(z).$$

Proof. We set

$$\alpha(z) \quad = \quad \frac{\pi(z)}{\det Q(z)} \quad = \quad \frac{1}{(\pi(z))^{r-2}\,\hat{\pi}(z)}$$

by Corollary 2.6. And

$$M_z^{-1} \quad = \quad \alpha(z)\,\mathrm{adj}\,Q(z).$$

Observe that for $\lambda \in \sigma(A)$, $\pi(\lambda) = 0$ and $\alpha(\lambda)$ is not defined for $r > 2$.

On the other hand, if $\hat{\pi}(\lambda) \neq 0$, then $\alpha(\lambda) = 0$ (resp. $\frac{1}{\hat{\pi}(\lambda)}$) for $r = 1$ (resp. 2). □

Remark. In view of Proposition A4, the discussion of [4, Section 5.2] should be modified.

Various illustrations are provided in [4], with $r = 2$.

A3. Singular square pencils [9, 10]

A square pencil $A + \lambda B$ is singular iff $\det(A + \lambda B) \equiv 0$ for any $\lambda \in \mathbb{C}$. The structure of singular pencils has been established by Kronecker ([10], Chapter 12 in Volume II), following the work of Weierstrass and Jordan on regular pencils ([10], Chapter 6 in Volume I).

A necessary condition for $A + \lambda B$ to be singular is that both A and B are singular. If A and B have a common null space (Ker $A \cap$ Ker $B \neq \{0\}$) then the generalized eigenproblem is *ill-posed*: the solutions are not continuous functions of the data. Such degenerate singular pencils represent a formidable computational challenge [9].

References

[1] F. Chaitin-Chatelin (2003), *Computing beyond analyticity. Matrix algorithms in Inexact and Uncertain Computing,* Cerfacs Technical Report TR/PA/03/110.

[2] F. Chaitin-Chatelin (2005), *The dynamics of matrix coupling with an application to Krylov methods,* in Z. Li et al. (eds.), *NAA 2004,* Lecture Notes in C.S. **3401**, pp. 14–25, Springer Verlag, Berlin; also available as Cerfacs Technical Report TR/PA/04/29 (2004).

[3] F. Chaitin-Chatelin (2004), *On Lidskii's algorithm to quantify the first order terms in the asymptotics of a defective eigenvalue. Part I,* Cerfacs Technical Report TR/PA/04/129.

[4] M. Ahmadnasab, F. Chaitin-Chatelin, N. Megrez (2005), *Homotopic Deviation in the light of Algebra,* Cerfacs Technical Report TR/PA/05/05.

[5] P. Lancaster, M. Tismenetsky (1997), **Theory of Matrices,** Academic Press.

[6] J. Moro, J. V. Burke, M. L. Overton (1997), *On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure,* SIAM J. on Matrix Analysis and Appl. **418:4**, pp. 793–817.

[7] F. Chaitin-Chatelin (2003), *The Arnoldi method in the light of Homotopic Deviation theory,* Cerfacs Technical Report TR/PA/03/15.

[8] F. Chaitin-Chatelin (2005), *Augmented Homotopic Deviation,* Working Notes, January 2005.

[9] J. Demmel, B. Kågström (1993), *The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: robust software with errors bounds and applications. Part I: Theory and Algorithms,* ACM Trans. Math. Softw. **19**, 160–174.

[10] F. Gantmacher (1960), **The Theory of Matrices,** Chelsea Publishing Co., New York.

The Cerfacs reports are available from http://aton.cerfacs.fr/algor/reports/