Nonnormality Estimation in Projection-Type System Realization Methods

F. S. V. BAZÁN *
Department of Mathematics,
Federal University of Santa Catarina,
Florianópolis, Santa Catarina
88040-900, Brazil
Email: fermin@mtm.ufsc.br

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Abstract

In this paper we investigate the influence of nonnormality on the spectral sensitivity of certain family of projection-type system realization methods based on estimation of Henrici's departure from normality. An immediate conclusion is that the system matrix of one of these methods is diagonally similar to the system matrix of Kung's method. This is then exploited to show that the spectral sensitivity of the methods decreases significantly under condition that are often met or easy to impose in practice, and when this is so, it is shown that the methods feature essentially the same spectral properties as Kung's method. As a result, a theoretical eigenvalue error bound is derived and illustrated by numerical simulations.

Keywords Eigenvalue sensitivity analysis, departure from normality, balanced realization

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1 Introduction

During the last thirty years a great deal of attention has been given to state-space models in balanced coordinates due to the role they have played in a number of important applications. Application areas include system identification, realization of systems, the construction of reduced-order models of complex systems, and sensitivity analysis [9, 10, 11, 15]. When computing reduced-order models, for example, balanced realization becomes important because it enables derivation of error bounds that allows one to assess the quality of the model [7, 8, 12]. On the other, desirable properties such as system stability are often attributed to balanced state space models [13].

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Increasing use of state-space models has also been observed when solving parameter estimation and harmonic retrieval problems, see, e.g., [14, 17, 20], and [2, 11, 14, 19] for certain variants of Kung's method that have become popular in several signal processing processing applications. As for Kung's method, it constructs state-space models in balanced coordinates from the singular value decomposition (SVD) of a block Hankel matrix containing noisy Markov parameters, fitting the data very well and handling clustered eigenvalues within a certain accuracy [10]. In contrast to this, however, very little is documented on sensitivity of system eigenvalues to small perturbations in the state transition matrix.

The departure from normality of a matrix plays an important role in scientific computing: among other things, it rules the accuracy of power methods for computing eigenvalue estimates, the convergence behavior of Krylov methods, and the sensitivity of eigenvalues to matrix perturbations; for an exhaustive discussion on the influence of nonnormality in scientific computing, the reader is referred to [5, Chap. 10]. Concerning eigenvalue computation, it is well known that while eigenvalues of normal matrices are easy to handle, that may not be the case when computing eigenvalues of nonnormal matrices. Therefore, measures to quantify nonnormality of matrices in connection with the spectral condition number of the eigenbasis are desirable.

The goal of this paper is to estimate a measure of nonnormality of transition matrices of certain family of projection-type realization methods and then derive eigenvalue error bounds. A particularly important feature of the family is that the transition matrix of one of these methods is diagonally similar to the transition matrix of Kung's method. Our approach relies on the fact that there exists a close relationship between eigenvalue condition numbers and measures of nonnormality [5, Chap. 10]. Based on this, we show that Henrici's departure from normality of the transition matrix of the above methods can be expressed in terms of quantities which describe well the spectral sensitivity of the methods, and then use the results for estimating the eigenvalue condition numbers.

The main conclusion of the paper is that the spectral sensitivity of the methods decreases significantly under conditions that are often met or easy to impose in practice, and when this is the case, it is seen that the projection-type methods feature essentially the same spectral properties as Kung's method. The analysis is restricted to the single-input single-output case with the assumption that all eigenvalues are distinct. Fundamental for the analysis is that if the Hankel data matrix is square, then the state transition matrix of Kung's method is balanced in the sense of Osborne [16].

The paper is organized as follows. Section 2 presents the proposed family of projection-type realization methods along with a brief review of Kung's method and a set of relationships and properties of system theory needed for the analysis. The notation used throughout is also included in this section. In Section 3, the connection between a particular projection method and Kung's method is made precise and the departure from normality of the system matrix of all methods is analyzed. The results are then exploited to derive an eigenvalue error bound which is described and discussed in Section 4. Numerical simulations that illustrate the theory are reported in Section 5. Finally, conclusions are provided in Section 6.

2 Projection-Type Realization Methods and Review of Kung's Method

In the remaining of the paper the following notation is used. The entries of a matrix A are denoted by $a_{i,j}$ or $[A]_{i,j}$. A^* denotes the conjugate transpose of A and $A^\dagger$ its Moore-Penrose pseudo-inverse. The column (resp. row) subspace of A is denoted by $\mathcal{R}(A)$ (resp. $\mathcal{R}(A^*)$) Also, as usual, $\|A\|_2$ and $\|A\|_F$ denote the spectral norm and the Frobenius norm of A, respectively. Finally, e_j denotes the j-th unit canonical vector of appropriate dimension, further notation will be introduced when needed.

In order to describe the proposed methods, consider a linear, discrete, time invariant system with p inputs and q outputs described in state-space form as:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{cases}, \tag{1}$$

where the state transition matrix (or system matrix) A is $n \times n$, the control matrix B is $n \times p$, C is $q \times n$ and D is $q \times p$. Assume that the system is stable, i.e., the eigenvalues λ_j of A fall inside the unit circle, and for simplicity assume that D is the zero matrix.

In system identification and or realization problems, one is concerned with the estimation of A B, and C (up to a similarity transformation) from either input/output data $\{u_k, y_k\}$ or Markov parameters h_k :

$$h_k = CA^{k-1}B, \quad k = 1, \ldots, h_L.$$

Like Kungs' method, the projection-type realization methods to be described in what follows, rely on the use of Markov parameters as data and start by forming an $(M \times q) \times (N \times p)$ block Hankel matrix $H(\ell)$ $(\ell \geq 1)$ with block entries in the position (i, j) equal to $h_{i+j+\ell-2}$. If the system is controllable and observable and $M, N \geq n$, it is well-known that this Hankel matrix has rank equal to n and a full-rank factorization of type

$$H(\ell) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{M-1} \end{bmatrix} A^{(\ell-1)}[B \; AB \; AB^2 \; \cdots \; A^{N-1}B] = \mathcal{O}A^{(\ell-1)}\mathcal{C}, \quad \ell \geq 1 \tag{2}$$

where $\mathcal{C}$ and $\mathcal{O}$ are the so-called extended controllability and extended observability matrices, respectively. It is worth noting that for single-input single-output systems the above full-rank factorization becomes

$$H(\ell) = V_M \, R \, \Lambda^{\ell-1} \, W_N, \tag{3}$$

where

$$W_N = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{N-1} \end{bmatrix}, \tag{4}$$

$$R = \text{diag}(r_1, r_2, \cdots, r_n), \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

and $V_M = W_M^T$.

Before continuing we emphasize that because our interest is to analyze the spectral sensitivity of the state transition matrix of the methods rather than the sensitivity of parameters B and C involved in the realization process, we will focus only on the construction and analysis of the corresponding state transition matrix. Observe that if one considers any solution F of the matrix equation

$$H(2) = FH(1), \tag{5}$$

because (2) is a full rank factorization of $H(1)$, it follows that $\text{rank}(F) = n$ and that the spectrum of F is described by

$$\lambda(F) = \{\lambda_1, \cdots, \lambda_n\} \bigcup \{0\}.$$

Additionally, if U is any orthonormal basis of $\mathcal{R}(H(1))$, the column space of $H(1)$ (and hence of $H(2)$), then the $n \times n$ matrix defined by

$$\check{A} = U^* F U, \tag{6}$$

is the projection orthogonal of F onto $\mathcal{R}(H(1))$, and $\lambda(\check{A})$, the spectrum of $\check{A}$, is formed by the eigenvalues of the system, i.e., one has

$$\lambda(\check{A}) = \{\lambda_1, \ldots, \lambda_n\}. \tag{7}$$

This suggests that one can construct infinitely many realizations of the form $\{\check{A}, \check{B}, \check{C}\}$ in which the state transition matrices are related to each other by a similarity transformation. As we will see, interesting spectral properties of matrices $\check{A}$ can be ensured provided that the solution F of (5) is properly chosen. From here on, realizations based on (6) will be called a projection-type realization.

Kung's method, on the other hand, constructs a realization $\{A_{\text{K}}, B_{\text{K}}, C_{\text{K}}\}$ by using estimates of an observable (resp. controllable) matrix obtained from the singular value decomposition of matrix $H(1)$. This can be described as follows. Let

$$H(1) = \mathcal{U}\Sigma\mathcal{V}^* = [\mathcal{U}_1 \; \mathcal{U}_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{V}_1^* \\ \mathcal{V}_2^* \end{bmatrix} \tag{8}$$

be a partitioned SVD of $H(1)$ where both $\mathcal{U}_1$ and $\mathcal{V}_1$ have n columns and $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ in which the nonzero singular values σ_i of $H(1)$ are ordered in decreasing form. Kung's method uses the SVD of $H(1)$, rewritten as

$$H(1) = (\mathcal{U}_1 {\Sigma_1}^{1/2})({\Sigma_1}^{1/2} \mathcal{V}_1^*),$$

and takes advantage of the multiple shift invariance property of $\mathcal{O}$ to define a state transition matrix as

$$A_{\mathrm{K}} = \Sigma^{-1/2}\overline{U}^{\dagger}\underline{U}\Sigma^{1/2} = \overline{\mathcal{U}}^{\dagger}\underline{\mathcal{U}}, \tag{9}$$

where $\overline{U}$ and $\underline{U}$ (resp. $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$) are formed by taking respectively the first N block rows and the last N block rows of $\mathcal{U}_1$ (resp. of $\mathcal{U}_1{\Sigma_1}^{1/2}$).

The realization $\{A_{\mathrm{K}}, B_{\mathrm{K}}, C_{\mathrm{K}}\}$ so obtained is known to satisfy the property that the associated controllable and observable Gramians are equal to Σ_1, in which case it is said that the system is balanced or that the state-space model is in balanced coordinates [15]. Observe that a state transition matrix can also be determined as

$$A_{\mathrm{K}} = \Sigma_1^{1/2}\underline{V}\overline{V}^{\dagger}\Sigma_1^{-1/2}, \tag{10}$$

where $\underline{V}$ and $\overline{V}$ are obtained from $\mathcal{V}_1^*$ by taking its first $(N-1)$ block columns and its last $(N-1)$ blocks columns, respectively.

3 Nonnormality Estimation

Our goal here is to estimate Henrici's departure from normality of the system matrix of a class of projection-type methods and that of Kung's method, focusing on single-input single-output systems with the assumption that the eigenvalues of the system are distinct. Using this assumption all results will apply automatically to system identification as well as to harmonic retrieval problems. We start by presenting a technical result.

Lemma 1 *Set $A = U^*FU$ where U is an orthonormal basis of $\mathcal{R}(H(1))$ and F is a companion matrix satisfying the matrix equation $H(2) = FH(1)$. Then we can rewritten $A = \overline{U}^{\dagger}\underline{U}$ where $\overline{U}$ (resp. $\underline{U}$) is formed by taking the first (resp. last) $N-1$ rows of U.*

Proof: First, observe from the full-rank matrix factorization (3) and the matrix equation $H(2) = FH(1)$ it follows that $FV_N = V_N\Lambda$. Now since that $\mathcal{R}(U) = \mathcal{R}(V_N)$, it follows that U is invariant under F and therefore

$$UA = UU^*FU = \mathcal{P}FU = FU,$$

where the last equation is because $\mathcal{P} = UU^*$ is the orthogonal projector onto $\mathcal{R}(H(1))$ and the columns of FU belong to such subspace. Equating the first $N-1$ rows of the matrix equation $UA = FU$ and then solving for A leads to $A = \overline{U}^{\dagger}\underline{U}$, as desired. ■

Observe that if in the previous lemma the orthonormal basis used in the definition of matrix A is chosen to be $\mathcal{U}_1$, the basis obtained from the SVD of $H(1)$, then the corresponding matrix A becomes diagonally similar to A_{K}, the system matrix of Kung's method. This observation is exploited in the following proposition to determine the Frobenius norm of matrix A_{K} from the entries of the system matrix of a well-chosen projection-type method.

Proposition 2 *Set* $\breve{A} = \mathcal{U}_1^* F \mathcal{U}_1$ *where* F *is a companion matrix satisfying the matrix equation* $H(2) = FH(1)$ *and* $\mathcal{U}_1$ *the basis of* $\mathcal{R}(H(1))$ *obtained from the SVD of* $H(1)$. *Then, provided* $M = N > n$ *we have*

$$\|A_K\|_F^2 = \sum_i |a_{i,i}|^2 + 2\sum_{i=1}^{n-1}\sum_{j>i} |a_{i,j}||a_{j,i}|, \tag{11}$$

where $a_{i,j}$ *are entries of* $\breve{A}$, *and* A_K *denotes the state transition matrix of Kung's method.*

Proof: Since for $M = N$ the Hankel matrix $H(1)$ is symmetric, its singular values satisfy $\sigma_j = |\gamma_j|$, $j = 1 : n$, where the γ_j are the non zero eigenvalues of $H(1)$. This implies that there exists a diagonal matrix J with entries $[J]_{j,j}$ equal to 1 or -1 such that $\mathcal{U}_1 = \mathcal{V}_1 J$. Using this relation and the definition of $A_{\rm K}$ (see (9)), because $\overline{U}^\dagger \underline{U} = \breve{A} = \mathcal{U}_1^* F \mathcal{U}_1$ by Lemma 1, we obtain

$$A_{\rm K} = \Sigma_1^{-1/2} J \mathcal{V}_1^* F \mathcal{V}_1 J \Sigma_1^{1/2} = J(\Sigma_1^{1/2}\mathcal{V}_1^* F^* \mathcal{V}_1 \Sigma_1^{-1/2})^* J = J \breve{A}_{\rm K}^* J,$$

where the last equality is for $(\mathcal{V}_1^* F^* \mathcal{V}_1)$ is the projection version of matrix $\underline{V}\overline{V}^\dagger$ in (10). This results shows that $\breve{A}$ is J-symmetric and thus

$$|[A_{\rm K}]_{i,j}| = |[A_{\rm K}]_{j,i}|, \quad 1 \le i, j \le n. \tag{12}$$

From this equality and the definition of $A_{\rm K}$, it follows that

$$0 = \left(\frac{\sigma_j^{1/2}}{\sigma_i^{1/2}}|a_{i,j}| - \frac{\sigma_i^{1/2}}{\sigma_j^{1/2}}|a_{j,i}|\right)^2 = \frac{\sigma_j}{\sigma_i}|a_{i,j}|^2 + \frac{\sigma_i}{\sigma_j}|a_{j,i}|^2 - 2|a_{i,j}||a_{j,i}|, \tag{13}$$

where the $a_{i,j}$'s are entries of matrix $\breve{A}$, and hence

$$|[A_{\rm K}]_{i,j}|^2 + |[A_{\rm K}]_{j,i}|^2 = 2|a_{i,j}||a_{j,i}|. \tag{14}$$

If the Frobenius norm of $A_{\rm K}$ is computed as

$$\|A_{\rm K}\|_F^2 = \sum_{i=1}^{n} |[A_{\rm K}]_{i,i}|^2 + \sum_{i=1}^{n-1}\sum_{j>i}^{n} |[A_{\rm K}]_{i,j}|^2 + |[A_{\rm K}]_{j,i}|^2,$$

the equality (11) is a consequence of (14). This concludes the proof. ■

Matrix $\breve{A}$ described above via pseudo-inversion is often used in estimating parameters from exponentially-damped noisy signals [20]. This matrix will sometimes be referred as to the transition matrix of Kung's method in non balanced coordinates.

An immediate consequence of (12) is that $A_{\rm K}$ is balanced in the 2-norm. Recall that a matrix is balanced in the α-norm if for any i, the α-norm of the i-th row equal the α-norm of the i-th column. Using results by Osborne [16], it follows that $\Sigma_1^{1/2}$ minimizes the Frobenius norm of all matrices $G = D^{-1}AD$ where D is a diagonal matrix, and so $\|A_{\mathrm{K}}\|_F \leq \|\breve{A}\|_F$. That is, the similarity transformation $G = D^{-1}AD$, with $D = \Sigma_1^{1/2}$, not only balances matrix $\breve{A}$ but also reduces the Frobenius norm. As a result the nonnormality A_{K} and all system matrices of projection-type methods as described in Lemma 1 can be now described. This is given in the following corollary where we use as measure of nonnormality Henrici's departure from normality [6, p. 314]. Recall also that for general $A \in \mathbb{C}^{n\times n}$, Henrici's departure from normality, denoted by $D(A)$, is defined as

$$\mathcal{D}(A) = \|A\|_F^2 - \sum_{j=1}^{n} |\lambda_j(A)|^2. \tag{15}$$

Corollary 3 *Assume that $M = N > n$. Then for any projection-type realization method whose system matrix results of projecting a companion matrix as described in Lemma 1 it holds*

$$\mathcal{D}(A_K)^2 \leq \mathcal{D}(A)^2. \tag{16}$$

Proof: We first prove that the inequality holds for the case where we consider the system matrix $\breve{A}$. In fact, a simple expression for $\mathcal{D}(A_{\mathrm{K}})^2$ can be given using the identity

$$|a_{i,j}|^2 + |a_{j,i}|^2 = 2|a_{i,j}||a_{j,i}| + (|a_{i,j}| - |a_{j,i}|)^2$$

in (11). Of course, making this (11) becomes

$$\mathcal{D}(A_{\mathrm{K}})^2 = \mathcal{D}(\breve{A})^2 - \sum_{i=1,n}^{n} \sum_{j>i}^{n} (|a_{i,j}| - |a_{j,i}|)^2. \tag{17}$$

This proves inequality (16) for the particular case where the system matrix is $\breve{A}$.

We now make the crucial observation that all system matrices of this class of projection-type realization methods are unitarily similar to $\breve{A}$. The assertion of the corollary follows then from this observation because all these system matrices share the same Henrici's departure from normality. ■

Two conclusions can be drawn from equality (17). The first is that $\mathcal{D}(A_{\mathrm{K}})^2$ is always smaller than $\mathcal{D}(\breve{A})^2$ and that matrix A_{K} can become much more close to being normal than matrix $\breve{A}$ provided that $|a_{i,j}|$ and $|a_{j,i}|$ differ by a large quantity. In terms of spectral sensitivity, this suggests that the eigenvalue problem related to A_{K} should be better conditioned than the eigenvalue problem related to A, despite the fact that both matrices share the same spectrum. This is postponed to the next section. The second conclusion is that the departure from normality of $\breve{A}$ and A_{K} can become small provided that $|a_{i,j}| \approx |a_{j,i}|$. This is important because in such case, both matrices $\breve{A}$ and A_{K} are ensured to behave almost similarly as normal matrices in the sense that their eigenvalues become relatively insensitive to small matrix perturbations. In other words, the condition $|a_{i,j}| \approx |a_{j,i}|$

ensures that $\breve{A}$ and A_K share essentially the same spectral properties, the same comment being valid if $\breve{A}$ is replaced by any system matrix as those described in Lemma 1. The following corollary provides an estimate for $\mathcal{D}(A_K)^2$.

Corollary 4 *Using the same assumptions of Corollary 3 it holds*

$$\mathcal{D}(A_K)^2 \leq n - 1 + \|f^\dagger\|_2^2 + \prod_{i=1}^{n} |\lambda_i|^2 - \sum_{i=1}^{n} |\lambda_i|^2 \tag{18}$$

where $f^\dagger$ denotes the minimum 2-norm solution of the linear system $H(1)f = H(2)e_N$.

Proof: We first observe that using Theorem 3 in [1] it follows

$$\mathcal{D}(A)^2 = n + \|f^\dagger\|_2^2 - \|p_1\|_2^2 - \sum_{i=1}^{n} |\lambda_i|^2,$$

where p_1 is the first column vector of the orthogonal projection onto $\mathcal{R}(V_N)$. The statement of the corollary is thus a consequence of Corollary 3 and of the fact that the right side of (18) is an upper bound for $\mathcal{D}(A)^2$, as seen from Lemma 7 in [1]. ■

In reference [1] it was proved that $\|f^\dagger\|_2$ decreases with the dimension M of the Hankel matrix and that this norm vanishes as M is going to infinite. Therefore, since the right hand of (18) is an upper bound for $\mathcal{D}(A)^2$, A being any system matrix of a projection-type method as in Lemma 1, we can conclude that small values of Henrici's departure from normality of all these matrices can be expected provided that M is large enough and the eigenvalues satisfy $|\lambda_j| \approx 1$. We end the section with the observation that the conditions $\|f^\dagger\|_2 \approx 0$ and $|\lambda_j| \approx 1$ are often met in connection with slightly damped systems.

4 Spectral Properties

Before proceeding, recall that for general $A \in \mathbb{C}^{n\times n}$ having simple eigenvalues $\lambda_j(A)$, the *condition number* of eigenvalue $\lambda_j(A)$ is given as

$$\kappa_j = \frac{\|u_i\|_2 \|v_i\|_2}{|u_i^* v_i|}, \tag{19}$$

where v_j and u_j are right and left eigenvectors of A associated with λ_j. It is well-known that $u_i^* v_i$ is always nonzero and that the real number κ_j controls the sensitivity of eigenvalue λ_j to perturbations on A (see, e.g., Wilkinson [21, p. 87]). The following result will be needed to prove our main result concerning the sensitivity spectral of matrix A_K and all matrices of the projected-type realization methods that use a companion matrix as described in Lemma 1.

Lemma 5 *The state transition matrix of Kung's method in non balanced coordinates A, has a spectral decomposition given as*

$$A = (\mathcal{U}_1^* V_N)\Lambda(\mathcal{U}_1^* V_N)^{-1} = (\mathcal{U}_1^* V_N)\Lambda(V_N^\dagger \mathcal{U}_1). \tag{20}$$

Consequently, the condition number κ_j of eigenvalue λ_j satisfies

$$\kappa_j = \|V_N e_j\|_2 \|V_N^\dagger e_j\|_2. \tag{21}$$

Proof: It is sufficient to observe that the companion matrix F used in the definition of the state transition matrix A satisfies the matrix equation $FV_N = V_N \Lambda$, and that the columns of $\mathcal{U}_1$ form an orthonormal basis for $\mathcal{R}(V_N)$. Details can be encountered in Bazán [1]. ■

Proposition 6 *Let $\breve{\kappa}_j$ and κ_j denote the condition number of eigenvalue λ_j regarded as eigenvalue of A_K and A, respectively. Then it holds that*

$$\breve{\kappa}_j \le \kappa_j, \quad j = 1, \ldots, n.$$

Proof: Let $\breve{v}_j$ and $\breve{u}_j$ denote right and left eigenvectors of matrix A_K associated with λ_j. From the definition of A_K and (20) it follows that they can be taken as

$$\breve{v}_j = \Sigma_1^{-1/2} \mathcal{U}_1^* V_N e_j, \quad \breve{u}_j^* = e_j^* V_N^\dagger \mathcal{U}_1 \Sigma_1^{1/2}. \tag{22}$$

These eigenvectors satisfy the normalization condition $\breve{u}_j^* \breve{v}_j = 1$, so $\breve{\kappa}_j = \|\breve{u}_j\|_2 \|\breve{v}_j\|_2$. From this observation and (22) it follows that using (22)

$$\begin{aligned} \|\breve{v}_j\|_2^2 &= e_j^* V_N^* \mathcal{U}_1 \Sigma_1^{-1} \mathcal{U}_1^* V_N e_j, \\ &= e_j^* V_N^* \mathcal{U}_1 \mathcal{V}_1^* \mathcal{V}_1 \Sigma_1^{-1} \mathcal{U}_1^* V_N e_j \\ &= e_j^* V_N^* \mathcal{U}_1 \mathcal{V}_1^* H(1)^\dagger V_N e_j \\ &= e_j^* V_N^* \mathcal{U}_1 \mathcal{V}_1^* W_N^\dagger R^\dagger V_N^\dagger V_N e_j \\ &= (e_j^* V_N^* \mathcal{U}_1)(\mathcal{V}_1^* W_N^\dagger e_j)/r_j, \end{aligned} \tag{23}$$

where $r_j = [R]_{j,j}$ and the last four equalities are for $\mathcal{V}_1 \Sigma_1^{-1} \mathcal{U}_1^* = H(1)^\dagger = W_N^\dagger R^\dagger V_N^\dagger$ and $V_N^\dagger V_N = I$. Similarly, from $\|\breve{u}_j\|_2$ it follows

$$\begin{aligned} \|\breve{v}_j\|_2^2 &= e_j^* V_N^\dagger \mathcal{U}_1 \Sigma_1 \mathcal{U}_1^* V_N^{*\dagger} e_j \\ &= e_j^* V_N^\dagger \mathcal{U}_1 \Sigma_1 \mathcal{V}_1^* \mathcal{V}_1 \mathcal{U}_1^* V_N^{*\dagger} e_j \\ &= e_j V_N^\dagger V_N R W_N \mathcal{V}_1 \mathcal{U}_1^* V_N^{*\dagger} e_j \\ &= r_j \, (e_j^* W_N \mathcal{V}_1)(\mathcal{U}_1^* V_N^{*\dagger} e_j), \end{aligned} \tag{24}$$

where, as before, we have used the fact that $\mathcal{U}_1 \Sigma_1 \mathcal{V}_1^* = H(1) = V_N R W_N$. From (23),(24), and the fact that $\|V_N e_j\|_2 = \|W_N e_j\|_2$ and $\|W_N^\dagger e_j\|_2 = \|V_N^\dagger e_j\|_2$, using the Cauchy-Schwartz inequality it then follows that

$$\breve{\kappa}_j^2 \le \kappa_j^2,$$

and the proof concludes. ■

An analysis of the condition numbers κ_j and their use to estimate the condition number of $H(1)$ can be encountered in [4, 3]. The main conclusion drawn there is that κ_j depends on eigenvalue separation, the closeness of the eigenvalues to the unit circle, and mainly on the dimension of the Hankel matrix in that moderate condition numbers are generally obtained whenever the Hankel matrix is large enough. Following the line of analysis in [4] an upper bound for $\breve{\kappa}_j$ can be immediately obtained.

Corollary 7 *For all $N > n$, the condition number $\breve{\kappa}_j$ satisfies*

$$1 \leq \breve{\kappa}_j < \left[1 + \frac{n - 1 + \|f^\dagger\|_2^2 \|p_1\|_2^2 + \prod_{i=1}^n |\lambda_i|^2 - |f_0|^2 - \sum_{i=1}^n |\lambda_i|^2}{(n-1)\delta_j^2} \right]^{(n-1)/2}, \; j = 1:n, \qquad (25)$$

where $f^\dagger$ denotes the minimum 2-norm solution of the linear system $H(1)f = H(2)e_N$, f_0 is the first component of $f^\dagger$, p_1 denotes the first column vector of the orthogonal projection onto $\mathcal{R}(V_N)$, and $\delta_j = \min_{\substack{j \\ j \neq i}} |\lambda_i - \lambda_j|$.

Proof: From [18]

$$\breve{\kappa}_j \leq \left[1 + \frac{\mathcal{D}(A_{\mathrm{K}})^2}{(n-1)\delta_j^2} \right]^{(n-1)/2}. \qquad (26)$$

Inequality (25) follows from using the expression for $\mathcal{D}(A_{\mathrm{K}})$ described in (17) where is used the exact value of $D(A)^2$, as described in Proposition 3 in reference [4]. ■

Observe that the right hand of inequality (25) can grossly overestimate the condition number $\breve{\kappa}_j$ since, as commented above, $\mathcal{D}(A_{\mathrm{K}})^2$ can become much smaller than $\mathcal{D}(A)^2$ when $|a_{i,j}|$ differs largely from $|a_{j,i}|$, a fact frequently seen when M is not large enough. Numerical examples illustrating this phenomenon are presented later. Observe also that if N is large enough and the eigenvalues in modulus are reasonably close to 1 but not extremely close to each other, then $\breve{\kappa}_j$ approaches 1 because under these conditions $\|f\|_2^2 \approx 0$. That is, if these conditions are satisfied, then the eigenvalues become relatively insensitive to small perturbation in the system matrix.

Remark A comment regarding the accuracy of Kung's method in the noisy data case is appropriated. Firstly it should be noted that because A_{K} is obtained from A via a diagonal similarity, the same set of eigenvalue are to be obtained from both matrices, at least theoretically, this result being valid regardless of whether the data are corrupted by noise or not. Secondly, it is convenient to invoke an eigenvalue perturbation analysis from matrix A, recently reported in [4], where the main conclusion is that if $\widetilde{A}$ is obtained from the SVD of a perturbed Hankel matrix $\widetilde{H}(1) = H(1) + E$ (containing noisy data: $\widetilde{h}_k = h_k + \epsilon_k$), and if $\|E\| < \sigma_n(H(1))$, whenever the dimension of the

Hankel matrix is large enough the eigenvalue error satisfies

$$|\lambda_j - \widetilde{\lambda}_j| \approx \sin\theta, \quad j = 1:n; \tag{27}$$

where $\sin\theta$ denotes the distance between the subspaces $\mathcal{R}(\mathcal{U}_1)$ and $\mathcal{R}(\widetilde{\mathcal{U}}_1)$ with $\widetilde{\mathcal{U}}_1$ the matrix of left singular vectors associated with the n largest singular values of $\widetilde{H}(1)$. Thus, since A_{K} and A share the same eigenvalues, it is concluded that the estimate (27) also holds for Kung's method in balanced coordinates, and that excellent eigenvalues estimates are to be expected provided that N is large enough and $\|E\|_1 \ll \sigma_n(H(1))$, as in this case $\sin\theta \approx 0$.

5 Numerical Simulation

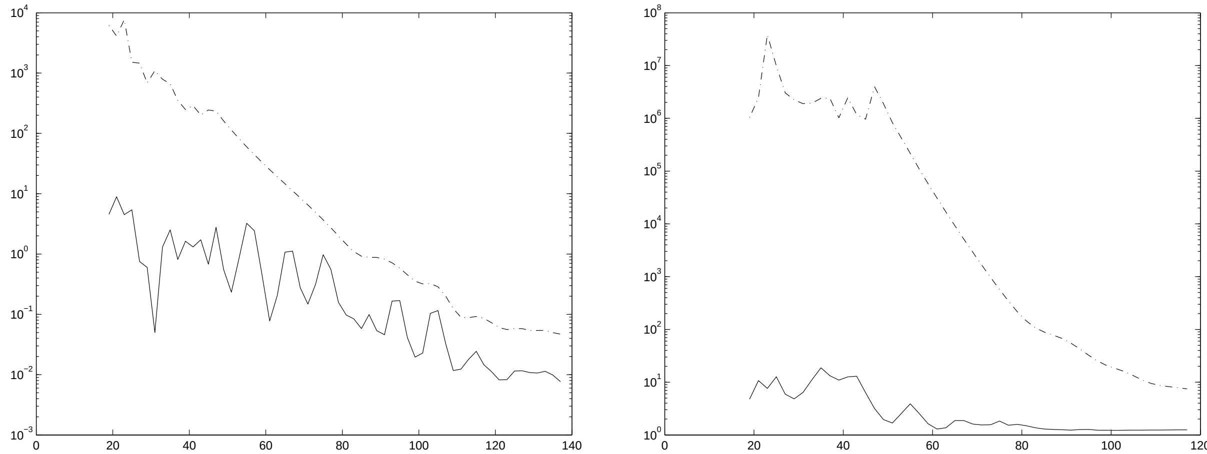
The goal here is to illustrate numerically the theory presented in the paper. For this we choose a system having closely spaced eigenvalues and then simulate two well differentiated cases in order to numerically assess the sensitivity of system eigenvalues to inaccuracies.

In the first case we use *exact* data and our goal is to illustrate how nonnormality of the state transition matrix affects the precision of eigenvalues extracted by using finite precision computations. For our illustration we compute Henrici's departure from normality of A_{K} and A, the eigenvalue condition numbers $\breve{\kappa}_j$, κ_j, and the eigenvalue error $|\lambda_j - \widetilde{\lambda}_j|$, where $\widetilde{\lambda}_j$ denotes the computed eigenvalue. All quantities were computed in MATLAB using square Hankel matrices of increasing orders to illustrate the effect of the dimension of the problem over the quantities computed. The data consists of samples of an impulse response (real) signal, h_k, $k = 1 : 1024$, synthesized from an eighty degree of freedom vibrating system, (see [2] for details), whose signal parameters (system eigenvalues λ_j and weights r_j) as well as the eigenvalue separations, are described in Table 1.

j	z_j , $\bar{z}_j$	$\lvert z_j\rvert$, $\lvert\bar{z}_j\rvert$	r_j , $\bar{r}_j$	δ_j^2
1	$0.9138 \pm 0.4032\imath$	0.9988	$-0.0013 \pm 0.0004\imath$	0.00151
2	$0.9908 \pm 0.1262\imath$	0.9988	$-0.0008 \pm 0.0032\imath$	0.00010
3	$0.9774 \pm 0.2062\imath$	0.9989	$-0.0002 \pm 0.0001\imath$	0.00548
4	$0.9457 \pm 0.3220\imath$	0.9990	$-0.0001 \pm 0.0001\imath$	0.00004
5	$0.9923 \pm 0.1162\imath$	0.9990	$0.0002 \pm 0.0010\imath$	0.00001
6	$0.9479 \pm 0.3157\imath$	0.9991	$-0.0003 \pm 0.0009\imath$	0.00004
7	$0.9596 \pm 0.2780\imath$	0.9991	$0.0079 \pm 0.0019\imath$	0.00155
8	$0.8269 \pm 0.5608\imath$	0.9991	$0.0015 \pm 0.0009\imath$	0.01991
9	$0.8977 \pm 0.4387\imath$	0.9991	$-0.0002 \pm 0.0006\imath$	0.00151

Table 1: Signal parameters corresponding to a vibrating system.

Figures 1-(a) and 1-(b) show that nonnormality and eigenvalue condition numbers strongly depend on the dimension of the problem, and that, as predicted in theory, the state transition



(b) (c)

Figure 1: (a) Henrici's departure from normality as a function of N. $D(A_{\mathrm{K}})^2$: solid line, $D(A)^2$: dash-dotted line. (b) Eigenvalue condition number as function of N. $\max_j \breve{\kappa}_j$:solid line, $\max_j \kappa_j$:dash-dotted line. In both cases the Y-axis is in a logarithmic (base 10) scale.

matrix in balanced coordinates A_{K} has in fact better spectral properties than the state transition matrix A. The eigenvalue error is computed by using several implementations of matrix A_{K}, namely the $\mathcal{U}_1$-projection version: $A_{\mathrm{K}} = \Sigma_1^{-1/2}\mathcal{U}_1^* C \mathcal{U}_1 \Sigma_1^{1/2}$, the $\mathcal{U}_1$ pseudo-inverse versions: $A_{\mathrm{K}} = \Sigma^{-1/2}\overline{U}^{\dagger}\underline{U}\Sigma_1^{1/2}$, and the corresponding counterparts using the right singular matrix $\mathcal{V}_1$. As a result of the computations it was confirmed that except for the projected based versions, all remaining $\mathcal{U}_1$-based versions using balanced or non balanced coordinates yield essentially the same eigenvalues. A similar comment applies for the eigenvalues obtained from the $\mathcal{V}_1$-based versions. The conclusion drawn from the results is that the projection-based versions tend to yield estimates of inferior quality compared with the remaining implementations either using matrix $\mathcal{U}_1$ or matrix $\mathcal{V}_1$. This difference in quality is evident for moderate values of N, as illustrated in Figure 2-(a).

The second case illustrates the performance of Kung's method in recovering system eigenvalues in a noise environment. For this, the exact signal is perturbed by using additive zero mean random noise, adjusted in such a way that $\|\text{noise}\|/\|\text{signal}\| = 0.05$, and then the Kung's matrix eigenvalues are computed. Mean values of 100 noisy realizations are reported. As a result of the computations it is observed again that the quality of the computed eigenvalues by using the projected version is in general inferior to that obtained from the other versions, although the difference is not as evident, see Fig 2-(b). Poor results are because the norm of the error matrix $\|E\|_2 = \|\widetilde{H}(1) - H(1)\|_2$ exceeded largely $\sigma_n(H(1))$, i.e., the condition $\|E\|_2 < \sigma_n(H(1))$ that insure the error estimate (27) was no longer valid.

Finally, the spectral stability of balanced and non balanced matrices associated with Kung's method, assuming that both matrices are corrupted by additive random noise, is illustrated. For this, both versions, the balanced and non balanced matrices, were corrupted by random matri-

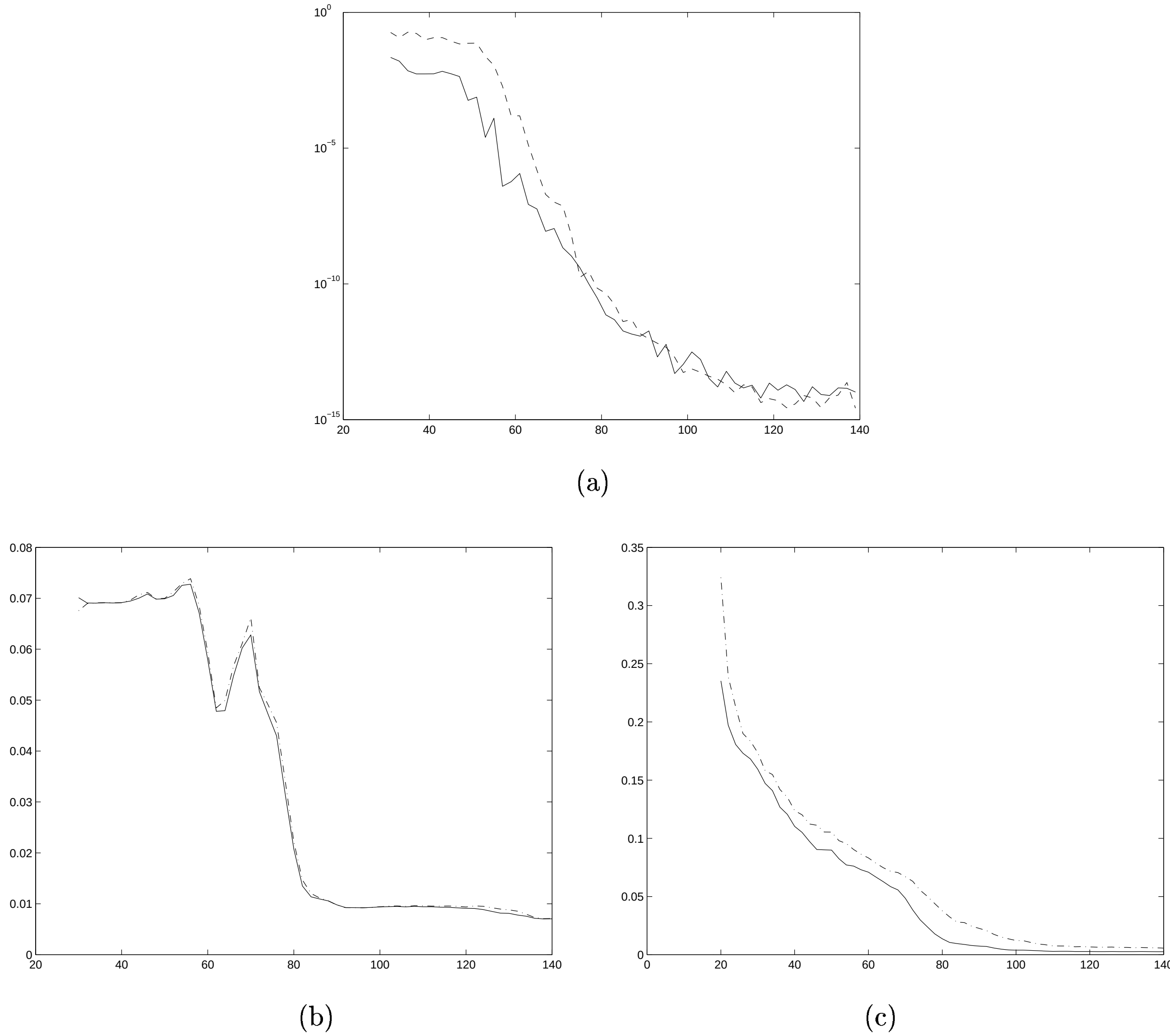


Figure 2: (a) Eigenvalue error as a function of N in the case where exact data are used and considered the effect of finite precision computations. Results from projected $\mathcal{V}_1$- based versions: dash-dotted line, Results from balanced pseudo-inverse $\mathcal{U}_1$-based versions: solid line. (b) Eigenvalue error as a function of N in the case where is considered additive noise in the signal h_k. Results from balanced $\mathcal{U}_1$ implementation : solid line, Results from non balanced $\mathcal{U}_1$ implementation: dash-dotted line. (c) Eigenvalue error as a function of N in the case where is considered additive noise in the transition matrices. Results from balanced $\mathcal{U}_1$ implementation : solid line, results from non balanced $\mathcal{U}_1$ implementation: dash-dotted line

ces such that $\|\text{state transition matrix}\|/\|\text{random matrix}\|_2 = 0.01$ and then their eigenvalues were extracted. Mean values of eigenvalue error obtained from 100 realizations, displayed in Fig 2-(c), illustrate that the balanced version of Kung's method features slightly better spectral stability than the non balanced one .

6 Conclusion

In this paper spectral properties of state transition matrices of certain projection-type realization methods were investigated based on the estimation of Henrici's departure from normality. It was shown that the system matrix of one of these methods is diagonally similar to the system matrix of Kung's method and proved that the spectral sensitivity of the methods decreases significantly under conditions that are reasonable in practice. Specifically, it was proved that if the Hankel matrix is large enough and the eigenvalues are reasonably close to the unit circle but not extremely close to each other, then the eigenvalues become relatively insensitive to small perturbations, and that in this case the projection-type methods feature essentially the same spectral properties as Kung's method.

Further, because both matrices in balanced and non balanced coordinates relate to each other via a diagonal similarity transformation, it was concluded that eigenvalue error bounds, that apply for state transition matrices in non balanced coordinates, also hold for the balanced case [4]. Experiments were provided to illustrate that the eigenvalue estimates obtained from the balanced case do not outperform so much those obtained from the non balanced ones (Fig. 2-b). The analysis of nonnormality and results of conditioning may be useful in system analysis when modifications of the state transition matrix are needed. Further research for assessing the spectral properties of system matrices in the multi-input multi-output case is needed.

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References

[1] F. S. V. Bazán. Conditioning of Rectangular Vandermonde Matrices with nodes in the Unit Disk. SIAM J. Matrix Analysis and Applications, 21(2) pp. 679-693, 2000.

[2] F. S. V. Bazán and C. Bavastri. An Optimized Pseudo-Inverse Algorithm (OPIA) For Multi-Input Multi-Output Modal Parameter Identification. Mechanical Systems and Signal Processing, 10:365-380, 1996.

[3] F. S. V. Bazán and Ph. L. Toint. Conditioning of Infinite Hankel Matrices of finite rank. Systems and Control Letters, 41:347-359, 2000.

[4] F. S. V. Bazán. Error analysis of signal zeros: a projected companion matrix approach. Lin. Alg. and Its Appl., 369, 153-167, 2003.

[5] F. Chaitin-Chatelin and V. Frayssé, Lectures on Finite Precision Computations, SIAM Philadelphia 1996.

[6] G. H. Golub and C. F. Van Loan. Matrix Computations. Johns Hopkins University Press, Baltimore, third edition, 1996.

[7] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds, International Journal of Control, 39: 11515-1193 (1984).

[8] K. Glover, R. F. Curtain, and J. R. Partington, Realization and approximation of linear infinite-dimensional systems with error bounds, SIAM J. Control Optim. 16(4): 863-898 1988)

[9] S. W. Gray and E. I. Verriest, On the sensitivity of generalized state-space systems, Proc. 28th IEEE Conf. Decision Contr., Tampa, FL, 1989.

[10] S. Y. Kung, A new identification and model reduction algorithm via singular value decomposition, Proc. 12th Asilomar Conf. on Circuits, Systems and Computers, Pacific Grove, CA, pp.705-714,1978.

[11] S. Y. Kung, K. S. Arun, and D. V. Bhaskar Rao, State-space and singular value decomposition-based approximation methods for the harmonic retrieval problem. Opt. Soc. Amer., Vol. 73, pp. 1799-1811, December, 1983.

[12] K. Lam and B. D. O Anderson, Li-impulse response error bound for balanced truncation, Systems Control Lett. 18:129-137 (1992).

[13] J. Lam, and S. Y. Hung, On the stability of Projections of Balanced Realizations, Lin. Alg. Appl. 257:163-182 (1987).

[14] F. Li and R. J. Vaccaro. Unified analysis for DOA estimation algorithms in array signal processing. Signal Processing 25:147-169, 1991.

[15] B. Moore, Principal component analysis in linear systems: Controllability, observability, and model reduction, IEEE Trans. Automat. Contr. AC-26 No. 1, pp. 17-32,1981.

[16] E. E. Osborne, On pre-conditioning of matrices, J. assoc. Comp. Math, 7,338-345 (1960).

[17] B. D. Rao, Relationship between matrix pencil and state-space based harmonic retrieval methods, IEE Trans. On Acoustics, speech and Signal Proc. Vol. 38, No. 1, Jan. 1990.

[18] R. A. Smith. The condition numbers of the matrix eigenvalue problem. Numer. Math, 10(1967), pp. 232-240.

[19] S. Van Huffel, H. Chen, C. Decanniere, and P. Van Hecke. Algorithm for time-domain NMR data fitting based on Total least squares, Journal of Magnetic Resonance (1994) 110:228-237.

[20] A. van der Veen, E. F. Deprettere and A. Lee Swindlehurst. Subspace-Based Signal Analysis Using Singular Value Decomposition. Proceedings of the IEEE, 81(9):1277-1309, September 1993.

[21] J. H. Wilkinson. The Algebraic Eigenvalue Problem. Oxford University Press, Oxford, Great Britain, 1965.