A short note on backward errors for the common eigenvector problem

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TR/PA/06/13

Abstract

In 1984, Dan Shemsh gave some conditions under which two square matrices A and B have a common eigenvector. He defined it as an element of the intersection of $\ker[A^k, B^l]$ for $0 < k, l < n$. Many works are based on his results, and several algorithms in different application fields also depend on these results (works of Al. George and K.D. Ikramov in 1999, and works of M. Tsatsomeros in 2001). However they basically remain formal results and difficult to use in the presence of round-off errors, because the Shemsh's formula is numerically unstable. A small perturbation can transform a couple of matrices with a common eigenvector into a couple which does not enjoy anymore this property as shown in this report.

Our work consists of studying the reliability of an approximate solution of the common eigenvector problem. Based on the notion of the backward error, we design a new approach to solve this problem. An important application of our work is shown in Algorithm 1, where we show that simultaneous triangularization of two matrices can be based on an algorithm that computes a common eigenvector for two matrices.

Keywords: Eigenvector, Eigenvalue, Backward error, Common invariant subspaces, Simultaneous triangularization.

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1 Mathematical background and motivation

1.1 Common invariant subspaces

For two integers k and n such that $0 < k \leq n$, let $\mathcal{D}_{k,n}$ be the set of strictly increasing sequences of k integers chosen in the interval $[1, ..., n]$. Let N denote the cardinality of $\mathcal{D}_{k,n}$:

$$N = \mid \mathcal{D}_{k,n} \mid = \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

The members of $\mathcal{D}_{k,n}$ are ordered lexicographically.

For $(s_1,\, s_2) \in (\mathcal{D}_{k,n})^2$ and $A \in \mathcal{M}_n(\mathbb{C})$, $A[s_1 \mid s_2]$ denote the sub-matrix of A in rows and columns indexed by s_1 and s_2 .

Definition 1 *: The k^{th} compound matrix of A is the $N \times N$ matrix whose entries are*

$$\det A[s_1 \mid s_2] \qquad (s_1, s_2) \in (\mathcal{D}_{k,n})^2.$$

This matrix is designated by $C_k(A)$.

Two recent papers by A. George [5] and M. Tsatsomeros [6] propose an algorithm to build a common invariant subspace of dimension $k > 1$ for two matrices A and B from a **common eigenvector** of their compound matrices $C_k(A)$ and $C_k(B)$. These works are based on the theorem given in [4] that constructs the common eigenvector for $C_k(A)$ and $C_k(B)$.

Theorem 1 (Dan Shemesh) *[4]*
The matrices X and Y have a common eigenvector if and only if

$$\mathcal{L} = \bigcap_{k,l=1}^{n-1} Ker[X^k, Y^l] \neq \{0\}.$$

The Shemesh theoretical condition is not easy to bring into use in floating point calculation for two reasons:

- First, it relies on an algorithm to compute the powers of matrices A and B. The matrix power is a sensitive map with respect to perturbation of the matrix, as shown in [7]. The algorithm requires also the computation of the intersection of the null-spaces of the matrices $[A^k, B^l]$. These algorithms are closely related to the problem of the rank determination, that involves a lot of thresholds, whose determination is delicate and crucial to ensure the backward stability of the algorithm.

- Second, the condition enables to decide when a common eigenvector exists, but does not provide an algorithm for computing it.

A possible application of the algorithms for the common eigenvector is the determination of a simultaneous basis P which brings both the matrix A and B to an upper triangular form as shown in Section 1.2.

1.2 Simultaneous triangulation of a pair of matrices

Let A and B in $\mathcal{M}_n(\mathbb{C})$. A nonzero complex vector x is a common eigenvector of A and B if there exist two complex numbers α and β such that:

$$\begin{cases} Ax = \alpha x, \\ Bx = \beta x. \end{cases} \qquad (\mathcal{P})$$

It is known that whenever the matrices A and B commute, they have at least one common eigenvector [1]. Also if the matrices A and B can be simultaneously brought to upper triangular form [3]; i.e. if it exists a nonsingular matrix P and two triangular matrices R and S such that:

$$\begin{cases} P^{-1}AP = R, \\ P^{-1}BP = S. \end{cases}$$

Then the first column of P is a common eigenvector of A and B. The following algorithm enables, in theory, the simultaneous triangularization provided that an algorithm to compute a common eigenvector exists.

Algorithm 1 Simultaneous triangulation

1: Compute v a common eigenvector for A_k, B_k

2: Complete v to compute an unitary base of $\mathbb{C}^n$, set Q the corresponding matrix

3: Compute

$$R = Q^{-1}AQ = \begin{pmatrix} \alpha & * \\ 0 & \\ \vdots & A_{k+1} \\ 0 & \end{pmatrix}, \; S = Q^{-1}BQ = \begin{pmatrix} \beta & * \\ 0 & \\ \vdots & B_{k+1} \\ 0 & \end{pmatrix}$$

4: Set $A_k = A_{k+1}$, $B_k = B_{k+1}$ Goto 1

2 Ill-poseness of the problem in the Hadamard sense

Set: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$, then for $\epsilon = 0$, A and B have $v = (1, 0)^T$ as a common eigenvector but for $\epsilon \neq 0$ they have no common eigenvector. Consequently the problem is not well posed in the Hadamard sense. It means that it is not possible to find a common eigenvector of a pair of matrices in the presence of round-off errors since the related perturbations are likely to transform a pair of matrices having a common eigenvector into a pair which does not enjoy this property. The topological reason beyond this result is shown in the theorem below :

Theorem 2 *:*
The set of matrices that do not have any eigenvector in common is dense in the set of all pairs of matrices i.e.

$$\bar{S} = M_n(\mathbb{C})^2$$

where $S = \{(A,B) \in M_n(\mathbb{C})^2 / A$ *and* B *does not have a common eigenvector*$\}$.

Proof

Let $(M,N) \in M_n(\mathbb{C})^2$. We have to find a set of matrices $(M_\epsilon, N_\epsilon) \in S$ that converge to (M,N) when $\epsilon \longrightarrow 0$.

If (M,N) do not have a common eigenvector then we take $M_\epsilon = M$ and $N_\epsilon = N$, else suppose $n \geq 2$ and let $[u_1, u_2, ..., u_r]$ be the set of the eigenvectors of M and set $H_i = \mathbb{C}u_i + \mathbb{C}Nu_i$ we have $\dim(H_i) \leq 2, \quad i = 1, \cdots, r$.

For $j = 1, \cdots, r$ choose a v_j not in H_j and set K the matrix that maps u_j with v_j:

$$Ku_j = v_j.$$

Set $M_\epsilon = M$, and $N_\epsilon = N + \epsilon K$ then $(M_\epsilon, N_\epsilon) \in S$ if not $\exists i_0$ such that u_{i_0} is a common eigenvector of M_ϵ and $N_\epsilon \Rightarrow \ \exists \lambda \in \mathbb{C}$:

$$(N + \epsilon K)u_{i_0} = \lambda u_{i_0} \Rightarrow Nu_{i_0} + \epsilon v_{i_0} = \lambda u_{i_0} \Rightarrow v_{i_0} \in H_{i_0}.$$

Contradiction, we then have $(M_\epsilon, N_\epsilon) \longrightarrow (M,N)$.
For $n = 2$ and $M,\ N$ have a common eigenvector, we can suppose that they have the form: $M = \begin{pmatrix} m_1 & m_2 \\ 0 & m_3 \end{pmatrix}$ and $N = \begin{pmatrix} n_1 & n_2 \\ 0 & n_3 \end{pmatrix}$ If we choose $M_\epsilon = M$ and $N_\epsilon = \begin{pmatrix} n_1 & n_2 + \epsilon\alpha \\ \epsilon & n_3 + \epsilon\beta \end{pmatrix}$ then as the eigenvectors of N_ϵ are a polynomial combination of the element of N_ϵ, we can find α and β such that N_ϵ and M_ϵ have no common eigenvector.

□

In the next section, we show how it is still possible to define a concept of approximate common eigenvector that is more appropriate from a computational point of view. This new definition of the problem is based on the notion of the backward error and can be used to design practical algorithms.

3 Backward error framework

The backward error plays an important role in numerical linear algebra, as it provides an answer to the question:
Is the computed solution $\tilde{x}$, the exact solution of nearby problem?
To define the backward error for problem $(\mathcal{P})$, we consider the perturbed common eigenvector problem:

$$\begin{cases} (A + \Delta A)\tilde{x} = \tilde{\alpha}\tilde{x}, \\ (B + \Delta B)\tilde{x} = \tilde{\beta}\tilde{x}. \end{cases} \qquad (\tilde{\mathcal{P}})$$

In what follows $\|.\|$ stands for the spectral 2-norm, set:

$$\epsilon = \sqrt{\frac{\|\Delta A\|^2}{\|A\|^2} + \frac{\|\Delta B\|^2}{\|B\|^2}}.$$

Then we can define the backward error for the common eigenvector problem as follows.

Definition 2 *Let $\tilde{x}$ be an approximation of the solution of the problem $(\mathcal{P})$, the backward error η associated with $\tilde{x}$ is given by:*

$$\eta(\tilde{x}) = \min_{\tilde{\alpha},\tilde{\beta}} \min \left(\epsilon : \left\{ \begin{array}{c} (A+\Delta A)\tilde{x} = \tilde{\alpha}\tilde{x} \\ (B+\Delta B)\tilde{x} = \tilde{\beta}\tilde{x} \end{array} \right. \right).$$

The following theorem gives an explicit expression of $\eta(\tilde{x})$.

Theorem 3 *The backward error is given by:*

$$\eta(\tilde{x}) = \sqrt{\frac{\|A\tilde{x} - \frac{(\tilde{x}^T A\tilde{x})}{\|\tilde{x}\|^2}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \frac{(\tilde{x}^T B\tilde{x})}{\|\tilde{x}\|^2}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}}.$$

Proof

We denote

$$\eta(\tilde{\alpha},\tilde{\beta}) = \min \left(\epsilon : \left\{ \begin{array}{c} (A+\Delta A)\tilde{x} = \tilde{\alpha}\tilde{x} \\ (B+\Delta B)\tilde{x} = \tilde{\beta}\tilde{x} \end{array} \right. \right).$$

We have

$$\left\{ \begin{array}{c} (A+\Delta A)\tilde{x} = \tilde{\alpha}\tilde{x} \\ (B+\Delta B)\tilde{x} = \tilde{\beta}\tilde{x} \end{array} \right. \Rightarrow \left\{ \begin{array}{c} \|\Delta A\|\|\tilde{x}\| \geq \|\tilde{\alpha}\tilde{x} - A\tilde{x}\| \\ \|\Delta B\|\|\tilde{x}\| \geq \|\tilde{\beta}\tilde{x} - B\tilde{x}\| \end{array} \right. .$$

So:

$$\frac{\|\Delta A\|}{\|A\|} \geq \frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|}{\|A\|\|\tilde{x}\|}, \quad \frac{\|\Delta B\|}{\|B\|} \geq \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|}{\|B\|\|\tilde{x}\|},$$

consequently

$$\epsilon \geq \sqrt{\frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}},$$

and

$$\eta(\tilde{\alpha},\tilde{\beta}) \geq \sqrt{\frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}}.$$

For

$$\Delta A = \frac{(\tilde{\alpha}\tilde{x} - A\tilde{x})x^T}{\|\tilde{x}\|^2}, \Delta B = \frac{(\tilde{\beta}\tilde{x} - B\tilde{x})x^T}{\|\tilde{x}\|^2}$$

$$\|\Delta A\| = \frac{1}{\|x\|^2}\max_{z\neq 0}\frac{\|(\tilde{\alpha}\tilde{x} - A\tilde{x})x^T z\|}{\|z\|} = \frac{\|(\tilde{\alpha}\tilde{x} - A\tilde{x})\|}{\|x\|^2}\max_{z\neq 0}\frac{|\tilde{x}^T z|}{\|z\|} = \frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|}{\|\tilde{x}\|}$$

$$\|\Delta B\| = \frac{1}{\|x\|^2}\max_{z\neq 0}\frac{\|(\tilde{\beta}\tilde{x} - B\tilde{x})x^T z\|}{\|z\|} = \frac{\|(\tilde{\beta}\tilde{x} - B\tilde{x})\|}{\|x\|^2}\max_{z\neq 0}\frac{|\tilde{x}^T z|}{\|z\|} = \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|}{\|\tilde{x}\|},$$

we get

$$\frac{\|\Delta A\|}{\|A\|} = \frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|}{\|A\|\|\tilde{x}\|}, \quad \frac{\|\Delta B\|}{\|B\|} = \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|}{\|B\|\|\tilde{x}\|},$$

then

$$\eta(\tilde{\alpha},\tilde{\beta}) = \sqrt{\frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}}$$

and

$$\eta(\tilde{x}) = \min_{\tilde{\alpha},\tilde{\beta}} \left(\sqrt{\frac{\|A\tilde{x} - \tilde{\alpha}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \tilde{\beta}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}} \right).$$

It is known that for the least square problems we have

$$\min_{\lambda} \|u_1 + \lambda u_2\| = \|u_1 - \frac{u_2^T.u_1}{\|u_2\|^2} u_2\|,$$

then:

$$\eta(\tilde{x}) = \sqrt{\frac{\|A\tilde{x} - \frac{(\tilde{x}^T A\tilde{x})}{\|\tilde{x}\|^2}\tilde{x}\|^2}{\|A\|^2\|\tilde{x}\|^2} + \frac{\|B\tilde{x} - \frac{(\tilde{x}^T B\tilde{x})}{\|\tilde{x}\|^2}\tilde{x}\|^2}{\|B\|^2\|\tilde{x}\|^2}}.$$

□

We compute the backward error using the spectral norm for matrices. This result could be extended to handle other matrix norms such as the one and the infinity norms. In order to numerically compute the common eigenvector, we could for example use an algorithm that minimizes the backward error, and use the backward error as a stopping criterion for this algorithm.

4 Conclusion

We proved that the common eigenvector of two matrices is not computable in the presence of round-off errors, because the set of matrices that do not have a common eigenvector is dense in $M_n(\mathbb{C})^2$. By introducing the backward error associated with this problem, we reformulated the problem as an unconstrained minimization problem. An important practical issue for the future is the choice of a good optimization algorithm to minimize the backward error.

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