The dynamics of spectral analysis by Homotopic Deviation.

Part I The spectral field.

Françoise Chaitin-Chatelin [*]

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Abstract : We consider the complex linear coupling $A(t) = A + tE$ for $A, E \in \mathbb{C}^{n\times n}$ and $t \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The deviation matrix E has rank $r \leq n$. When $r < n$ the analysis of $\sigma(A(t))$ as $|t| \to \infty$ leads to the consideration of (at most) $n - r$ reduction matrices B_ξ of order n defined at each ξ in a subset H of the closed frontier set $\bar{F}(A, E)$.

The forward information flow from $\sigma(A)$ to $\sigma(B_\xi)$ for all $\xi \in H$ (as $|t| \to \infty$) is complemented by the backward flow which results from all the couplings $B_\xi(s) = B_\xi + sE$ for $s \in \hat{\mathbb{C}}$, $st = 1$ ($|t| \to 0 \Leftrightarrow |s| \to \infty$). Then the original spectrum $\sigma(A)$ is compared with the spectral information contained in the coupling (A, E).

Keywords : Complex coupling, synthesis, Homotopic Deviation, complex intensity, observation point, communication matrix, homotopic polynomial, frontier set, frontier multiplicity, homotopic multiplicity, normwise observability, spectral observability.

1 An introduction to Homotopic Deviation

We are given a deviation matrix $E \in \mathbb{C}^{n\times n}$ of rank $r \leq n$. Given $A \in \mathbb{C}^{n\times n}$ with spectrum $\sigma(A)$, we are interested in the transformation of $\sigma(A)$ under the **complex**

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[*] Université Toulouse 1 and CERFACS,42 avenue G. Coriolis 31057 Toulouse Cedex 1, France. `E-mail: chatelin@cerfacs.fr`

coupling of A and E. The matrix A is the original source of spectral information. Together with A, the deviation E produces the coupling (A, E) which contains the resulting spectral information $\sigma(A, E)$. The original spectrum $\sigma(A)$ is to be compared to $\sigma(A, E)$.

The coupling (A, E) is, by assumption, taken to be *linear* of the form $A(t) = A + tE$, where $t \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The parameter t expresses the *intensity* of the coupling; it varies in the *closed complex* plane. The point at ∞ defines the completed coupling denoted $A(\infty)$. Observe that the notation $A(\infty)$ represents the concept of *synthesis* between A and E, and not a matrix. This idea can be made explicit. Consider $st = 1$, $s = 1/t \in \hat{\mathbb{C}}$, then $A(t) = A + tE = t(E + sA) = \frac{E(s)}{s} \Leftrightarrow E(s) = \frac{A(t)}{t}$ for any $s, t \in \mathbb{C}\backslash\{0\}$ related by $st = 1$.

The analysis of $\sigma(A(t))$ is based on the following *homotopic* factored form for the resolvent $(A(t) - zI)^{-1} = R(t, z)$, where z is an *observation* point chosen in $re(A) = \mathbb{C}\backslash\sigma(A)$:

$$R(t, z) = (A - zI)^{-1}[I + tE(A - zI)^{-1}]^{-1} \tag{H}$$

The theory is non trivial when the deviation matrix E is singular with $r = \text{rank}\, E < n$. The two factors $U, V \in \mathbb{C}^{n\times r}$ (with rank r) for the SVD representation $E = UV^H$ play an essential role. For $r < n$, they define *three* levels at which the spectral information is processed by matrix computation. These levels correspond to the orders $n + r$, n and r of the intervening matrices, with $1 \leq r < n < n + r$. The complete analysis consists of three steps:

1- The first step considers $A(t) = A + tE$ and the limit of $\sigma(A(t))$ as $|t| \to \infty$. This step creates at most $n - r$ secondary sources of information as matrices B_ξ of order n and rank $\leq r$, defined at the points $\xi \in re(A)$ where $\lim_{|t|\to\infty} R(t, \xi)$ does not exist. Such points constitute the frontier set $F(A, E) \subset re(A)$.

2- The second step extends the definition of the frontier set in $\mathbb{C}$. This defines $\bar{F}$. When this is possible, B_ξ is defined at certain eigenvalues of A in $H \subset \overline{F}$

3- In the third step, at each $\xi \in H$, the matrix B_ξ is coupled with E by $B_\xi(s) = B_\xi + sE$ and $st = 1$. As $|s| \to \infty$, this realises a backward coupling $|t| \to 0$, or equivalently, a decoupling betwen A and E.

At this point, the original spectral information $\sigma(A)$ can be compared with the spectral information $\sigma(A, E)$ produced by the coupling (A, E). But more is potentially happening, which leads to complement the explicit spectral field $t \mapsto \sigma(A(t))$ by the *implicit* field of evolution. This field is obtained by replacing the pencil $A - zI$ by the quadratic polynomial $z^2E - z(EA + AE) + AEA$. Part II is devoted to this study.

Part I of this report presents all three steps about the spectral field. The last two steps are new [9]. The first step has already been presented in various works [1, 3, 5, 6, 7, 8, 10]. Therefore we only give a brief survey in the next Section

2 A survey of Homotopic Deviation on $A(t) = A + tE$ for $z \in re(A)$, $t \in \hat{\mathbb{C}}$.

The theory uses several new algebraic tools which are motivated by the properties of $R(t, z)$ as $|t| \to \infty$. It is understood that $R(t, z)$ is expressed in the homotopic form (H), which restricts z to $re(A) = \mathbb{C}\backslash\sigma(A)$.

Let $\pi(z) = \det(zI - A)$ be the characteristic polynomial for A. Similary, $\pi(t, z) = \det(A - zI + tE)$, $\pi(0, z) = (-1)^n\pi(z)$.

For $r \leq n$, and $z \in \mathbb{C}$, we consider the augmented matrix $\hat{A}(z) = \begin{pmatrix} zI - A & -U \\ V^H & 0 \end{pmatrix}$ of order $n + r \leq 2n$.

Definition 2.1 [8] $\hat{\pi}(z) = \det \hat{A}(z)$ *is the homotopic polynomial for* (A, E).

This polynomial has degree $\hat{d}$, $0 \leq \hat{d} \leq n - r = g =$ geometric multiplicity of $0 \in \sigma(E)$, with $\hat{d} = 0$ necessarily when $r = n$. The role of this polynomial $\hat{\pi}$ for $R(t, z)$ will be the analogue for (A, E) to that of π for $R(0, z) = (A - zI)^{-1}$ in the absence of deviation $(t = 0)$.

We set $\hat{Z} = \{z \in \mathbb{C}\ ;\ \hat{\pi}(z) = 0\}$.

We also introduce the ratio $\theta(z) = \dfrac{\hat{\pi}(z)}{\pi(z)}$. The rational function $\theta(z)$ may not be defined for $\lambda \in \sigma(A)$.

2.1 New algebraic tools for $z \in re(A)$

For $z \notin \sigma(A)$ we define the *communication* matrix [5]:

$$M_z = V^H(zI - A)^{-1}U.$$

Its eigenvalues are $\mu_{iz} \in \sigma(M_z)$, $i = 1, \cdots, r$. And $\det M_z = \theta(z) = \prod_{i=1}^{r} \mu_{iz}$, $\det(I - tM_z) = \prod_{i=1}^{r}(1 - t\mu_{iz}) = (-1)^n \dfrac{\pi(t, z)}{\pi(z)}$, [7].

For $r < n$, the *frontier* set [6] is $F(A,E) = \{z \in re(A);\ \theta(z) = 0\}$. It contains the *critical* set $F_c(A,E) = \{z \in F(A,E);\ \rho(M_z) = 0\}$ which can be empty. For $r = 1$, $F = F_c$.

Definition 2.2 *For* $st = 1$, *the pencil* $Q(z) = E + s(A - zI)$ *is the reverse pencil for* $P(z) = A - zI + tE = tQ(z)$ *for* $t \in \mathbb{C}$.

$\det P(z) = \pi(t,z)$. We set $\det Q(z) = \psi(s,z)$. Thus $\psi(s,z)$ is the revere polynomial for $\pi(t,z)$.

Proposition 2.1 *For* $z \in re(A)$, $\pi(t,z)$ *is a polynomial in* t *of degree* e_z *such that*

i) $e_z = r$ *for* $z \in re(A) \backslash F(A,E)$.

ii) $e_z = r - a_z$ *for* $z \in F(A,E)$, *where* a_z, $1 \le a_z \le r$, *is the algebraic multiplicity of* $0 \in \sigma(M_z)$.

Proof. e_z is the exponent of the dominant power of t in $\pi(t,z)$. Because $\pi(t,z) = (-1)^n \pi(z) \prod_{i=1}^{r} (1 - t\mu_{iz})$ for $z \in re(A)$, it is clear that $e_z = r$ for $z \notin (F(A,E) \cup \sigma(A))$. This is true almost everywhere in $\mathbb{C}$ if $\hat{\pi} \not\equiv 0$, that is, if $F(A,E) \neq re(A)$. ☐

Lemma 2.2 $\pi(t,z) = t^{e_z} \psi(s,z)$ *for* $z \in re(A)$.

Proof. Follows from Proposition 2.1. ☐

2.2 Characterization of $F(A,E)$

The following result [8] is crucial:

$$F(A,E) = \hat{Z} \cap re(A).$$

For $1 \le \hat{d} \le g$, the frontier set is discrete as is $\hat{Z}$.

When $\hat{d} = 0$, the nature of $\hat{Z}$ depends whether $\hat{\pi}(0) = \det \begin{pmatrix} -A & -U \\ V^H & 0_r \end{pmatrix} \neq 0$ or not. For $r = n$, $\hat{\pi}(0) = \det U \ \det V^H \neq 0$, and $\hat{Z} = \emptyset$ confirms that $F(A,E) = \emptyset$. When $r < n$, then $\hat{Z} = \mathbb{C}$ ($\hat{\pi}(0) = 0$) or $\emptyset$ ($\hat{\pi}(0) \neq 0$) yields $F = re(A)$ or $\emptyset$. The exceptional case $\hat{d} = 0$ can be interpreted further when $0 \notin \sigma(A)$. $\det \begin{pmatrix} -A & -U \\ V^H & 0 \end{pmatrix} = \Big(\det(-A)\Big) \det(-V^H A^{-1} U)$. Therefore $\hat{Z} = \emptyset$ iff $\hat{d} = 0$ and $V^H A^{-1} U$ is invertible; $\hat{Z} = \mathbb{C}$ otherwise ($\hat{d} = 0$ and $0 \in \sigma(V^H A^{-1} U)$).

Unless otherwise stated, we assume below that $r < n$ and $\hat{d} \ge 1$, so that $\hat{Z}$ is neither $\mathbb{C}$ nor $\emptyset$.

2.3 Existence and analyticity of $R(t,z)$ for $z \in re(A)$

For $z \in re(A) \backslash F(A,E)$, $R(t,z)$ is analytic in t around $0(|t| < 1/\rho(M_z)))$ and around $\infty(|t| > \rho(M_z^{-1})))$. Moreover $\lim\limits_{|t|\to\infty} R(t,z) = R(\infty,z) = R(0,z)[I + UM_z^{-1}V^H R(0,z)]$ is nonzero for $r < n$: the achieved coupling $A(\infty)$ produces an effet at z. Observe that for $r = n$, $F(A,E) = \emptyset$ and $R(\infty,z) = 0$, since $[I + UM_z^{-1}V^H R(0,z)] = [I - UM_z^{-1}M_zU^{-1}] = 0$.

For $z \in F(A,E)$, the existence of $R(\infty,z)$ vanishes. In particular, when z is critical, $R(t,z)$ is a matrix polynomial in t of degree $\leq r$, defined for $t \in \mathbb{C}$. By construction, $R(t,z)$ has a pole of order r at $|t| = \infty$.

The function $z \mapsto R(z,t)$ is meromorphic in $re(A)$, for $t \in \hat{\mathbb{C}}$. Its singularities for $|t| = \infty$ are the zeros in $\sigma(M_z) \cap re(A)$. The Cauchy integral (defined for z around $\xi \in F(A,E)$ and for $|t|$ large enough) of the function $R(t,z)$ will be considered in Section 3. This will extend to (A,E), in several aspects, the classical notion of spectral projection for A [4].

2.4 The algebraic reduction (top-down n to r at $z \in re(A)$)

It is clear that, when $\rho(M_z) = 0$, $\pi(t,z) = \det(A + tE - zI_n) \neq 0 \quad \forall t \in \mathbb{C}$. A critical observation point z cannot be an eigenvalue of any $A(t)$, $t \in \mathbb{C}$: it repels the spectral field $t \mapsto \sigma(A(t))$.

On the other hand, any non critical z given in $re(A)$ is an eigenvalue $\lambda(t)$ in $\sigma(A(t))$ for $t \in \mathbb{C}$ iff $t\mu_z = 1$ for some $0 \neq \mu_z \in \sigma(M_z)$.

This result allows us to connect the computation of $\sigma(A(t))$ at level n and the computation of $\sigma(M_z)$ at level r by means of $t \in \hat{\mathbb{C}}$. Any observation point $z \in re(A)$ is an eigenvalue of r matrices $A(t_i)$ where $t_i = 1/\mu_{iz} \in \hat{\mathbb{C}}$, $\mu_{iz} \in \sigma(M_z)$, $i = 1, \cdots, r$.

2.5 $\lim\limits_{|t|\to\infty} \sigma(A(t)) = \{\infty, \mathrm{Lim}\}$ for $r < n$.

When $r = n$, all eigenvalues $\lambda(t)$ in $\sigma(A(t))$ escape to ∞.

When $r < n$, some eigenvalues $\lambda(t)$ may have a limit in $\mathbb{C}$ as $|t| \to \infty$. The set Lim is the set of these limits. How can we characterize Lim ?.

Let us consider $s = 1/t$, $st = 1$ for s and $t \in \hat{\mathbb{C}}$. $A(t) = A + tE = t(E + sA) = tE(s) = \frac{1}{s}E(s)$. Therefore $\lambda(t) \in \sigma(A(t))$ and $\alpha(s) \in \sigma(E(s))$ are related as $|t| \to \infty$ by

$$\lambda(t) = \frac{\alpha(s)}{s}, \qquad |s| = \frac{1}{|t|} \to 0.$$

The numbers $\xi \in \mathbb{C}$ in Lim correspond to the zero eigenvalues $\alpha(0)$ in $\sigma(E)$ which are such that $\alpha(s) = \xi s + o(s)$: $\alpha(s)$ converges to $0 = \alpha(0)$ with order 1 ($\xi \neq 0$) or order > 1 ($\xi = 0$). And $\xi = \alpha'(0) \in \mathbb{C}$.

The numbers in Lim can be characterized by the

Proposition 2.3 $\xi \in \text{Lim} \iff \xi = \frac{\mathrm{d}\,\alpha}{\mathrm{d}\,s}(0) \in \mathbb{C} \iff 0$ *is an eigenvalue of the reverse pencil* $Q(\xi) = E + s(A - \xi I) \iff \psi(0, \xi) = 0$.

Proof. $\alpha(s) - \xi s - o(s) \in \sigma(E + s(A - \xi I)$ for s small enough implies that $\psi(0, \xi) = 0$. $\square$

When $\xi \in \text{Lim}$, we say that ξ is an eigenvalue of $A(\infty)$. Lim may be non empty when $r < n$. This may be the case when there exist zero eigenvalues of E which have (trivial) Jordan blocks of dimension 1. These simple eigenvalues are called *trivial*. They may not always exist, and in that case, no theoretical prediction about Lim is available to-day.

What can be said about Lim when trivial eigenvalues for E do exist?

The fundamental works of Puiseux (1850) and Lidskii (1965) have paved the way towards a computational approach [6,8]. We set $g = g_1 + g_2 = n - r$, where g_1 (resp. g_2) represents the number of Jordan blocks for $0 \in \sigma(E)$ of dimension 1 (resp ≥ 2).

We assume that $g_1 \geq 1$. This is verified when 0 is semi-simple: $g = g_1$ (assumption $(\Sigma) \iff \det V^H U \neq 0$). But, in general, when 0 is defective (case $(\Delta) \iff \det V^H U = 0 \iff g_2 > 0$) it is possible that $g_1 = 0$: all Jordan blocks may have a dimension ≥ 2.

When $g_1 \geq 1$ we denote by P_1 the eigenprojection for E on the eigenspace Im P_1 spanned by the g_1 eigenvectors defining the trivial blocks. We set $\Pi_1 = P_1 A P_{1 \restriction \text{Im } P_1}$ which is a matrix of order g_1. We define the *kernel* matrix [6]

$$\tilde{\Pi}(z) = \begin{pmatrix} \Pi_1 - z I_{g_1} & R \\ L & \Gamma \end{pmatrix}$$

of order g. The 4 blocks Π_1, Γ, L and R are derived from the data A and the left and right eigenvectors for $0 \in \sigma(E)$.

Let P_g represent the eigenprojection on Ker E. Then $\tilde{\Pi} = \begin{pmatrix} \Pi_1 & R \\ L & \Gamma \end{pmatrix}$ of order g represents the map $P_g A P_{g \restriction \text{Ker } E}$, partitionned into four blocks according to $g = g_1 + g_2$. Moreover, the g_2 eigenvectors starting non trivial blocks are ordered by non decreasing dimension of their block.

Finally, we define the *kernel* polynomial $\tilde{\pi}(z) = \det \tilde{\Pi}(z)$ of degree $\tilde{d}$, $0 \leq \tilde{d} \leq g_1$. Set $\tilde{Z} = \{z \in \mathbb{C}, \tilde{\pi}(z) = 0\}$, card $\tilde{Z} = \tilde{d}$ for $\tilde{d} \geq 1$. When $\tilde{d} = 0$, either $\tilde{Z} = \emptyset$ $\left(\det \begin{vmatrix} \Pi_1 & R \\ L & \Gamma \end{vmatrix} \neq 0\right)$ or $\tilde{Z} = \mathbb{C}$ ($\tilde{\pi} \equiv 0$). In the latter case, consequences for Lim are new. For all g_1 trivial eigenvalues $\alpha(s)$, the derivative $\alpha'(0)$ can take an *arbitrary* value $\xi \in \mathbb{C}$. The situation is beyond the algebraic perturbation analysis of Puiseux. And the result $\tilde{Z} = \mathbb{C} \supset \mathrm{Lim}$ does not bring any useful information about Lim . Information about Lim $\cap\, re(A)$ can come from $F(A, E)$ only.

One can show [8] that for $\tilde{d} \geq 1$, Lim $\supset \tilde{Z}$ under the assumption $g_1 \geq 1$. When $g_1 = 0$, $\tilde{Z}$ does not exist, and there is no available theory to-day to predict Lim ; each case has to be treated specifically.

Two remarks are in order about $\tilde{\Pi}(z)$.

1. When $g = g_1$ (Σ), then Γ does not exist and Π_1 becomes the Galerkin projection $\Pi = PAP_{\restriction \mathrm{Ker}\, E}$, where $P = P_g$ is the total eigenprojection for E on Ker E associated with the semi-simple 0 [5, 10].

2. When $g_2 \geq 1$ (Δ) and when Γ is invertible $(\det \Gamma \neq 0)$, then $\tilde{d} = g_1$. $\tilde{Z}$ can be interpreted as the spectrum of $\Omega = \Pi_1 - L\Gamma^{-1}R$: Ω is the Schur complement of Γ in $\tilde{\Pi}(0)$. This interpretation was given by Moro et al. (1997, see [6, 8]).

   The matrix Ω can be written under the form $\Omega = P_1AQP_{1\restriction \mathrm{Im}\, P_1}$, where Q is a projection matrix of rank $n-1$ which entangles the spectral information in A and E in a complex fashion, due to the defectiveness of $0 \in \sigma(E)$. When 0 is semi-simple, then $Q = I_n$ and $P_1 = P$, thus $\Omega = \Pi$.

Definition 2.3 *When min*$(g_1, \tilde{d}) \geq 1$*, the non empty set* $\tilde{Z}$ *defines the kernel set of* (A, E).

The role of the kernel polynomial $\tilde{\pi}$ (which is defined when $g_1 \geq 1$) is to extract simplicity (via Π_1 of order $g_1 \leq g$) from a larger matrix $\tilde{\Pi}$ of order $g = g_1 + g_2$.

The identity $\tilde{Z} =$ Lim holds always under (Σ). It holds generically when Γ is invertible $(\tilde{d} = g_1)$. Non generically, the inclusion $\tilde{Z} \subset$ Lim holds when $\min(g_1, \tilde{d}) \geq 1$. Finally $\tilde{Z}$ does not exist when $g_1 = 0$.

We set $K(A, E) = \tilde{Z} \cap re(A)$ and $\Lambda(A, E) =$ Lim $\cap\, re(A)$. We study further the inclusions $\Lambda \subset F$ and $K \subset \Lambda$ $(\min(g_1, \tilde{d}) \geq 1)$ which are valid in $re(A)$ in the Section 3.

We mention for future reference the following result [3, 6]:

Proposition 2.4 *When* $F(A, E) = re(A)$*, then either i)* $F = F_c$ *and* $\Lambda = \emptyset$, $\sigma(A) = \sigma(A(t)) =$ Lim *for all* $t \in \mathbb{C}$*, or ii)* F_c *is discrete, consisting of at most* $n-1$ *distinct points in* $\mathbb{C}$.

3 The two homotopic inductions (bottom-up r to n and r to $\hat{n} = n + r$) at $\xi \in F(A, E)$

3.1 The frontier reduction matrix B_ξ at $\xi \in F(A, E)$

Let ξ be a given frontier point in $F(A, E) = F$ isolated by a Jordan curve Γ traced in $re(A) \backslash F$.

Definition 3.1 *The frontier reduction matrix associated with* (A, E) *at* $\xi \in F(A, E)$ *is defined by*

$$B_\xi = \frac{-1}{2i\pi} \int_\Gamma R(\infty, z) dz.$$

Let $P_{0\xi} = P_0(M_\xi)$ represent the spectral projection for M_ξ associated with $0 \in \sigma(M_\xi)$. Its rank is a_ξ, the algebraic multiplicity of $0 \in \sigma(M_\xi)$, $1 \leq a_\xi \leq r$.

Lemma 3.1 $B_\xi = R(0, \xi) U P_{0\xi} V^H R(0, \xi)$

Proof. Follows from the formula for $R(\infty, z)$. $\square$

Proposition 3.2 *For* $|t| > \frac{1}{m_\Gamma}$ *with* $m_\Gamma = \min\limits_{z \in \Gamma} |\mu_{iz}| = \left(\max\limits_{z \in \Gamma} \rho(M_z^{-1}) \right)^{-1}$, $B_\xi = \frac{-1}{2i\pi} \int_\Gamma R(t, z) dz$.

Proof. For $t \in \mathbb{C} \backslash \{0\}$, $R(t, z) = R(0, z)[I_n - U(\frac{1}{t} I_r - M_z)^{-1} V^H R(0, z)$ and $\sigma(M_z - \frac{1}{t} I_r) = \{\mu_{iz} - 1/t,\ i = 1, \cdots, r\}$. For $|t| > \max\limits_{z \in \Gamma} \rho(M_z^{-1})$, $R(t, z)$ is analytic in t for $z \in \Gamma$. The result follows classically [4]. $\square$

3.2 The inductive role of the homotopic polynomial $\hat{\pi}$

It appears that the frontier reduction B_ξ generalizes for the coupling (A, E) the notion of spectral projection for A. The zeros of $\hat{\pi}$ in $re(A)$ where $R(\infty, z)$ does not exist replace the zeros of π where $R(0, z)$ does not exist.

Even though the homotopic polynomial $\hat{\pi}$ has degree $\hat{d} \leq g < n$, it represents the determinant of the *augmented* matrix $\hat{A}(z)$ of order $\hat{n} = n + r > n > r$. An intermediate induction from r to n is made possible at each $\xi \in F(A, E)$ by the matrix B_ξ which is a secondary source of spectral information. It is a source of algebraic induction which does not necessarily use any direct information from $\hat{\pi}$ or

$\hat{A}(z)$. Rather, it may use only the information provided by M_ξ, using the coincidence in $re(A)$

$$0 \in \sigma(M_\xi) \Longleftrightarrow \hat{\pi}(\xi) = 0.$$

We shall see some computational consequences later.

Even though the polynomial $\hat{\pi}$ is only implicit for B_ξ, its role is fundamental. For example, its degree $\hat{d}$, $0 \leq \hat{d} \leq n - r$ is an important parameter. When $r = n$, then $\hat{d} = 0$ and $F = \emptyset$: there cannot exist any source of induction. When $r < n$ the case $\hat{d} = 0$ is exceptional: induction is possible either nowhere or at any $z \in re(A) = F$. In general, $1 \leq \hat{d} \leq n - r < n$ and there exists a finite number of sources of induction.

We expect that the situation will vary with the respective sizes of the 4 parameters $\hat{d}, \ g = n - r, \ r$ and $\left[\frac{n}{2}\right]$. There are 3 possibilities:

- for $r \leq \left[\frac{n}{2}\right]$, then $r < \hat{d} \leq g$, or $\hat{d} \leq r \leq g$,
- for $r > \left[\frac{n}{2}\right]$, then $\hat{d} \leq g < r$.

A complete analysis is beyond the scope of this report.

3.3 The structure of B_ξ

We have defined a_ξ, $1 \leq a_\xi \leq r$, to be the *algebraic* multiplicity of $0 \in \sigma(M_\xi)$ for $\xi \in F(A, E)$. Similary, let $\hat{m}_z$, $0 \leq \hat{m}_z \leq \hat{d}$, be the multiplicity of z as a root of $\hat{\pi}$ in $\hat{Z}$.

For $\hat{d} \geq 1$, then $\hat{m}_z \geq 1$. For $\hat{d} = 0$, then $\hat{m}_z$ is either 0 ($\hat{Z} = \emptyset$) or $\neq 0$ for any $z \in \mathbb{C}$. For $\xi \in F(A, E)$ which is assumed to be discrete nonempty, $\hat{m}_z \geq 1$ represents the *homotopic* multiplicity of ξ as a root of $\hat{\pi}$ in $re(A)$. This is the case when $\hat{d} \geq 1$. Without further assumption on $\xi \in F(A, E) \neq \emptyset$ we have $\min(a_\xi, \hat{m}_\xi) \geq 1$.

We set $U_\xi = R(0, \xi) U P_{0\xi}$ and $V_\xi^H = P_{0\xi} V^H R(0, \xi)$, where each matrix has a rank between 0 and a_ξ. Therefore $B_\xi = U_\xi V_\xi^H$ has rank δ_ξ, $0 \leq \delta_\xi \leq a_\xi$.

Definition 3.2 *The frontier multiplicity of $\xi \in F(A, E)$ is $\delta_\xi = rank\, B_\xi$.*

How are the frontier and homotopic multiplicities δ_ξ and $\hat{m}_\xi$ of ξ related? In the most general situation, when the induction at ξ stays at the intermediate level n, they are *not* related a priori. Any of the following 3 possibilities may be the case:

$$0 \leq \delta_\xi < \hat{m}_\xi, \ \ \delta_\xi = \hat{m}_\xi \geq 1, \ \ \delta_\xi > \hat{m}_\xi \geq 1.$$

Definition 3.3 *The induction power of B_ξ is weaker than (resp. equal to, stronger than) that of $\hat{\pi}(\xi) = 0$ iff $0 \leq \delta_\xi < \hat{m}_\xi$ (resp. $\delta_\xi = \hat{m}_\xi$, $\delta_\xi > \hat{m}_\xi$). When $\delta_\xi = \hat{m}_\xi$, the intermediate induction $r \mapsto n$ is said to be exact. Otherwise it is weak or strong.*

When the intermediate induction is exact the dimension δ_ξ of Im B_ξ is exactly equal to $\hat{m}_\xi$, the multiplicity of ξ in $\hat{\pi}(z)$. The cases $\delta_\xi \geq \hat{m}_\xi$ forbid that $\delta_\xi = 0$, that is $B_\xi = 0$.

Lemma 3.3 *B_ξ is nilpotent iff $\mathcal{G}_\xi = P_{0\xi} V^H R(0,\xi)^2 U P_{0\xi}$ of order r is nilpotent.*

Proof. $\sigma(B_\xi) = \{0\} \cup \sigma(\mathcal{G}_\xi)$, with $\mathcal{G}_\xi = V_\xi^H U_\xi$. A nilpotent B_ξ is 0 when it is semi-simple. ☐

The matrix B_ξ is not necessarily idempotent because, in general, $\sigma(\mathcal{G}_\xi) \neq \{0, 1\}$, and B_ξ may even be nilpotent. Not surpringly, the spectral analysis of (A, E) is much *richer* than the classical one for A [4].

When ξ is critical, $a_\xi = r$ and $P_{0\xi} = I_r$. Therefore $B_\xi = R(0,\xi)ER(0,\xi)$ with $\delta_\xi = r$. Exact intermediate induction at ξ critical is possible only when $r \leq \hat{d}$, hence $r \leq [n/2]$.

3.4 The inclusion $\Lambda(A, E) \subset F(A, E)$

Definition 3.4 *The actual multiplicity α_ξ of $\xi \in F(A, E)$ is given by the number (according to multiplicity) of eigenvalues $\lambda(t) \in \sigma(A(t))$ which converge to ξ as $|t| \to \infty$.*

Lemma 3.4 *α_ξ is the multiplicity of 0 as an eigenvalue of $Q(\xi) = E + s(A - \xi I) \Longleftrightarrow \psi(0, \xi) = 0$.*

Proof. Clear by Proposition 2.3. ☐

Corollary 3.5 *$\xi \in \Lambda(A, E)$ iff*

$$\lim_{|t| \to \infty} t^{-e_\xi} \pi(t, \xi) = \lim_{s \to 0} \psi(s, \xi) = \psi(0, \xi) = 0$$

Proof.
$$\begin{aligned} \pi(t,\xi) &= \det(tQ(\xi)) = t^{e_\xi} \det Q(\xi). \\ &= t^{e_\xi} \psi(s, \xi) \quad \text{by Lemma 2.2.} \end{aligned}$$
$e_\xi = r - a_\xi$, $\xi \in \Lambda \subset F \subset \hat{Z}$. ☐

The rate of the growth at $|t| = \infty$ for $\pi(t, \xi)$ is less than t^{e_ξ} when $\xi \in \Lambda(A, E) = \text{Lim} \setminus \sigma(A)$. When $\xi \in \Lambda(A, E) \cap F_c$ is a critical point, $e_\xi = 0$ and $\lim\limits_{|t| \to \infty} \pi(t, \xi) = 0$. The following is clear:

$$\xi \in \Lambda(A, E) \Longleftrightarrow \alpha_\xi \geq 1 \quad \text{and} \qquad \xi \in F \setminus \Lambda \Longleftrightarrow \alpha_\xi = 0.$$

Proposition 3.6 $0 \leq \alpha_\xi \leq \min(\delta_\xi, \hat{m}_\xi)$ *for* $\xi \in F(A, E) \neq \emptyset$.

Proof. The inequality $\alpha_\xi \leq \hat{m}_\xi$ is clear since $F \neq \emptyset$. The second inequality is proved in 2 steps:

1) $\alpha_\xi \leq a_\xi$ follows from the representation $A - zI + tE = [I + tER(0, z)](A - zI)$ for $z \in re(A)$.

2) $\alpha_\xi \leq \delta_\xi$ follows from the lift of spectral information realized by B_ξ and based on $P_{0\xi}$. ☐

Proposition 3.7 *At any* $\xi \in \Lambda(A, E)$, $1 \leq \alpha_\xi = \delta_\xi \leq \hat{m}_\xi$.

Proof. Direct consequence of Proposition 3.2 for $\alpha_\xi \geq 1$. The total multiplicity α_ξ of eigenvalues $\lambda(t)$ close to ξ for $|t|$ large enough equals δ_ξ. ☐

Corollary 3.8 *At any* $\xi \in \Lambda(A, E)$, *intermediate induction is 2-fold, either exact or weak. It is 3-fold at any* $\xi \in F \setminus \Lambda$, *either strong, exact or weak.*

3.5 About the kernel set $K(A, E)$ inside $\Lambda(A, E)$

When $\min(g_1, \tilde{d}) \geq 1$, the kernel set $\tilde{Z}$ is not empty inside Lim . If $K(A, E) = \tilde{Z} \cap re(A)$ is itself non empty, it contains limit points in $\Lambda(A, E)$, therefore

$$K(A, E) \subset \Lambda(A, E) \subset F(A, E).$$

Obviously, $\alpha_\xi \geq 1$ at the kernel points: they are the roots of $\tilde{\pi}(z)$ which lie inside $\Lambda \subset re(A)$.

We assume that $\tilde{Z}$ and $\hat{Z}$ are neither empty nor continuous. Both are supposed to be discrete. We set $d_0 = \text{card}\, K(A, E)$, $0 \leq d_0 \leq \tilde{d}$ for $\tilde{d} \geq 1$.

Proposition 3.9 *When* $\tilde{Z}$ *and* $\hat{Z}$ *are discrete, the polynomials* $\tilde{\pi}(z)$ *and* $\hat{\pi}(z)$ *have exactly* d_0 *common roots in* $re(A)$.

Proof. Clear from the inclusions $K \subset \Lambda \subset F$. We recall that $0 \leq d_0 \leq \min(\tilde{d}, \hat{d})$ with $\tilde{d} \leq g_1 \leq g$ and $\hat{d} \leq g$. $\square$

More can be said under stronger assumptions.

I) We suppose first that $0 \in \sigma(E)$ is semi-simple (Σ). Then $\text{Lim} = \sigma(\Pi)$, $\tilde{d} = \hat{d} = g$ and $K(A,E) = \Lambda(A,E)$. One shows [5, 10] that for $\xi \in K = \Lambda \subset F$, $\gamma_\xi = \dim \text{Ker } M_\xi = \dim \text{Ker } (\Pi - \xi I_g)$.

This result sheds light on the *geometric* aspect of the transmission of spectral information which takes place at ξ under (Σ).

There are two useful consequences about *critical* points ξ where M_ξ is nilpotent, with degree nilpotency ν_ξ, $1 \leq \nu_\xi \leq r$.

1. If ξ is critical then
   - i) $r \leq \left[\frac{n}{2}\right] \Longrightarrow 1 \leq \nu_\xi \leq r$,
   - ii) $r > \left[\frac{n}{2}\right] \Longrightarrow 2 \leq \nu_\xi \leq r$, and M_ξ is necessarily defective ($M_\xi \neq 0$).
2. If $r \leq \left[\frac{n}{2}\right]$ and $\xi \in K(A,E)$ with $\gamma_\xi = r$, then ξ is critical with $\nu_\xi = 1$. Hence $M_\xi = 0$ and $\gamma_\xi = \delta_\xi = \alpha_\xi = r$.

An illustration of item 2 is given in [10] where the coupling parameter is the complex admittance in an acoustic problem.

II) When $0 \in \sigma(E)$ is defective, the theory is much less advanced. Assuming that Γ is invertible, Ω exists, and $\tilde{d} = g_1$, $\hat{d} < g$. A preliminary result [8] identifies γ_ξ with $\dim \text{Ker } (\Omega - \xi I)$, in the case $\gamma_\xi = 1$, and $\xi \in K(A,E) = \sigma(\Omega) \cap re(A)$.

In general, when Γ is singular, the conditions which rule the values of $\tilde{d} < g_1$ and $\hat{d} < g$ are based on two different sets of data (Π_1, L, Γ, R for $\tilde{d}$, and $A, U, 0, V$ for $\hat{d}$).

Proposition 3.10 *If $\tilde{d} > \hat{d} \geq 1$, then $\tilde{Z} \cap \sigma(A) \neq \emptyset$.*

Proof. We use the two inclusions $\tilde{Z} \subset \text{Lim}$ and $\Lambda \subset F$. There cannot exist more than $\hat{d}$ points in Λ. Therefore at least $\tilde{d} - \hat{d}$ points in $\tilde{Z}$ are eigenvalues of A. $\square$

Proposition 3.11 *Under the condition $\tilde{d} > \hat{d} = 0$, exactly one of the following two statements is the case for $r > 1$:*

i) $\tilde{Z} \subset \sigma(A)$, or

ii) $\tilde{Z} \subset \text{Lim}$, when $F = re(A) \neq F_c$ discrete.

	$F(A,E)=\emptyset$	$F(A,E)=re(A)$	
		$F=F_c$	$r>1,\ F\neq F_c$
$\tilde{Z}=\emptyset$	$\tilde{Z}=\Lambda=F=F_c=\emptyset$ Lim $\subset\sigma(A)$	$\tilde{Z}=\Lambda=\emptyset$ Lim $=\sigma(A)$	$\Lambda\subset F$
$\tilde{Z}=\mathbb{C}$	Lim $\subset\sigma(A)\subset\mathbb{C}$	Lim $\subset\sigma A\subset\mathbb{C}$	$\Lambda\subset F$
	no induction	induction everywhere in *re*(A)	

Table 3.1: $\tilde{d}=\hat{d}=0$

Proof. We suppose that $\hat{d}=0$. If $\hat{Z}=\emptyset$ then $\Lambda=\emptyset$ and $\tilde{Z}\subset\sigma(A)$. If $\hat{Z}=\mathbb{C}$, then $F=re(A)$. Either $F_c=re(A)$ and $\sigma(A)=\text{Lim}$ so that $\tilde{Z}\subset\sigma(A)$. Or F_c is discrete and $\tilde{Z}\subset\text{Lim}$. Case *i*) corresponds to $\hat{\pi}(0)\neq 0$, or to $\hat{\pi}(0)=0$ with $F=F_c$. $\square$

When $r=1$, the identity $F=F_c$ entails that the possibility *ii*) does not exist. We end this paragraph by looking at the coincidence $\tilde{d}=\hat{d}=0$. The new feature comes from the possibility that $\hat{Z}=\mathbb{C}$, which does not inform about Lim . The 6 possibilities are listed in the Table 3.1

3.6 The matrix pencils associated with $\hat{\pi}$ and $\tilde{\pi}$

The augmented matrix $\hat{A}(z)$ is the homotopic matrix pencil of order $\hat{n}=n+r$

$$\hat{A}(z)=\begin{pmatrix}-A & -U\\ V^H & 0\end{pmatrix}+z\begin{pmatrix}I_n & 0\\ 0 & 0_r\end{pmatrix}=\hat{A}+z\hat{I}_n$$

This pencil is *regular* when $\hat{\pi}\not\equiv 0$, and it has ∞ as an eigenvalue of multiplicity $\hat{n}-\hat{d}\geq 2r$. It is then strictly equivalent to a block diagonal with 2 blocks, one of order $\hat{n}-\hat{d}$ corresponding to ∞ and the other of order $\hat{d}$ corresponding to $\hat{Z}$. The pencil is *singular* when $\hat{d}=\det\hat{A}=0$. The homotopic pencil rules the structure of the family of pencils of order n, depending on the parameter z in $\mathbb{C}$, defined by $P(z)=A-zI+tE$, see [1,2,8]. The pencil $P(z)$ is regular for all $z\in re(A)$. The case when $z=\lambda\in\sigma(A)$ will be addressed in Section 5.

The kernel matrix $\tilde{\Pi}(z)$ is the *kernel* pencil of order g

$$\begin{aligned}\tilde{\Pi}(z) &= \begin{pmatrix} \Pi_1 & R \\ L & \Gamma \end{pmatrix} - z \begin{pmatrix} I_{g_1} & 0 \\ 0 & 0_{g_2} \end{pmatrix} \\ &= \tilde{\Pi} - z\hat{I}_{g_1}\end{aligned}$$

Similarly, the pencil is regular when $\tilde{\pi} \not\equiv 0$, and has ∞ as an eigenvalue of multiplicity $g - \tilde{d} \geq g_2 \geq 0$. Its $\tilde{d}$ finite eigenvalues correspond to $\tilde{Z}$.

The pencil is singular when $\tilde{d} = \det \tilde{\Pi} = 0$.

3.7 Summary

The above study has shown that *two* inductions play a role in Homotopic Deviation. The intermediate induction $r \mapsto n$ is *explicit* in the computation. The complete induction $r \mapsto \hat{n} = n + r$ is most often *implicit*: the meta-level $\hat{n} = n + r$ plays a hidden role via the homotopic polynomial $\hat{\pi}(z)$ of degree $\hat{d} \leq n - r < n$. It is therefore easy to overlook the role of *complete* induction to explain the computational behaviour of $\sigma(A(t))$ as $|t| \to \infty$. Theory shows that the inclusion $\Lambda \subset F$ cannot be explained without a reference to the meta-level $n + r$ at which $\hat{\pi}(z) = \det \hat{A}(z)$ operates.

Given $r < n$, at the (intermediate) level n, the 3 control parameters α_ξ, δ_ξ, a_ξ satisfying $0 \leq \alpha_\xi \leq \delta_\xi \leq a_\xi \leq r$ cannot, in general, give a complete description of $\Lambda \subset F$. The complete analysis requires the additional 2 control parameters $\hat{m}_\xi \leq \hat{d}$ and α_ξ such that $0 \leq \alpha_\xi \leq \min(\delta_\xi, \hat{m}_\xi)$ for $\xi \in F \backslash \Lambda$, and $1 \leq \alpha_\xi = \delta_\xi \leq \hat{m}_\xi$ for $\xi \in \Lambda$.

When $\tilde{Z} \cap re(A) = K(A, E)$ is not empty, the kernel polynomial $\tilde{\pi}(z)$ is another tool at our disposal. For any $\xi \in K(A, E)$, $\alpha_\xi \geq 1$. This brings in a **new** parameter $\tilde{d}$, the degree of $\tilde{\pi}$.

In the coupling (A, E), *three* polynomials are in action, respectively π, $\hat{\pi}$ and $\hat{\pi}$ associated with three matrix pencils of order n, $\hat{n} = n + r$ and $g = g_1 + g_2$, $g_1 \geq 1$. If $g_1 = 0$, then only two polynomials/pencils survive.

The matrix pencils are

- the characteristic pencil: $-A + zI_n$,
- the homotopic pencil: $\hat{A} + z\hat{I}_n$, and
- the kernel pencil: $\tilde{\Pi} - z\hat{I}_{g_1}$, for $g_1 > 0$.

By contrast with the first, the last two pencils may be singular. We observe that $\hat{A}$ uses as additional data the two factors U, V of the SVD for $E = UV^H$. These 2 factors contain the complete metric information about E related to its r *nonzero* singular values. On the other hand, $\tilde{\Pi}$ uses only the geometric information provided by the left and right sets of eigenvectors associated with $0 \in \sigma(E)$. $\tilde{\Pi}$ does not contain any *metric* information about E.

Moreover, the factors U and V are used *explicitly* in $\hat{A} = \begin{pmatrix} -A & -U \\ V^H & 0 \end{pmatrix}$, with no interaction with A. By comparison, in $\tilde{\Pi}$, the 4 blocks encode the purely geometric information about the eigenvectors of E and E^H associated with 0 in an *implicit* way. First the eigenvectors are sorted by *increasing* size of the Jordan blocks they generate, provided that $g_1 \geq 1$. Simplicity can be extracted if at least one simple 0 exists in $\sigma(E)$. Second, the 4 blocks in $\tilde{\Pi}$ represent the Galerkin approximation $P_g A P_g$ restricted to Ker E. The geometric information about $0 \in \sigma(E)$ is therefore entangled with A in a highly complex fashion. The importance of the difference in construction between $\hat{A}$ and $\tilde{\Pi}$ cannot be overestimated.

When the homotopic and kernel pencils are regular, they provide inner (in $\mathbb{C}$) and outer (in $re(A)$) localizations for Lim :

- inner: $\tilde{Z} \subset \text{Lim}$,
- outer: $\text{Lim} \cap re(A) \subset \hat{Z} \cap re(A)$.

The possible extension of the notions Λ and F at an observation point $\lambda \in \sigma(A)$ will be studied in Section 5.

4 The two limit cases ($r = 1$, $g = n - 1$) and ($r = n - 1$, $g = 1$)

We first address the case $r = 1$

4.1 $r = 1$, evolution of $\hat{d}$ and $\tilde{d}$ in $\{1, \cdots, n-1\}$

We are given A, $E = uv^H$, for u, $v \in \mathbb{C}^n$. The augmented matrix $\hat{A}(z)$ has order $\hat{n} = n + 1$ and determinant $\hat{\pi}(z) = v^H \text{adj}\,(zI - A)u$.

Following Gantmacher [11, vol I, p.85], we write

$$\text{adj}\,(zI - A) = z^{n-1} I + z^{n-2} B_1 + \cdots + B_{n-1}$$

where the B_k, $k = 1$ to $n-1$, can be computed iteratively, starting from $B_0 = I$, $p_0 = 0$, $B_1 = A - p_1 I = A - (\text{tr } A)I$, $B_k = AB_{k-1} - p_k I = A^k - p_1 A^{k-1} - \cdots - p_k I$, until $B_{n-1} = (-1)^{n-1}\text{adj } A$. Moreover, $AB_{n-1} = p_n I$ with $p_n = (-1)^{n-1}\det A$.

Therefore $\hat{\pi}(z) = \sum_{k=0}^{n-1} b_k z^{n-1-k}$ with $b_k = v^H B_k u$. We set $a_k = v^H A^k u$, then $a_0 = b_0 = v^H u$, $b_k = a_k - \sum_{i=1}^{k} p_i a_{k-i}$ for $k \geq 1$.

Proposition 4.1 *The degree $\hat{d}$ of $\hat{\pi}$ equals $n-1-k \geq 1$ for $k = 0$ to $n-2$ iff the $k+1$ conditions:*

$$a_j = 0,\ j = 0 \text{ to } k-1,\ a_k \neq 0$$

are satisfied.

Proof. Straightforward induction on k. It uses the linear dependence of b_k on a_j, $j = 0$ to k. □

We look first at the case $\hat{d} = n-1 \iff v^H u \neq 0 \iff (\Sigma)$. Then $g_1 = g = n-1$, $g_2 = 0$, $\Pi = \Pi_1$ and $\tilde{\Pi}(z) = \Pi - zI_{n-1}$. Therefore $\tilde{d} = d = n-1$,

Proposition 4.2 *Under (Σ), we assume that $\hat{\pi}(\lambda)\tilde{\pi}(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$. Then $\tilde{Z} = \hat{Z} = \text{Lim} \subset re(A)$. The two polynomials $\hat{\pi}$ and $\tilde{\pi}$ of same degree $n-1$ are proportional.*

Proof. We use the inclusions $\tilde{Z} \subset \text{Lim}$ and $F(A, E) = \hat{Z} \cap re(A)$.

We assume that $\tilde{\pi}(\lambda)\hat{\pi}(\lambda) \neq 0$ for any λ such that $\pi(\lambda) = 0$. Then $\hat{Z} \cap re(A) = \hat{Z}$ and $\tilde{Z} \cap re(A) = \tilde{Z}$, with $\tilde{Z} \subset \text{Lim} \subset \hat{Z}$ in $re(A)$. The identity follows from the fact that card $\tilde{Z} = \text{card } \hat{Z} = n-1$.

The 2 polynomials $\tilde{\pi}$ and $\hat{\pi}$ of same degree and same roots are proportional. The coefficient of z^{n-1} in $\hat{\pi}$ is $v^H u \neq 0$. □

Next we suppose that $a_0 = v^H u = 0 \iff (\Delta) \iff E^2 = 0$. $\hat{d} = n-2$ iff $\gamma = v^H A u = a_1 \neq 0$. Now $g_2 = 1$ and $g_1 = n-2$, so that $g = g_1 + g_2$. $\tilde{\Pi}(z) = \begin{pmatrix} \Pi_1 - zI_{n-2} & d^H \\ l & \gamma \end{pmatrix}$ of order $g = g_1 + 1$ has determinant

$$\tilde{\pi}(z) = \gamma \det(\Pi_1 - zI_{n-2}) - d^H \text{adj } (\Pi_1 - zI_{n-2}) l.$$

The coefficient of z^{n-2} is $(-1)^{n-2}\gamma$. Therefore $\hat{d} = \tilde{d} = n-2$ iff $a_0 = 0$, $a_1 \neq 0$.

Proposition 4.3 *When $v^H u = 0$ and $v^H A u \neq 0$, the condition $\hat{\pi}(\lambda)\tilde{\pi}(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$ is equivalent to $\tilde{Z} = \hat{Z} = \text{Lim} \subset re(A)$. The two polynomials $\hat{\pi}(\lambda)$ and $\tilde{\pi}(\lambda)$ of same degree $n-2$ are proportional.*

Proof. See Proposition 4.2. $\square$

The coincidence $\hat{d} = \tilde{d}$ at the values $n-1$ and $n-2$ does not extend generally for values $\leq n-3$, which are obtained when $v^H u = v^H A u = 0$. In this situation, $\tilde{\pi}(z) = -d^H \text{adj}\,(\Pi_1 - zI_{n-2})l$.

The value $\tilde{d}$ is determined by Proposition 4.1, where u, v, A are replaced by $-l, d, \Pi_1$. For $\tilde{d} \leq n-3$ and $\hat{d} \leq n-4$ ($a_0 = a_1 = a_2 = 0$) it is possible that $\tilde{d} > \hat{d}$.

Therefore Propositions 3.8 and 3.9 may apply, yielding one of the 2 possibilities:

- $\tilde{d} > \hat{d} \geq 1$, then $\tilde{Z} \cap \sigma(A) \neq \emptyset$,
- $\tilde{d} > \hat{d} = 0$, then $\tilde{Z} \subset \sigma(A)$.

When $\tilde{d} = \hat{d} = 0$, the 4 possibilities are given by the first two columns of Table 3.1.

Remark 4.1 The possibility that $\tilde{d} > \hat{d}$ has an important algorithmic value for computing $\sigma(A)$. This is the mathematical mechanism which explains why Krylov methods are so robust. These ideas will be developped elsewhere.

4.2 $r = 1$, $F = F_c \subseteq re(A)$

When $r = 1$, all $\hat{d}$ frontier points are critical for $1 \leq \hat{d} \leq n-1$. At each $\xi \in F(A, E)$, the source of induction is $B_\xi = R(0,\xi)ER(0,\xi)$ with $\delta_\xi = 0$ or 1, $\sigma(B_\xi) = \{0, g_\xi = v^H R(0,\xi)^2 u\}$. When $g_\xi = 0$, then either $B_\xi = 0$ and $\delta_\xi = 0$ or $B_\xi^2 = 0$ and $\delta_\xi = 1$. The particular structure of B_ξ (due to ξ critical) will have important consequences for the backward flow as $|s| \to \infty$. They will be developped in due course (Section 6.4).

4.3 $g = 1$, $\hat{d} \in \{0, 1\}$

When $0 \in \sigma(E)$ is *multiple*, but not derogatory, there is a *unique* Jordan block of size m, the algebraic multiplicity of 0, $m > 1$. Therefore $\hat{d} \leq 1$, $\hat{n} = 2n-1$, and $g_2 = g = 1$, $g_1 = 0$. The homotopic polynomial $\hat{\pi}$ has degree 1 at most, and the kernel polynomial $\tilde{\pi}$ does not exist.

By way of contrast, when $0 \in \sigma(E)$ is *simple* with $g = m = 1 = g_1$, then $g_2 = 0$, and $\tilde{d} = 1$.

The list of all 10 possibilities for Lim is given in Table 4.1, where $\xi \in re(A)$ and $\rho \in \mathbb{C}$ is the root of $\tilde{\pi}$ under (Σ).

	$\hat{d} = 1$		$\tilde{d} = 0$		
	$F = \emptyset$	$F = \{\xi\}$	$F = re(A)$		$F = \emptyset$
			$F = F_c$	$F \neq F_c$ $n \geq 3$	
(Δ) no $\tilde{\pi}$	$\Lambda = \emptyset$	$\Lambda \subset \{\xi\}$	$\Lambda = \emptyset$ Lim $= \sigma(A)$	Lim ?	$\Lambda = \emptyset$ Lim ?
(Σ) $\hat{d} = 1$	$\tilde{Z} = \{\rho\}$ $\subset$ Lim	$\tilde{Z} = \{\rho\}$ $= \Lambda$	$\rho \in$ Lim $= \sigma(A)$	$\rho \in$ Lim	$\rho \in$ Lim $\Lambda = \emptyset$

Table 4.1: Information about Lim when $g = 1$.

Inspection of Table 4.1 indicates the cases (marked Lim ?) when 0 is defective (Δ) where no information about Lim can be derived from the frontier set $F(A, E)$. We observe that, under (Σ), $\tilde{\pi}(z) = \rho - z$ where $\rho = \frac{1}{y^H x} y^H A x$ is the Rayleigh quotient for A based on the normalized right (x) and left (y) eigenvectors for the simple 0 in $\sigma(E)$, $\|x\|_2 = \|y\|_2 = 1$. By assumption $y^H x \neq 0$, and $\frac{1}{y^H x} \geq 1$ is the condition number for $0 \in \sigma(E)$.

5 The observation point is $\lambda \in \sigma(A)$

5.1 Various notions of observability at $\lambda \in \sigma(A)$

When the observation point is an eigenvalue $\lambda \in \sigma(A)$, $(A - \lambda I)^{-1}$ is not defined and there are 4 possibilities deriving from this situation [3].

First $M_\lambda = \lim_{z \to \lambda} M_z$ may exist (resp. not exist) defining normwise-nonobservability $\|\cdot\|-$no (resp. $\|\cdot\|-$observability).

Second, when M_λ does not exist, one may look at the various limits $\lim_{z \to \lambda} \mu_{iz}$, $i = 1, \cdots, r$, for the eigenvalues in $\sigma(M_z)$. When there exists at least one limit value $\mu_\lambda \in \mathbb{C}$, λ is said to be partially spectrally-observable. Otherwise it is completely σ-observable.

When all r limits exist, that is when

$$\sigma_\lambda = \lim_{z \to \lambda} \sigma(M_z) \in \mathbb{C}^r,$$

λ is said to be σ-nonobservable ($\sigma-$no).

Table 5.1 gathers the 5 possibilities for the existence/nonexistence (resp/partial existence) of M_λ (resp. σ_λ).

<table>
<tr><td>$\|\cdot\|-$</td><td>M_λ</td><td>yes</td><td colspan="3">no</td></tr>
<tr><td>$\sigma-$</td><td>σ_λ</td><td colspan="2">yes</td><td>partial</td><td>no</td></tr>
</table>

Table 5.1:

Lemma 5.1 *Let $\lambda \in \sigma(A)$ be partially σ-observable with $\mu_z \to \mu_\lambda \neq 0$ as $z \to \lambda$. Then $\pi(\frac{1}{\mu_\lambda}, \lambda) = 0$. If $\mu_\lambda = 0$, $\pi(\infty, \lambda) = 0$.*

Proof. For z near λ, $t\mu_z = 1$ and $\pi(t, z) = 0$. By continuity of π when $z \to \lambda$, we get $\pi(\frac{1}{\mu_\lambda}, \lambda) = 0$: λ is an eigenvalue of $A(1/\mu_\lambda)$. When $\mu_\lambda = 0$, then $|\frac{1}{\mu_\lambda}| = |t| = \infty$. $\square$

5.2 Study of $\hat{\sigma} = \hat{Z} \cap \sigma(A)$ [9]

When $\lambda \in \hat{\sigma} = \hat{Z} \cap \sigma(A)$, then λ is a common root for π and $\hat{\pi}$. We denote m_λ and $\hat{m}_\lambda$ the respective multiplicities of $\lambda \in \hat{\sigma}$ relative to π and to $\hat{\pi}$.

The ratio $\theta(\lambda) = \frac{0}{0}$ is indeterminate, withe 3 possibilities: either not defined, or when defined, equal to 0 or not.

The frontier set $F(A, E)$ defined by $\theta(z) = 0$ for $\pi(z) \neq 0$, can be extended by continuity into $\sigma(A)$ to include the *frontier* eigenvalues $\lambda \in \overline{\sigma} \subset \sigma(A)$ for which $\theta(\lambda) = \lim_{z \to \lambda} \theta(z) = 0$.

Proposition 5.2 *The frontier eigenvalues in $\overline{\sigma} \subset \hat{\sigma}$ are characterized by $\hat{m}_\lambda > m_\lambda \geq 1$ for $\lambda \in \hat{\sigma}$.*

Proof. Clear. $\square$

Definition 5.1 *The closure of the frontier set in $\mathbb{C}$ is $\bar{F}(A, E) = F(A, E) \cup \overline{\sigma}$.*

The set $\sigma_1 = \hat{\sigma} \backslash \overline{\sigma}$ is characterized by $1 \leq \hat{m}_\lambda \leq m_\lambda$ for $\lambda \in \hat{\sigma}$. It consists of eigenvalues λ such that $\theta(\lambda) \neq 0$ in $\hat{\mathbb{C}}$.

Proposition 5.3 *A frontier eigenvalue $\lambda \in \overline{\sigma}$ is at most partially σ-observable.*

Proof. By definition of $\bar{\sigma}$, $\theta(\lambda) = 0$. It is possible that some μ_{iz} have no limit as $z \to \lambda$. Indeed the convergence $\prod_{i=1}^{r} \mu_{iz} \to 0$ does not forbid compensation between values which $\to 0$ and $\to \infty$. $\square$

Let a_λ represent the number of values $\mu_{iz} \in \sigma(M_z)$ for $z \neq \lambda$ which converge to 0 as $z \to \lambda$.

For $\lambda \in \overline{\sigma}$ then $1 \leq a_\lambda \leq r$. We define the subset of *critical* eigenvalues in $\overline{\sigma}$ as $\sigma_c = \{\lambda \in \sigma(A);\ a_\lambda = r\} \subset \overline{\sigma}$. Critical eigenvalues exist iff $\sigma_c \neq \emptyset$, they are spectrally nonobservable. The closure of $F_c(A, E)$ in $\mathbb{C}$ is therefore

$$\overline{F}_c(A, E) = F_c(A, E) \cup \sigma_c.$$

When λ is spectrally nonobservable ($\sigma-$no), we can relate $\pi(t, \lambda)$ and $\psi(s, \lambda)$ by continuity as follows, assuming that $\hat{\pi} \not\equiv 0$.

Lemma 5.4 *When $\hat{\pi} \not\equiv 0$ and $\lambda \in \sigma(A)$ is $\sigma-$no, the dominant term in $\pi(t, \lambda)$ can be taken to be t^r by continuity.*

Proof. $e_z = r$ for $z \in re(A) \backslash F(A, E)$. Therefore $\pi(t, z)$ has degree r for almost all z in $\mathbb{C}$ when $\hat{\pi} \not\equiv 0$. $\square$

Corollary 5.5 *When $\hat{\pi} \not\equiv 0$, and $\lambda \in \sigma(A)$ is $\sigma-$no, then $\pi(t, \lambda) = t^r \psi(s, \lambda)$ by continuity.*

Proof. Clear. $\square$

We look now at what computation actually says. We write $\pi(z) = (z - \lambda)^{m_\lambda} q(z)$ with $q(\lambda) = \prod_{\mu \neq \lambda, \mu \in \sigma(A)} (\lambda - \mu) \neq 0$.

Proposition 5.6 $\dfrac{\partial^{m_\lambda}}{\partial^{m_\lambda} z} \pi(t, \lambda) = (-1)^n q(\lambda) \prod_{\mu_{i\lambda} \neq 0} (1 - t\mu_{i\lambda})$ *at any $\lambda \in \sigma(A)$ which is $\sigma-$no.*

Proof. For $z \in re(A)$ and λ $\sigma-$no, we get $\dfrac{\pi(t, z)}{(z - \lambda)^{m_\lambda}} = (-1)^n q(z) \prod_{i=1}^{r} (1 - t\mu_{iz})$. Thus $\lim_{z \to \lambda} \dfrac{\pi(t, z)}{(z - \lambda)^{m_\lambda}} = \dfrac{\partial^{m_\lambda}}{\partial^{m_\lambda} z} \pi(t, \lambda) = (-1)^n q(\lambda) \prod_{\mu_{i\lambda} \neq 0} (1 - t\mu_{i\lambda})$.

The partial derivative of order m_λ with respect to z is a polynomial of degree $e_\lambda = r - a_\lambda$, $0 \leq e_\lambda \leq r$, in $t \in \mathbb{C}$, for $z = \lambda \in \sigma(A)$ $\sigma-$no. $\square$

Corollary 5.7 *If $\sigma_\lambda = \{0\}$, $\dfrac{\partial^{m_\lambda}}{\partial^{m_\lambda} z} \pi(t, \lambda) = (-1)^n q(\lambda) \neq 0$, independently of $t \in \mathbb{C}$.*

Proof. $a_\lambda = r$ and $e_\lambda = 0$. $\square$

5.3 Study of $\mathrm{Lim} \cap \sigma(A)$ [6, 9]

Looking forward, as $t \neq 0$, at the evolution of $\lambda(t)$ enables us to qualify the eigenvalues in $\sigma(A)$, counted with their algebraic multiplicity, as

- *invariant*, $\lambda \in \sigma^i \iff \lambda = \lambda(t)$ for all $t \in \mathbb{C}$,
- *evolving*, $\lambda \in \sigma^e \iff \lambda \neq \lambda(t)$ for almost all $t \in \mathbb{C}$.

Sorting in Lim (the set of limits in $\mathbb{C}$ of $\lambda(t)$ as $|t| \to \infty$) the points which belong to $\sigma(A)$, we denote σ^f the set of such *final* eigenvalues. They can belong to either σ^i or σ^e.

Therefore [6]

$$\begin{aligned} \mathrm{Lim} \cap \sigma(A) &= (\sigma^i \cap \sigma^f) \cup (\sigma^e \cap \sigma^f) \cup (\sigma^i \cap (\sigma(A) \backslash \sigma^f)) \\ &= \sigma^f \cup \tau \end{aligned}$$

The set τ consists of the invariant eigenvalues in σ^i which are *not* final limits when $|t| \to \infty$.

Final eigenvalues have the following property:

Proposition 5.8 *All final eigenvalues $\lambda \in \sigma^f$ satisfy $0 \in \sigma_{E,A-\lambda I}$, the set of finite eigenvalues for the reverse pencil $Q(\lambda) = E + s(A - \lambda I)$.*

Proof. $\lambda \in \sigma^f$ iff $\alpha(s) = \lambda s + o(s) \in \sigma(E + sA)$. That is $\alpha(s) - (\lambda s + o(s)) \in \sigma(E + s(A - \lambda I))$. The result follows when $s \to 0$. □

Definition 5.2 *When 0 is an eigenvalue of the reverse pencil $Q(\lambda)$, we say that λ is an eigenvalue of $A(\infty)$, as well as of $A = A(0)$.*

Corollary 5.9 *When $\lambda \in \sigma^i$ then $P(\lambda)$ is singular. If moreover $\lambda \in \sigma^f$, then 0 is an eigenvalue of $A(\infty)$.*

Proof. Let $\mu \neq 0$ be an eigenvector for λ in $\mathrm{Ker}\,(A - \lambda I)$. $\lambda \in \sigma^i \iff \mathrm{Ker}\, E \cap \mathrm{Ker}\,(A - \lambda I) \neq \{0\} \iff \pi(t, \lambda) = 0$ for any $t \in \mathbb{C}$. □

In analogy with Definition 5.1, we define the closure of $\Lambda(A, E)$ in $\mathbb{C}$ as $\overline{\Lambda}(A, E) = \Lambda(A, E) \cup \sigma^f$.

The inclusion $\Lambda(A, E) \subset F(A, E)$ valid in $re(A)$ can be extended into $\sigma(A)$ by the

Theorem 5.10 *The inclusion $\sigma^f \subset \overline{\sigma}$ holds in $\sigma(A)$.*

Proof. Follows from the continuity, when $z \to \lambda$, of the coincidence $0 \in \sigma(M_z) \iff \hat{\pi}(z) = 0$. □

We look now at critical eigenvalues in σ_c.

Proposition 5.11 *The following inclusion holds:* $\sigma_c \subset \{\sigma^i, \bar{\sigma}\}$. *Therefore* $\sigma_c \cap \sigma^f \subset \sigma^i \cap \sigma^f$.

Proof. We have to show that $\sigma_c \subset \sigma^i$. We recall that for $z \in F_c$, $\pi(t,z) = \pi(z)$ for all $t \in \mathbb{C}$. By continuity as $z \in F_c \to \lambda \in \sigma_c$ we get $\pi(t,\lambda) = \pi(\lambda) = 0$ for all t: that is, $\lambda(t) = \lambda$ for all t. □

We notice important differences between σ_c and σ^i. An invariant eigenvalue in σ^i can be observable or not, whereas a critical eigenvalue in σ_c is necessarily spectrally unobservable. Also $\pi(t,\lambda) = \pi(\lambda) = 0$ for $\lambda \in \sigma^i$, But we know nothing more. For $\lambda \in \sigma_c$, Corollary 5.7 holds.

5.4 Algebraic induction at a normwise nonobservable eigenvalue in $\overline{\sigma}_{\|\cdot\|-no} \subset \overline{\sigma}$.

We suppose that $\lambda \in \overline{\sigma}$ is normwise unobservable: $M_\lambda = \lim\limits_{z \to \lambda} M_z$ exists and $P_{0\lambda}$ of rank a_λ, $1 \leq a_\lambda \leq r$ denotes the spectral projection associated with $0 \in \sigma(M_\lambda)$.

The notion of frontier reduction matrix B_ξ defined at a frontier point $\xi \in F(A,E)$ in Section 3 can be extended to any $\| \cdot \|$-nonobservable frontier eigenvalue.

We partition the set $\overline{\sigma}$ of frontier eigenvalues into $\overline{\sigma}_{\|\cdot\|-no}$ ($\| \cdot \|$-observable) and $\overline{\sigma}_{\|\cdot\|-no}$ ($\| \cdot \|$-nonobservable)

$$\overline{\sigma} = \overline{\sigma}_{\|\cdot\|-o} \cup \overline{\sigma}_{\|\cdot\|-no}.$$

We consider the spectral projection P_λ for A and the reduced resolvent $S_\lambda = S(0,\lambda) = \lim\limits_{z \to \lambda}(I - P_\lambda)R(0,z)$.

We define $D_\lambda = (A - \lambda I)P_\lambda$ which satisfies $D^l = 0$ where $l \geq 1$ is the ascent of λ (size of the largest Jordan block) [4].

Lemma 5.12 $M_\lambda = \lim\limits_{z \to \lambda} M_z$ *exists iff the following* l *algebraic conditions are satisfied:*

$$V^H P_\lambda U = 0, \;\; V^H D_\lambda^k U = 0, \;\; k = 1, \cdots, l-1.$$

When M_λ *exists, then* $M_\lambda = V^H S_\lambda U$.

Proof. Follows from the Laurent expansion of M_z around λ, [5]. □

$R(0,\lambda)$ does not exist, but one can consider the reduced resolvent for $t \in \mathbb{C}$

$$\begin{aligned} S(t,z) &= (I - P_\lambda)R(t,z) \;\; \text{and} \\ S(t,\lambda) &= \lim_{z \to \lambda}(I - P_\lambda)R(t,z) \\ &= S_\lambda[I_n - tU(I_r - tM_\lambda)^{-1}V^H S_\lambda], \end{aligned}$$

which exists if $(I_r - tM_\lambda)^{-1}$ exists. When M_λ is invertible, then

$$\lim_{|t|\to\infty} S(t,\lambda) = S(\infty,\lambda) = S_\lambda[I_n + UM_\lambda^{-1}V^H S_\lambda].$$

Definition 3.1 of B_ξ for $\xi \in F(A,E)$ can be complemented by the

Definition 5.3 *The frontier reduction matrix associated with* (A,E) *at* $\lambda \in \overline{\sigma}_{\|\cdot\|-no}$ *is defined by*

$$B_\lambda = \frac{-1}{2i\pi}\int_\Gamma S(\infty,z)dz.$$

Lemma 5.13 $B_\lambda = S_\lambda U P_{0\lambda} V^H S_\lambda = (S_\lambda U P_{0\lambda})(P_{0\lambda} V^H S_\lambda)$

Proof. Follows from the formula $S(\infty,z) = S_\lambda[I + UM_z^{-1}V^H S_\lambda]$. $\square$

Proposition 5.14 *Set* $T_\Gamma = \max_{z\in\Gamma} \rho(M_z^{-1})$. *For* $|t| > T_\Gamma$, $\frac{-1}{2i\pi}\int_\Gamma S(t,z)dz = B_\lambda$.

Proof. Replace $R(t,z)$ by $S(t,z)$ in the proof of Proposition 3.2. $\square$

Definition 5.4 *The frontier multiplicity of* $\lambda \in \overline{\sigma}_{\|\cdot\|-no}$ *is* $\delta_\lambda = rank\, B_\lambda$, $0 \le \delta_\lambda \le a_\lambda$. *The actual multiplicity of a frontier eigenvalue* $\lambda \in \overline{\sigma}$ *is the number* α_λ *of eigenvalues* $\lambda(t)$ *converging to* λ *as* $|t| \to \infty$, $0 \le \alpha_\lambda \le a_\lambda$.

It is clear that $\alpha_\lambda \ge 1 \iff \lambda \in \sigma^f$.

Lemma 3.3 and Propositions 3.4, 3.5 can be extended readily to any $\lambda \in \overline{\sigma}_{\|\cdot\|-no}$. This casts light on $\sigma^f \cap \overline{\sigma}_{\|\cdot\|-no}$.

5.5 The normwise observable part $\overline{\sigma}_{\|\cdot\|-o}$ of the frontier spectrum $\overline{\sigma}$

When M_λ does not exist, no algebraic induction ($r \mapsto n$) is possible. We have at our disposal the two parameters α_λ and a_λ, $0 \le \alpha_\lambda \le a_\lambda$ and $1 \le a_\lambda \le r$ for $\lambda \in \overline{\sigma}_{\|\cdot\|-o}$ which satisfies $\hat{m}_\lambda > m_\lambda \ge 1$.

5.6 About the nature of the family of pencils $P(z) = (A - zI) + tE$, where the parameter $z \in \mathbb{C}$

We consider the polynomial $\pi(t, z) = \det(A - zI + tE)$ which has degree $< n$ in t and constant coefficient $\det(A - zI)$.

We recall that for $z \in re(A)$,

$$\pi(t, z) = (-1)^n \pi(z) \det(I - tM_z)$$

The pencil $P(z)$ is regular for $\pi(t, z) \not\equiv 0$ in t. This is the case when $z \in re(A)$. What is the situation for $\lambda \in \sigma(A)$?

If $\lambda \in \sigma^e$ (resp. σ^i) then $\pi(t, z) \not\equiv 0$ ($\equiv 0$). P_λ is singular at all invariant eigenvalues of A.

For $z \in re(A)$, the Weierstrass canonical form of $P(z)$ depends on whether z is frontier or not. When $z \in re(A) \backslash F(A, E)$, then $P(z)$ has exactly r finite eigenvalues, and the infinite eigenvalue is semi-simple of multiplicity $g = n - r$. Such regular pencils are said to have *index* 1 (referring to the infinite eigenvalue).

We now assume that z is not critical in $F(A, E)$. The canonical form consists of the *two* diagonal blocks given below, with $\varepsilon \in \{0, 1\}$:

$$\begin{pmatrix} 1 & t\varepsilon & & 0 \\ & 1 & \ddots & \\ & & \ddots & t\varepsilon \\ 0 & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\mu_{1z}} & \varepsilon & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ 0 & & & \frac{1}{\mu_{r_z z}} \end{pmatrix}$$

The nonzero values μ_{iz}, $i = 1, \cdots, e_z = r - a_z \geq 1$ are the nonzero eigenvalues of M_z, for z non critical. The first block of order $g + a_z \leq n - 1$ corresponds to the infinite eigenvalue of $P(z)$, and the second block of order $e_z = r - a_z$ to the finite eigenvalues of $P(z)$.

When $z \in re(A) \backslash F(A, E)$, $a_z = 0$ and $e_z = r$, $\varepsilon = 0$ in the first block. The structure of $P(z)$ does *not* depend on the observation point z.

When z is critical, $a_z = r$ and there is a unique block corresponding to the infinite eigenvalue: $\pi(t, z) \neq 0$ for all $t \in \mathbb{C}$. Now if z critical belongs to $\Lambda(A, E)$ more can be said: z is a finite eigenvalue of $A(\infty)$, that is $\psi(0, z) - 0$. This result goes beyond the theory of Weierstrass. It signals a change in the nature of $R(t, z)$ as $|t| \to \infty$.

Weierstrass form can be extended without difficulty to any $\lambda \in \sigma^e$ which is not critical. When $z \to \lambda$ and $|\mu_z| \to \infty$ then $\frac{1}{\mu_z} \to 0$. For λ non critical in $\sigma^e \cap \overline{\sigma}$, then $1 \leq a_\lambda < r$.

When λ is critical in σ_c, $a_\lambda = r$, and λ is invariant. $P(\lambda)$ is singular. One should shift to the Kronecker form [11]. The relevance of HD theory to analyze the Kronecker form for the singular pencil $P(\lambda)$ when $\lambda \in \sigma^i$ will be discussed elsewhere.

5.7 The comparison between $\hat{Z}$ and Lim

The two sets Lim $= (\Lambda(A,E) \cup \sigma^f) \cup \tau$ and $\hat{Z} = (F(A,E) \cup \overline{\sigma}) \cup \sigma_1$ satisfy the partial inclusion:

$$\overline{\Lambda}(A,E) = \Lambda \cup \sigma^f \subset \overline{F}(A,E) = F \cup \overline{\sigma}.$$

What can be said about the two residual sets τ and σ_1 which are subsets of $\sigma(A)$?

They are neither eigenvalues of $A(\infty)$ ($\tau \cap \sigma^f = \emptyset$) nor frontier eigenvalues of A ($\sigma_1 \cap \overline{\sigma} = \emptyset$). No more can be said.

We have completed the forward spectral analysis of $\sigma(A(t))$ as $t \in \hat{\mathbb{C}}$ for all possible observation points z in $\mathbb{C}$. The final result can be stated under the form of

Theorem 5.15 *The forward spectrum of the coupling (A,E) is the closed spectrum $\overline{\Lambda}(A,E) = \Lambda \cup \sigma^f = \mathrm{Lim}\backslash\tau$. It is contained in the closed frontier set $\overline{F}(A,E) = F \cup \overline{\sigma} = \hat{Z}\backslash\sigma_1$.* Lim *contains the kernel set $\tilde{Z}$. In summary,*

$$\mathrm{Lim}\backslash\tau \subset \hat{Z}\backslash\sigma_1 \quad \textit{and} \quad \tilde{Z} \subset \mathrm{Lim}.$$

Remark 5.1 Theorem 5.15 can be used to extend Table 3.1 and Table 4.1 to the closed frontier set $\overline{F} = \hat{Z}\backslash\sigma_1$ defined by $\hat{\pi}$.

6 The backward flow of spectral information

6.1 The sources B_ξ of backward information

Definition 6.1 *The (algebraic) induction set ($r \mapsto n$) for (A,E) is the subset $H(A,E) = F(A,E) \cup \overline{\sigma}_{\|\cdot\|-no}$ of the closed frontier set $\overline{F}(A,E)$.*

At each point ξ in H, a reduction matrix B_ξ exists which, in the presence of the deviation matrix E, creates by linear coupling the family $B_\xi(s) = B_\xi + sE$, for $s = 1/t \in \hat{\mathbb{C}}$.

The spectrum $\sigma(B_\xi(s))$ can be studied by the Homotopic Deviation theory expounded in the 5 preceeding Sections. The specific structure of B_ξ adds new features to the general theory. To ease the notation, we drop the subscript ξ below, whenever there is no ambiguity.

For example we set $B' = U'V'^H$ with $U' = R(0,\xi)UP_{0\xi}$ and $V'^H = P_{0\xi}V^H R(0,\xi)$, $G' = V'^H U'$, $\pi'(z) = \det(zI - B')$.

6.2 The Homotopic Deviation theory applied to B' with the deviation matrix E, $s \in \hat{\mathbb{C}}$.

Three polynomials are in action: first π' the characteristic polynomial for B', then $\hat{\pi}'(z) = \det \hat{B}'(z)$ and $\tilde{\pi}'(z) = \det \tilde{\Pi}'(z)$, where $\hat{B}'(z) = \hat{B}' + z\hat{I}_n$, $\quad \tilde{\Pi}'(z) = \tilde{\Pi}' - z\hat{I}_{g_1}$.

They define the two inclusions $\tilde{Z}' \subset \mathrm{Lim}\,'$ in $\mathbb{C}$ and $\Lambda(B', E) \subset F(B', E)$ in $re(B')$.

These sets carry additional information about the spectral content of the coupling (A, E).

6.3 Temporary conclusion

The structure of the frontier reduction matrices B_ξ at $\xi \in F(A, E)$ suggests to consider, more generally, for $z \in re(A)$, th family of *evolution* matrices

$$C_z = R(0, z)ER(0, z).$$

They have the *fixed* rank r, and are *nonzero*. They differ from B_ξ at $\xi \in F(A, E)$ unless ξ is critical ($P_{0\xi} = I_r$).

It is natural to consider the coupling

$$C_z(s) = C_z + sE$$

for the *reverse* intensity s, $st = 1$.

We set $\Delta(z) = z^2E - z(EA + AE) + AEA = (A - zI)E(A - zI)$ for $z \in \mathbb{C}$.

$$\begin{array}{rcl} C_z(s) & = & R(0, z)[E + s\Delta(z)]R(0, z) \\ & = & s(E + tC_z) = sR(0, z)[\Delta(z) + tE]R(0, z). \end{array}$$

Set $\delta(t, z) = \det(\Delta(z) + tE)$. Then for $z \in re(A)$, $\det(\frac{1}{s}C_z(s)) = \det(E + tC_z) = \frac{1}{\pi^2(z)}\delta(t, z)$.

Part II of this work will be devoted to the study of the zeros of $\delta(t, z)$.

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