Computing beyond classical logic:
SVD computation in nonassociative Dickson algebras(*)

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Dedicated to Gregory J. Chaitin for his 60th birthday

CERFACS Technical Report TR/PA/07/54

Abstract : This short note puts the fundamental work of G. Chaitin into an historical perspective about the multisecular evolution of the art of computing. It recalls that each major step forward in the creation of new numbers was met by strong opposition. It shows, by way of an example taken from SVD computation in nonassociative Dickson algebras, why classical logic cannot account for certain results which carry complexified information.

Keywords : Chaitin, Omega, Turing thesis, quaternions, octonions, nonassociative Dickson algebra, classical logic, SVD computation, nonclassical singular value, induction, information, Life, evolution.

(*)Published in **Randomness & Complexity : from Leibniz to Chaitin** (C. Calude ed.), pp.13-23, World Scientific, Singapore, 2007.

1 Introduction

The scientific œuvre of Gregory Chaitin revolves around computation and displays a remarkable unity of thought. More than 4 decades ago, Chaitin began to explore the limits of computation within the paradigm of a Turing machine. This led him to the celebrated Omega number [1,2] which expresses the ultimate in uncomputability à la Turing.

The Turing « thesis » about computability is an axiomatic definition of what can be computed (by a machine) within the limits of classical logic, the rational logic

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based on elementary arithmetic. This axiom is now accepted by computer scientists and logicians as a universal rule for computation. Therefore the work of Chaitin, which questions this claim to universality from *within*, has aroused passionate and antagonistic reactions, positive and negative.

One of the main reasons for the irrational passion stirred by his work is that it is rooted at a most fundamental level. Few questions reach deeper into human understanding than « What can rational computation achieve ? ».

From the point of view of classical logicians, the theoretical findings of Chaitin about computation are unacceptable. Not because the mathematics are wrong —the proofs are impeccable—, but because the conclusions are viewed as heretical. Some of these logicians have expressed their criticisms in a forceful way [13]. However, the absolute faith that computer scientists put in the universal validity of the axiom of Turing is not equally shared by everyone in the scientific community. Highly successful books by Penrose [11, 12] and Wolfram [15] testify to the necessity to explore other computational routes. An extension of the classical logic based on quantum coherence was already advocated by D. Deutsch in the 1980's [9]. This quantum logic led to the development of quantum computing. Experiments have shown that such a computation is physically realizable at the atomic level (Zeilinger).

There are converging indications that new kinds of logic are required to understand the real world which extends around and inside us. This has not, however, mellowed the criticisms against the pioneering insights of Gregory Chaitin raised by orthodox logicians [10] and other conservative philosophers of Science [14].

This is not at all surprising. There are many historical cases of the rejection, by the vast majority of mathematicians, of radically new ideas, which, much later, were recognized as fundamental to the advance of mathematical understanding [7,8]. Among the best known examples, one finds *new kinds of numbers :* i) *negative* numbers (12^{th}-15^{th} Century), ii) *complex* numbers (16^{th}-19^{th} Century), iii) *quaternions* (1843). Before being finally incorporated into the mathematical corpus, each new kind of numbers was met by skepticism at best, and its significance was passionately debated [3, 4, 7, 8].

This is all too understandable : each new number contradicted a commonly shared opinion of the time, implicitly taken as a universal law of computation. These opinions were respectively the following :

i) all equations have either positive solutions, or no solution (Middle Ages),

ii) any nonzero number has a positive square (late Renaissance),

iii) the multiplication of numbers is commutative (early 19^{th} Century).

The discovery of each of these new numbers was a major step forward in the evolution of the art of computing in the western world. This advance, which spanned over seven centuries, was instrumental in the axiomatic clarification of the foundations of mathematics which occurred at the dawn of the 20^{th} Century. Thereafter, even associativity became an optional feature for multiplication.

2 Nonassociativity of multiplication

The notion of associativity was invented by Hamilton in July 1844 when he realized that the multiplication of two octonions was *not* associative. The *octonions* had been discovered 6 months earlier by his classmate J. T. Graves, a lawyer at the University of London, in an effort to derive an 8 squares theorem on the model of Hamilton's quaternions. Such a discovery was extremely ahead of its time. The non-commutative quaternions were then hardly accepted by mathematicians. Their use was to be aggressively questioned by eminent American physicists (Gibbs-Heaviside) [8].

Despite this opposition, the *associative* algebras of Clifford (1878), which extend the quaternions, have been successful tools for the development of algebraic geometry and theoretical physics until to-day [4].

Two kinds of nonassociative algebras participated in the success : the algebras of Lie and of Pascual Jordan. This very success did cast a shadow on the role of other nonassociative algebras, such as Graves' octonions, in the analysis of computation. The 8D-octonions are the smallest of the nonassociative Dickson algebras.

3 Nonassociative Dickson algebras

3.1 Presentation of Dickson's doubling process

The three associative Dickson algebras A_k, $k = 0$ to 2, define successively the reals, $A_0 = \mathbb{R}$, the complexes $A_1 = \mathbb{C}$, and the quaternions $A_2 = \mathbb{H}$.

The nonassociative algebras A_k extend, for $k \geq 3$, the quaternions in a way different from Clifford's. Multiplication and conjugation are inductively defined so that

$$1_{k+1} = (1_k, 0), \quad \tilde{1}_{k+1} = (0, 1_k)$$

$$A_{k+1} = A_k \times 1_{k+1} \oplus A_k \times \tilde{1}_{k+1}, \quad k \geq 0,$$

where 1_k is the real unit of A_k [4].

This inductive process defines, from $A_0 = \mathbb{R}$ at the beginning, an endless chain of *complexified* algebras A_k of dimension 2^k, $k \in \mathbb{N}^*$.

The process was observed by Dickson around 1912, and presented for $k = 2$ as a computational way to induce the multiplication table for the octonions, which had been given independently by Graves (1844) and Cayley (1845), from Hamilton's multiplication table for the quaternions (1843).

It is conventional wisdom that the lack of associativity is a severe limitation for computation in A_k, $k \geq 3$. Nothing could be further from reality, as this was shown in [3,4]. Nonassociativity creates computational *opportunities* which are well exemplified by the Singular Value Decomposition (SVD) for the left multiplication map $L_a : x \mapsto a \times x, \ x \in A_k$ (Section 4).

Remark : Vectors in Dickson algebras have been called hypercomplex numbers in the 19$^{\text{th}}$ Century. And computation on hypercomplex numbers is classically known as « hypercomputation » [3,4,7]. This mathematical notion should not be confused with a recent version of computation designed by computer scientists to overcome some of Turing's limitations (see `http ://en.wikipedia.org/wiki/Hypercomputation`.)

3.2 Alternative vectors in A_k, $k \geq 4$

Let $[x, y, z] = (x \times y) \times z - x \times (y \times z)$ denote the *associator* for x, y, z in A_k, $k \geq 3$.

An alternative vector a in A_k satisfies the weakened associativity condition :

$$-[a, a, x] = ||a||^2 x + a \times (a \times x) = 0$$

for any x in A_k. The condition is identically satisfied for $k \leq 3$, but not for $k \geq 4$. All canonical basis vectors $e_i, \ i = 0$ to $2^k - 1$, are alternative for arbitrary k. Among them, the two vectors $1 = e_0$ and $\tilde{1} = e_{2^{k-1}}$ have stronger properties. They span the subalgebra $\mathbb{C}_{\tilde{1}} = \text{lin}(1, \tilde{1})$ isomorphic to $\mathbb{C}$. Any pair of vectors (x, y) in $\mathbb{C}_{\tilde{1}}$ satisfies

$$[x, x, y] = [x, y, y] = 0.$$

The vectors in $\mathbb{C}_{\tilde{1}}$ are *fully* alternative for $k \geq 4$ [3].

3.3 The splitting $A_k = \mathbb{C}_{\tilde{1}} \oplus \mathcal{D}_k$, $k \geq 2$

Let be given a in A_k. It can be represented as the sum

$$a = \alpha + \beta\tilde{1} + c$$

where $h = \alpha + \beta\tilde{1} \in \mathbb{C}_{\tilde{1}}$ is the fully alternative *head* and c is the *tail* : c belongs to the subspace $\mathcal{D}_k = \mathbb{C}_{\tilde{1}}^{\perp}$ of vectors with zero component on 1 and on $\tilde{1} = \tilde{1}_k$. These vectors are called « doubly pure ». Such a splitting plays an important role in non classical SVD calculations in A_k, $k \geq 3$ (Section 4).

4 SVD computation in $\mathcal{D}_k$ and A_k, $k \geq 3$

The notion of singular values for a matrix (or a linear map) plays an essential role in matrix computations, in particular for backward analysis when the data are uncertain [7]. It dates back to Camille Jordan (1873).

For $a \in A_k$, the singular values of the map L_a are the non-negative square roots of the eigenvalues of the symmetric map $L_a^T L_a$. For $k \leq 3$, $\overline{a} \times (a \times x) = ||a||^2 x$ for any x : there is a *unique* singular value $||a||$ for L_a, and $||a|| = 0$ iff $a = 0$. But this need not be true anymore for $k \geq 4$, unless a is alternative.

4.1 $c \in \mathcal{D}_k$ is doubly pure, $k \geq 4$.

Let be given c in $\mathcal{D}_k$, $k \geq 4$. There are between 1 and 2^{k-2} distinct nonnegative singular values for L_c for $k \geq 5$. For $k = 4$, the number reduces to 1 or 3. The multiplicities are multiples of 4. The euclidean norm $||c||$ is always one of the singular values, corresponding to the 4D-singular subspace $\mathbb{H}_c$ (isomorphic to $\mathbb{H}$) spanned by $1, c, \tilde{c}$ and $\tilde{1}$, with $\tilde{c} = c \times \tilde{1}$ [3].

c is a zerodivisor iff $\mathrm{Ker} L_c \neq \{0\}$, that is iff 0 is one of the singular values for L_c. It can be proved [3] that a zerodivisor is necessarily doubly pure. When the non alternative vector c is not a zerodivisor, c^{-1} is uniquely defined by $c^{-1} = -\frac{c}{||c||^2}$. Therefore $L_c^{-1} \neq L_{c^{-1}} = -\frac{1}{||c||^2} L_c$, and $\frac{\|L_c\|}{\|c\|} = \|c\| \|L_{c^{-1}}\|$ represents the largest normalized singular value.

4.2 Deriving the SVD of a in A_k from that of the tail c in $\mathcal{D}_k$, for $k \geq 4$

Let c be given in $\mathcal{D}_k$ such that $||c|| = 1$. The spectrum of $-L_c^2$ is denoted $\sigma_c = \sigma(-L_c^2)$, and λ is any eigenvalue in σ_c, $\lambda \geq 0$. Let $\lambda = 1^4$ denote the eigenvalue associated with the 4D-eigenspace $\mathbb{H}_c$. The notation $\lambda \neq 1^4$ means that either $\lambda \neq 1$, or, if $\lambda = 1$, its multiplicity is ≥ 8. We set $N_\lambda = \alpha^2 + \beta^2 + \lambda$, for $\lambda \in \sigma_c$, thus $N_\lambda \geq ||h||^2 > 0$ for $h \neq 0$.

Theorem 4.1 *For $a = \alpha + \beta\tilde{1} + c$, $c \in \mathcal{D}_k$, $L_a^T L_a = (\alpha^2 + \beta^2) I - L_c^2$. The eigenvalues of $L_a^T L_a$ are N_λ, for $\lambda \in \sigma_c$ with the same multiplicities.*

Proof. Direct computation of $(\alpha - \beta\tilde{1} - c) \times (\alpha x + \beta\tilde{1} \times x + c \times x)$, $x \in A_k$. One checks that $c \times (\tilde{1} \times x) + \tilde{1} \times (c \times x) = 0$ for $x \in \mathbb{H}_c$ and $x \in \mathbb{H}_c^\perp$. The conclusion follows. □

The *classical* derivation of the SVD for L_a from that for L_c yields a generalization of Pythagoras theorem to $||h|| \neq 0$ and to the singular values for L_c, $c \neq 0$:

$$a = h + c \Longrightarrow N_\lambda = ||h||^2 + \lambda > 0 \text{ for } h \neq 0.$$

We have discovered (2005) that the nonassociative nature of multiplication in Dickson algebras for $k \geq 3$ enables us to perform a *nonclassical* derivation, which is a computational artifact in A_k, $k \geq 3$ [3].

4.3 Nonclassical derivation from c to a, $k \geq 3$

The nonclassical mode of derivation is defined in [3,Section 9]. It uses the block-diagonal form of $L_a^T L_a$ (with blocks of order 4) written in the eigenbasis for $-L_c^2$. In this nonclassical approach, the order in which the addition to c of α and $\beta\tilde{1}$ is performed *matters*. From an SVD point of view, addition is not always associative in A_k, $k \geq 3$, as we shall see.

When $\alpha\beta \neq 0$, there are 3 different routes to go from c to a in $\mathbb{C}_{\tilde{1}}$, as sketched on Figure 1 : one can reach a either directly (diagonally) or sideways through $d = \beta\tilde{1}+c$, or through $e = \alpha + c$. When $\alpha\beta = 0$, the route is unique.

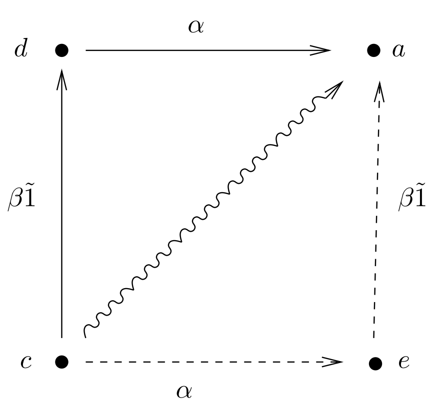

FIG. 1 – Three routes from c to a in $\mathbb{C}_{\tilde{1}}$ for $\alpha\beta \neq 0$

Section 9 in [3] uses (implicitly) the direct route $a = h + c$, which yields the same results as the sided one through e : $a = \beta\tilde{1} + e$. The two routes through d and e give different results for $\alpha\beta \neq 0$.

We define $\xi \ : \ (s,t) \in \mathbb{R} \times \mathbb{R}^+ \mapsto \xi(s,t) = ((s-t)^2, (s+t)^2) \in (\mathbb{R}^+)^2$ where $t \geq 0$, $s \in \mathbb{R}$.

Theorem 4.2 *For $a = \alpha + \beta\tilde{1} + c$, $c \in \mathcal{D}_k$, $k \geq 3$, the nonclassical SVD derivation yields the nonnegative values listed below in two columns :*

λ	*via d*	*direct or via e*
1^4	$N_1 = \alpha^2 + \beta^2 + 1$	N_1
$0 < \lambda \neq 1^4$	$N_\lambda \pm 2\beta\sqrt{\lambda}$ $= \alpha^2 + \xi(\beta, \sqrt{\lambda})$	$N_\lambda \pm 2\beta\sqrt{\lambda + \alpha^2}$ $= \xi(\beta, \sqrt{\lambda + \alpha^2})$
0	$N_0 = \alpha^2 + \beta^2$	$N_0 \pm 2\beta\alpha = (\alpha \pm \beta)^2$

Table 1

Proof. Based on [3, Lemma 9.3 and Proposition 9.4]. □

For each λ in σ_c and for $\alpha\beta \neq 0$, there are $1, 3$ or 4 different singular values when they are computed non classically in A_k, $k \geq 3$. All results for $0 < \lambda \neq 1^4$ differ from the exact value N_λ given in Theorem 4.1 when $\beta \neq 0$, with common mean. We now take a fresh look at the logical paradox arising from the existence of split zerodivisors for $\beta^2 = \lambda + \alpha^2$ [3].

5 Is the nonclassical SVD derivation absurd ?

5.1 The conventional analysis

From the point of view of classical logic, the nonclassical SVD results are plainly **wrong**, since for $0 < \lambda \neq 1^4$ they do not agree with the exact value N_λ. Moreover, when $\beta^2 = \alpha^2 + \lambda$, they contradict the theoretical result that zerodivisors necessarily belong to $\mathcal{D}_k$. At face value, nonclassical SVD seems **absurd**, and it should be rejected by any sane mathematician. Or should it not ?

Should we think twice ? In the 16^{th} Century, $\sqrt{-1}$ was a complete mystery, which appeared totally absurd at first sight. It took three centuries of painful reflections by some of the greatest minds like Euler and Cauchy to master its meaning as the « imaginary » unit i. Once tamed, $i = \sqrt{-1}$ found its way in almost all engineering calculations of the 19^{th} Century which dealt with wave propagation (light, sound, electricity,magnetism,...).

Warned by history, we should be extremely cautious. We should not jump hastily to the « obvious » conclusion. Could it be possible that nonclassical SVD computation *serves a purpose* from a computational point of view, and that it delivers useful *information* ?

5.2 Induction and nonclassical singular values

To the vector $a = \alpha + \beta\tilde{1} + c$ in A_k, we associate $\varphi = (\alpha + c, \beta\tilde{1})$ in A_{k+1}, for $k \geq 3$. We still assume that $||c|| = 1$, $c \in \mathcal{D}_k$, and $\beta \neq 0$. Observe that $||\varphi|| = ||a|| = \sqrt{N_1}$.

Theorem 5.1 *The eigenvalues of $L_\varphi^T L_\varphi$ are given in Table 1 by the leftmost column. Their values equal the nonclassical eigenvalues for $L_a^T L_a$, computed in A_k via $d = \beta\tilde{1} + c$. Their multiplicities are multiplied by 2.*

Proof. Let $\varphi = (\alpha + c, \beta\tilde{1})$, and $v = (x, y)$. Direct computation of $\overline{\varphi}(\varphi \times v)$ shows that $L_\varphi^T L_\varphi$ has the 2×2 block representation

$$L = \left(\begin{array}{c|c} M & G \\ \hline -G & M \end{array}\right)$$

where $M = (\alpha^2 + \beta^2)I - L_c^2 = L_a^T L_a$, $G = -G^T = \beta[c, -, \tilde{1}]$. Now $Gx = \beta[c, x, \tilde{1}]$ is 0 for $x \in \mathbb{H}_c$ by associativity, and equals $2\beta\widetilde{c \times x}$ for $x \in \mathbb{H}_c^\perp$.

It is easily shown that L has a block diagonal structure, with blocks of order 8, derived from the eigenstructure for $-L_c^2$, see [3,Section 11].

For $\lambda = 1^4$ or 0, the corresponding blocks $N_\lambda I_8$ are diagonal. For $0 < \lambda \neq 1^4$, the blocks are of the form $N_\lambda I_8 + 2\beta\sqrt{\lambda}J$, with $J = \left(\begin{array}{c|c} 0 & K \\ \hline -K & 0 \end{array}\right)$ and $K = \left(\begin{array}{c|c} 0 & -I_2 \\ \hline I_2 & 0 \end{array}\right)$. K is antisymmetric, $K^T = -K$, $K^T K = -K^2 = I_4$. Its eigenvalues are $\pm i$ and its singular values are 1 quadruple. Thus the eigenvalues of J are given by the quadruple pair ± 1. And the eigenvalues of $L_\varphi^T L_\varphi$ are $N_\lambda \pm 2\beta\sqrt{\lambda}$ for $0 < \lambda \neq 1^4$. □

We have been able to interpret half of the seemingly meaningless singular values in A_k by the singular values of $(a + c, \beta\tilde{1})$ in the complexified algebra $A_{k+1} = A_k \oplus A_k \times \tilde{1}_{k+1}$.

This is not a complete surprise. The interpretation of the nonclassical singular values by induction from A_k to A_{k+1} mimics, for $k \geq 3$, the interpretation of $\sqrt{-1}$ from $\mathbb{R}$ to $\mathbb{C}$ ($k = 0$). What seemed at first impossible or absurd at a given level (dimension 2^k) can be resolved and understood easily at the next level (dimension 2^{k+1}).

However, this is just the tip of the iceberg, since any a in A_k can induce 4 or 8 different vectors in A_{k+1}. A more complete study can be found in [6]. It sheds light on the role of nonclassical SVD in the process of **creation** by hypercomputation.

6 Conclusion

The moral of this story about computation with hypercomplex numbers has already been given by Leibniz more than 300 years ago : « There is hardly any paradox without its proper role ». And history tells us that extreme caution should be used before judging, based on past experience, that certain computations are absurd or impossible. Computation in nonassociative Dickson algebras begs for an extension of classical logic. It calls for a dynamical logic where the results of a computation can be right *and* wrong, depending on the *point of view*.

For example, in A_k, $k \geq 3$, $d = \beta\tilde{1} + c$ in $\mathcal{I}m\ A_k$ is alternative iff c is alternative in $\mathcal{D}_k$. Thus d cannot be a zerodivisor in A_k when we assume c to be alternative. For $|\beta| = ||c||$, $\varphi = (c, \beta\tilde{1})$ is a zerodivisor in $\mathcal{D}_{k+1}$ [3]. This property is indicated by the *nonclassical* singular values : one is 0, the other is $2||c||$. These 2 values are **wrong** in relation with a, in A_k, but they are the **exact** singular values for L_φ in $\mathcal{D}_{k+1}$. The exact classical singular value relative to a is, of course, $\sqrt{2}||c|| = ||a|| = ||\varphi||$, but it is mute about the 2 other singular values for L_φ.

This internal dynamical relativity of viewpoints created by *induction* exists for each level k. The limit as $k \to \infty$ defines an evolution which is clearly beyond the reach of any Turing machine [6].

If one wants to understand the manifested world, the moving, flexible world that one sees and experiences, it is necessary to scrutinize the way *information* is being dynamically processed during computation. This necessity was sensed by Gregory Chaitin already in the mid 1960's when he conceived of his Algorithmic Information Theory (AIT). His theory explores the limits of formal axiomatic reasoning based on the Turing paradigm. As was mentioned in the introduction, Chaitin exposes the limitations from *within* the paradigm. It is clear that Dickson's hypercomputation lies *outside* the paradigm, shedding a complementary light on the limitations from *without*.

Time will come when it will be obvious that the Turing thesis is a straight jacket imposed on computation to make it mechanical. Time will come when the message of Chaitin about the limitations of purely rational computation and of axiomatic reasoning will be received by everyone [2].

There are many ways out of the evolutive dead-end that would result from any axiomatically constrained computation, such as the one that was imagined in the 20^{th} Century by Hilbert (1900) and Turing (1936).

A few such examples were mentioned in the introduction. We presented in some detail another example set in the framework of nonassociative Dickson algebras, for which an extension of classical logic beyond Turing is meaningful from the point of view of information : it takes into account the duality of viewpoints based on induction. Computation in Dickson algebras defines its own internal dynamics for evolution by successive complexification. The internal complexity differs from, yet is complementary to, the descriptive complexity of AIT. In Algorithmic Information, one considers the complexity from the viewpoint of an observer who *simulates* the phenomenon by programme, but is not a player in the evolution.

Nonassociative Dickson algebras appear as a natural framework for nonlinear computation of the kind required by Life itself. Hypercomputation helps us understand some of Life's computing mechanisms which are not revealed by associativity.

Even more than physics, biology, and Life sciences in general, are in desperate need for new computational logics. Logics which can explain how information is being processed by living organisms during their evolution. Chaitin is one of the forerunners in this quest.

Acknowledgment. The author wishes to thank Gregory Chaitin for countless illuminating discussions over the past 14 years. The views expressed above about the Turing thesis on computability are entirely hers.

Références

[1] G. Chaitin (2005) **Meta Math ! The quest for Omega,** Pantheon Books, New York

[2] G. Chaitin (2006) *The limits of reason,* Scientific American **294**, 74-81, March 2006.

[3] F. Chaitin-Chatelin (2005). *Inductive multiplication in Dickson algebras*, Technical Report TR/PA/05/56, CERFACS, Toulouse, France, 2005.

[4] F. Chaitin-Chatelin (2006). *Calcul non linéaire dans les algèbres de Dickson*, Technical Report TR/PA/06/07, CERFACS, Toulouse, France, 2006.

[5] F. Chaitin-Chatelin (2006). *Computation in nonassociative Dickson algebras*, Invited talk, July 2006, Symposium in honour of Prof. Jacqueline Fleckinger, Univ. Toulouse 1.

[6] F. Chaitin-Chatelin (2007). *About an organic logic ruling the continuous evolution of SVD measurements with Dickson hypercomplex numbers.* Cerfacs Technical Report TR/PA/07/55, CERFACS, Toulouse, France 2007.

[7] F. Chaitin-Chatelin, E. Traviésas-Cassan (2005) *Qualitative Computing*, Chapter 5 in **Accuracy and Reliability in Scientific Computing** (Bo Einarsson ed.), p.77-92, SIAM Philadelphia.
Also Cerfacs Report TR/PA/02/58.

[8] M.J. Crowe (1994) **History of vector analysis,** Dover Publ. Inc., New York.

[9] D. Deutsch (2000) **The fabric of reality,** Penguin, London.

[10] T. Franzén (2005) **Gödel's Theorem : an incomplete guide to its use and abuse,** A. K. Peters, Wellesley, MA

[11] R. Penrose (1994) **Shadows of the mind : a search for the missing science of Consciousness,** Oxford University Press, New York.

[12] R. Penrose (2004) **The road to reality : a complete guide to the laws of the Universe,** Jonathan Cape, London

[13] P. Raatikainen (2001) Review of *The unknowable* and *Exploring Randomness* (by G. Chaitin), Notices of the AMS **48**, 992-996.

[14] P. Raatikainen (2007) Review of [10], Notices of the AMS **54**, 380-383.

[15] S. Wolfram (2002) **A new kind of Science,** Wolfram Media, Champaign, IL

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