About an organic logic ruling the continuous evolution of SVD measurements
with Dickson hypercomplex numbers.

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Abstract : The report is a follow-up for [3,4,5]. It studies the properties of the several families of measures which can be associated with a vector in a nonassociative Dickson algebra A_k, $k \geq 3$.

These measures are derived from classical and non classical SVD computations in A_k and A_{k+1}. The notion of organic logic is introduced to organize the various computational landscapes which emerge from these measures.

Keywords : Nonassociative Dickson algebras, classical and nonclassical SVD computations, measurement loop, evolution, logistic, classical logic, organic logic.

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> Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe.
>
> Jacques Hadamard.

1 Introduction

1.1 Paradoxes in Computation

Logical paradoxes are commonly feared as exposing a logical "mistake". This bad reputation is largely undeserved. As Leibniz said about 3 centuries ago: "Il n'est guère de paradoxe sans utilité". We have argued in [3,4,5,7] that 0 and $\sqrt{-1}$ are the two paradoxes which lie at the foundations of modern Calculus. They paved the way for the necessity of accepting new kinds of numbers, repectively the negative and complex numbers.

Accepting 0 as a number was the first step towards a rational analysis of the elusive notion of infinity. The great Euler did not let ∞ stand in his way. He was bold enough to try to give meaning to *divergent series.* For example the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ for $|x| < 1$. Choosing $x = 2$ outside the domain of convergence, Euler proposed to assign the "value" $\frac{1}{1-2} = -1$ to the divergent series $\sum_{n=0}^{\infty} 2^n$. This seemingly absurd claim (an infinite sum of positive numbers is regarded as negative) remained an oddity until it was given a rigorous meaning, almost two centuries later, in the framework of the ring of 2-adic numbers created by K. Hensel.

During the 19th Century there were a few other attempts to create new numbers which did not succeed to convince the mathematical community. The commonly shared opinion, at the dawn of the 20th Century, was that the construction of numbers had been satisfactorily completed with $\mathbb{C}$, the topological closure of the algebraic numbers. Hamilton's quaternions (1843) which allow the noncommutative multiplication of 4D vectors, were seen as an exotic curiosity with no computational significance [3,4]. They were not interpreted as a clue offered by Computation that multiplication on vectors of 4 dimensions could be a fertile notion for continuous analysis as well as for Number Theory. As we now understand Computation [2], the structure of the Hurwitz ring of integral quaternions (1896) explains the elegant proof of the Bachet-Lagrange theorem of 4 squares proposed by Euler (1773).

A consequence of this prevalent opinion was that sets of vectors of n dimensions were given primarily an *additive* structure, with the concept of linear vector space.

In the 20^{th} Century, the development of Physics favoured the concept of *associative* Clifford algebras as an extension of the quaternions. Nonassociative algebras such as the algebras of Lie and P. Jordan are used to explain, but not to compute. Common wisdom has it that nonassociativity is too poor a notion to provide interesting calculations. Nothing could be further from reality as we have already shown in [3,4]. In retrospect, it should not be surprising that relaxing the associativity condition for × opens new computational avenues.

1.2 Hypercomputation in Dickson algebras A_k, $k \in \mathbb{N}$

The Dickson algebras A_k of dimension 2^k, $k \geq 0$, provide *genuinely nonlinear* algebraic structures for Computation. They can be constructed by induction from $A_0 = \mathbb{R}$, yielding $A_1 = \mathbb{C}$ and $A_2 = \mathbb{H}$. For $k \geq 3$, they are *nonassociative*, and nonalternative with zerodivisors for $k \geq 4$ [3,4].

The inductive connection $A_{k+1} = A_k \oplus A_k \times \tilde{1}_{k+1}$ for $\tilde{1}_{k+1} \in A_k^{\perp}$ provides these algebras with a high potential for matrix computation [3]. The existence of zerodivisors creates computing paradoxes which modify such a well-established notion as that of the euclidean norm for a vector. These paradoxes challenge classical logic [3,4,5]. They call for the emergence of a new computational logic. The *organic logic* (section 10) is the emergent logic which organizes computations in nonassocaitive Dickson algebras.

1.3 Presentation of SVD in A_k, $k \in \mathbb{N}$

We develop in this report a systematic study of the inductive SVD (Singular Value Decomposition) which was partially addressed in previous works [3,4,5].

To avoid repetition, the reader is refered to [3] for definitions, notations and theoretical background. According to the splitting $A_k = \mathbb{C}_{\tilde{1}} \oplus \mathcal{D}_k$ which is nontrivial for $k \geq 2$, any vector a in A_k, $k \geq 2$, can be represented as the two sums

$$a = \alpha + \beta\tilde{1} + c = h + c,$$

where $h = \alpha + \beta\tilde{1} \in \mathbb{C}_{\tilde{1}}$ is the fully alternative head, and $c \in \mathcal{D}_k$ is the doubly pure tail.

For $c \neq 0$, we consider the normalization $a = ||c|| \left(\frac{h}{||c||} + \frac{c}{||c||} \right)$. Thus, without loss of generality, we assume that $||c|| = 1$.

Let $\sigma_c = \sigma(-L_c^2)$ denote the spectrum of $-L_c^2$. Any $\lambda \in \sigma_c$ is nonnegative with multiplicity $\equiv 0 \pmod 4$. We set $N_\lambda = \alpha^2 + \beta^2 + \lambda = N(h) + \lambda$, $\lambda \in \sigma_c$.

Let $\lambda = 1^4$ denote the eigenvalue $1 = N(c)$ with multiplicity 4, and 4D-eigenspace $\mathbb{H}_c = \{1, c, \tilde{c}, \tilde{1}\} \cong \mathbb{H}$. We set $\hat{\sigma}_c = \sigma_c - \{1^4\} \neq \emptyset$ for $k \geq 3$.

For $0 < \lambda \neq 1^4$, the eigenspace $E_\lambda \subset \mathbb{H}_c^\perp$ is a direct sum of 4D-subspaces of the form $V_{x_\lambda} = \mathbb{H}_c \times x_\lambda$, $x_\lambda \in E_\lambda$. For $\lambda = 0$, E_0 is a direct sum of 2D-subspaces of the form $W_{x_0} = \mathbb{C}_{\tilde{1}} \times x_0$, for $x_0 \in E_0$.

Given the spectral decomposition for $-L_c^2$ corresponding to the tail $0 \neq c \in \mathcal{D}_k$, and given the head $0 \neq h \in \mathbb{C}_{\tilde{1}}$, we analyze in this report the spectral decompositions, for $k \geq 2$, of the following maps

i) $L_a^T L_a$ with $a = \alpha + \beta\tilde{1} + c \in A_k$,

ii) $L_{\varphi_l}^T L_{\varphi_l}$, $l = 0$ to 7, where φ_l is one of the 8 vectors in A_{k+1} induced by the 3 orthogonal vectors $\alpha 1 = \alpha, \beta\tilde{1}, c$, according to the Table 1.1 below, for $\alpha\beta \neq 0$, $c \neq 0$, where $l' \equiv l \pmod 4$

l'	0	1	2	3
$\varphi_l = \varphi_{l'}$	$(c + \beta\tilde{1}, \alpha)$	$(c + \alpha, \beta\tilde{1})$	(c, h)	$(a, 0)$
$\varphi_l = \varphi_{l'+4}$	$(\alpha, c + \beta\tilde{1})$	$(\beta\tilde{1}, c + \alpha)$	(h, c)	$(0, a)$

Table 1.1: $\alpha\beta \neq 0$

When $\alpha\beta = 0$, then Table 1.1 has only 2 (or 1) different columns because of the following identifications:

$\alpha = 0, \beta \neq 0$	$\varphi_1 = \varphi_2$	$\varphi_0 = \varphi_3$
	$\varphi_5 = \varphi_6$	$\varphi_4 = \varphi_7$
$\alpha \neq 0, \beta = 0$	$\varphi_0 = \varphi_2$	$\varphi_1 = \varphi_3$
	$\varphi_4 = \varphi_6$	$\varphi_5 = \varphi_7$
$\alpha = \beta = 0$	$\varphi_0 = \varphi_1 = \varphi_2 = \varphi_3$	$\varphi_4 = \varphi_5 = \varphi_6 = \varphi_7$.

Altogether, there are 2,4 or 8 different induced vectors depending on the location of h in $\mathbb{C}_{\tilde{1}}$. For all l, $||\varphi_l|| = ||a||$.

We observe that for $l' = 0$ to 3, $\varphi_l = \varphi_{l'+4}$ is the *reverse* of φ_l: the left and right parts are exchanged. We set $\varphi_{l'+4} = \varphi_{l'}^R$ (R for reverse) and define the 4 couples $C(\varphi_{l'}) = \{\varphi_{l'}, \varphi_{l'}^R\}$ for $l' = 0$ to 3.

2 SVD for $a \in A_k$ derived from $c \in \mathcal{D}_k$, $k \geq 2$

Let $k \geq 2$. We consider c given in $\mathcal{D}_k$, $\|c\| = 1$. Therefore $a = \alpha + \beta\tilde{1} + c \in \mathbb{H}_c$ represents an eigenstate for $-L_c^2$ associated with $N(c) = 1^4$. In the quaternionic framework $\mathbb{H}_c$, a is represented by the vector $(\alpha, \beta, 1, 0)$ with a *zero* component on $\tilde{c}$. Similarly $\tilde{a} = (-\beta, \alpha, 0, 1)$ in $\mathbb{H}_c$.

Lemma 2.1 *For $c \in \mathcal{D}_k$ and $x \in A_k$, one gets*

$$\begin{aligned} c \times (\tilde{1} \times x) + \tilde{1} \times (c \times x) = 0, \\ (x \times c) \times \tilde{1} + (x \times \tilde{1}) \times c = 0. \end{aligned}$$

Proof. 1)For $x \in \mathbb{H}_c$, the result follows by associativity:
$c \times (\tilde{1} \times x) = (c \times \tilde{1}) \times x = -(\tilde{1} \times c) \times x$
and $(x \times c) \times \tilde{1} = x \times (c \times \tilde{1}) = -x \times (\tilde{1} \times c) = -(x \times \tilde{1}) \times c.$
2)For $x \in \mathbb{H}_c^\perp$, then $c \times (\tilde{1} \times x) = -c \times \tilde{x} = \tilde{x} \times c = -\widetilde{x \times c} = \widetilde{c \times x} = -\tilde{1} \times (c \times x)$, and $(x \times \tilde{1}) \times c = \tilde{x} \times c = -\widetilde{x \times c}$. ☐

Proposition 2.2 *For $a = \alpha + \beta\tilde{1} + c, \in \mathcal{D}_k$, then $L_a^T L_a = N(h) I_{2^k} - L_c^2 = M_a$.*

Proof. Direct computation for $x \in A_k$ of
$L_a^T L_a x = (\alpha - \beta\tilde{1} - c) \times (\alpha x + \beta\tilde{1} \times x + c \times x) =$
$(\alpha^2 + \beta^2)x - c \times (c \times x) + (\alpha\beta - \alpha\beta)\tilde{1} \times x + (\alpha - \alpha)c \times x - \beta[\tilde{1} \times (c \times x) + c \times (\tilde{1} \times x)]$. ☐

Corollary 2.3 *The singular values of L_a for $a = h + c$ are given by $\sqrt{N_\lambda} = \sqrt{N(h) + \lambda}$, $\lambda \in \sigma_c$*

Proof. Clear from $L_a^T L_a = N(h)I - L_c^2$. ☐

For $k = 2$ or 3, the corollary reduces to the theorem of Pythagoras for the euclidean norms of the two orthogonal vectors c and $h : N(a) = N(h) + N(c)$.

Corollary 2.3 extends, for $k \geq 4$, the theorem to all the singular values for L_c and L_a. The quantities $\sqrt{N_\lambda}$ represent the exact singular values for L_a.

We introduce the

Definition 2.1 *The classical SVD derivation in A_k, $k \geq 2$, is ruled by Proposition 2.2*

We have discovered in 2005 [3] that the nonassociativity for multiplication in A_k, $k \geq 3$, opens the possibility for a different derivation of the SVD for L_a from that for L_c. This other derivation yields results which do not all obey the Pythagorean rule when $\beta \neq 0$.

Definition 2.2 *The different SVD derivation in A_k, $k \geq 3$, to be defined in Section 3 is called nonclassical.*

3 Nonclassical SVD derivation in A_k, $k \geq 3$

3.1 Nonclassical singular values for L_a

The nonclassical derivation is defined in [3, Section 9]. It uses the block-diagonal form of $L_a^T L_a$ (with blocks of order 4) written in the eigenbasis for $-L_c^2$. In this nonclassical approach, the order in which the addition of α and $\beta\tilde{1}$ to c is performed matters from the SVD point of view: *addition* is seen as *nonassociative* through the SVD filter. When $\alpha\beta \neq 0$, there are 3 different routes to go from c to a in $\mathbb{C}_{\tilde{1}}$ (see Figure 1 in [5]). One can reach a directly along the diagonal $h : a = c + h$, or side ways through $d = c + \beta\tilde{1}$, or through $e = c + \alpha$. When $\alpha\beta = 0$, the route is unique.

For $s \geq 0$, $t \in \mathbb{R}$, we define the map
$\zeta : (s,t) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \zeta(s,t) = ((s-t)^2, (s+t)^2) \in (\mathbb{R}^+)^2$.

Observe that $\zeta(s,-t)$ and $\zeta(s,t)$ are symmetric with respect to the (first) bisector in $(\mathbb{R}^+)^2$.

Theorem 3.1 *For $a = \alpha + \beta\tilde{1} + c$, $c \in \mathcal{D}_k$, $k \geq 3$, the nonclassical SVD derivation yields the nonnegative eigenvalues for $L_a^T L_a$ listed in the two columns below, according to $\lambda \in \sigma_c$ and to the chosen route :*

$\lambda \in \sigma_c$	via $d = c + \beta\tilde{1}$	direct or via $e = c + \alpha$
1^4	$N_1 = \alpha^2 + \beta^2 + 1$	$N_1 = \|\|a\|\|^2$
$0 < \lambda \neq 1^4$	$N_\lambda \pm 2\beta\sqrt{\lambda} =$ $\alpha^2 + \zeta(\sqrt{\lambda}, \beta) \geq \alpha^2$	$N_\lambda \pm 2\beta\sqrt{\lambda + \alpha^2} =$ $\zeta(\sqrt{\lambda + \alpha^2}, \beta) \geq 0$
0 $(k \geq 4)$	$N_0 = \alpha^2 + \beta^2$	$N_0 \pm 2\beta\alpha = (\alpha \pm \beta)^2$

Table 3.1:

Proof. It is based on [3, Lemma 9.3] where the roles of a and c have been exchanged.

1) The results for the direct route $a = c + h$ are given by Proposition 9.4 in [3]. They are listed in the 2nd column.

2) To get the 1^{st} column, we examine $-L_d^2$ by letting $\alpha = 0$ in Proposition 9.4. Therefore, with the notations of [3] :

$$\begin{aligned} -L^2_{d\restriction B_i} &= (\beta^2+1)I_4 \quad \text{for} \quad i=1, \\ &= (\beta^2+\lambda)I_4 - 2\beta\sqrt{\lambda}\left(\begin{array}{c|c} 0 & I_2 \\ \hline I_2 & 0 \end{array}\right) \quad \text{for} \quad i=2, \\ &= \beta^2 I_2 \quad \text{for} \quad i=3. \end{aligned}$$

Then we use $L_a^T L_a = \alpha^2 I - L_d^2$.

3) The results via e follow from the fact that $L_e^T L_e = \alpha^2 I - L_c^2$ with eigenvalues $\lambda + \alpha^2$.

Then we apply Lemma 9.3 to $a = \beta\tilde{1} + (\alpha + c)$. The results for $L_a^T L_a$ are obtained from the values for $-L_d^2$ where λ is replaced by $\lambda + \alpha^2$. For $\lambda = 0 (i = 3)$, 0 is replaced by α^2 so that $L_a^T L_{a\restriction B_3} = (\alpha^2+\beta^2)I_2 + 2\alpha\beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\square$

Some remarks are in order.

For $\beta = 0$, the the results in Table 3.1 agree with the exact eigenvalues N_λ given in Corollary 2.3.

For $\alpha = 0$, the first and second columns in Table 3.1 agree with each other.

For $\alpha\beta \neq 0$, all four nonclassical values obtained for $0 < \lambda \neq 1^4$ are different from the exact value N_λ, they share the common arithmetic mean N_λ. Not surprisingly, the roles of 1 and $\tilde{1}$ are different, and account for the resulting nonassociativity of addition.

The difference between classical and nonclassical singular values for $0 < \lambda \neq 1^4$ exposes the limit of linear matrix computation in a *nonassociative* framework. Some of the nonclassical singular values are not, however, as "wrong" as they appear in A_k. Their values can be interpreted as "correct" by induction in A_{k+1}, see Section 4 and 5, and also [3,4,5]. The two minimum values for a given λ are obtained for $\beta^2 = \lambda$ via d (resp. $\beta^2 = \lambda + \alpha^2$ via e or directly) as the respective values $|\alpha|$ and $\sqrt{\alpha^2 + 4\lambda}$ (resp. 0 and $2\sqrt{\lambda + \alpha^2}$: a is a split zerodivisor [3]).

3.2 Summary for $k \geq 3$

By definition, the classical and nonclassical SVD derivations in A_k may differ for $k \geq 3$ only ($\hat{\sigma}_c \neq \emptyset$). They yield the same results for all k when $\beta = 0$, α arbitrary. They differ for $\beta \neq 0$. In this case, nonclassical derivation provides two types (resp. one type) of answers when $\alpha \neq 0$ (resp. $\alpha = 0$).

Let us consider in the complex plane $\mathbb{C}_{\tilde{1}}$ the two axes which partition $\mathbb{C}_{\tilde{1}}$ into 3 disjoint sets (Figure 3.1)

I : the real axis $\mathbb{R}$ $(\beta = 0)$
II : the imaginary axis $\mathbb{R}^* \times \tilde{1}$ $(\beta \neq 0,\ \alpha = 0)$
III : the domain outside the axes $(\alpha\beta \neq 0)$

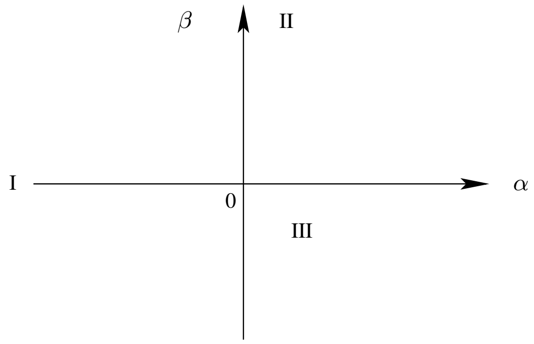

Figure 3.1: Partition of $\mathbb{C}_{\tilde{1}}$

Classical and nonclassical eigenvalues for $L_a^T L_a$ are listed below, depending on $\lambda \in \sigma_c$ and on the location of the head $h = \alpha + \beta\tilde{1}$ in $\mathbb{C}_{\tilde{1}}$:

	I	II	III
$0 < \lambda \neq 1^4$	$\alpha^2 + \lambda > 0$	$\beta^2 + \lambda > 0,$ $(\beta \pm \sqrt{\lambda})^2 \geq 0$	$\lambda + \alpha^2 + \beta^2, \alpha^2 + (\beta \pm \sqrt{\lambda})^2$ $(\sqrt{\lambda + \alpha^2} \pm \beta)^2 \geq 0$
$\lambda = 1^4$	$\alpha^2 + 1 \geq 1$	$\beta^2 + 1 > 1$	$1 + \alpha^2 + \beta^2 > 1$
$\lambda = 0$	$\alpha^2 \geq 0$	$\beta^2 > 0$	$\alpha^2 + \beta^2 > 0,\ (\alpha \pm \beta)^2 \geq 0$

Table 3.2:

There are 6 (resp. 9) possibilities when c is not (resp. is) a zerodivisor, which yield 1,3 or 5 values. We observe that the values depend continuously on the components α, β of the head. However the multiplicities are not preserved when $\beta \to 0$ (interfaces II/I and III/I) and $\alpha \to 0$ (interface III/II).

The metric interpretation of Table 3.2 is deferred to Section 4.3, where we consider a in the 3D-subspace $lin(1, \tilde{1}, c)$ of A_k.

3.3 Counting the singular values

Let d represent the number of distinct eigenvalues in $\hat{\sigma}_c = \sigma_c - \{1^4\}$: $d = 1$ for $k = 3$, $d = 1$ or 3 for $k = 4$ and $1 \leq d \leq 2^{k-2}$ for $k \geq 5$.

Therefore the total number of different eigenvalues in σ_c is d (resp. $d+1$) iff $mult(1) \geq 8$ (resp. $=4$). Counting multiplicities, we have $t = 2^k$ eigenvalues.

We set $m_1 = mult(1) - 4$ and $m_0 = mult(0)$. Generically, for $k \geq 5$, m_1 and m_0 are either 0 or multiples of 4. For $k = 3$, $m_1 = 4$ and $m_0 = 0$, and for $k = 4$, $m_1 > 0$, and $m_0 \geq 0$.

Definition 3.1 *D is the maximum number of different classical and nonclassical singular values for L_a. T is the total number of such values counting multiplicities.*

The maximum number D is achieved for $\alpha\beta\lambda \neq 0$ since the quantities $|\beta|\sqrt{\lambda}$ and $|\beta|\sqrt{\lambda + \alpha^2}$ are positive and distinct for $\lambda \in \hat{\sigma}_c$. The values of D depend on $\alpha, \beta \in \mathbb{R}$, m_0, $m_1 \in \mathbb{N}$ and on d. They are generic for $k \geq 5$, and correspond to 6 possibilities.

Proposition 3.2 *The values of D are listed below:*

	I		II	III
$m_1 > 0$	d	$m_0 > 0$	$2d$	$4d$
$m_1 = 0$	$d+1$	$m_0 = 0$	$2d+1$	$4d+1$

Table 3.3: $d \mapsto D$, $k \geq 3$

Proof. Use Table 3.2.

In III, for $m_0 > 0$, $D = 1 + 4(d-1) + 3 = 4d$ and for $m_0 = 0$, $D = 1 + 4d$.

In II, the factor 4 and term 3 (for $m_0 > 0$) are replaced respectively by 2 and 1.

The rest follows. □

The values of T depend on $\alpha, \beta \in \mathbb{R}$, on $t = 2^k$ and on $m_0 \in \mathbb{N}$, with $0 \leq m_0 \leq 2^k - 4(k-1)$ for $k \geq 3$ [3, Proposition 11.4]. There are 3 possibilities for $k \geq 3$.

Proposition 3.3 *The values of T are listed below:*

I	II	III
t	$2t - m_0 - 4$	$4t - m_0 - 12$

Proof. Use Proposition 3.2.

For III, $T = (2^k - 4 - m_0)4 + 4 + 3m_0 = 4t - m_0 - 12$.

For II,$T = (2^k - 4 - m_0)2 + 4 + m_0 = 2t - m_0 - 4$. □

Corollary 3.4 *The values listed below are valid for D when c is alternative, and for T more generally when c is not a zerodivisor.*

	I	II	III	condition
D	1	3	5	$\sigma_c = \{1\}$
T	t	$2(t-2)$	$4(t-3)$	$m_0 = 0$

Proof. Clear since $m_1 > 0$ and $d = 1$ for D, with $m_0 = 0$ for T. □

Proposition 3.5 *For $k \geq 4$ and $\beta \neq 0$, T can take integer values in the following intervals:*
II : $[t + 4(k-2),\ 2(t-2)]$,
III : $[3t + 4(k-4),\ 4(t-3)]$

Proof. We use the bounds $0 \leq m_0 \leq t - 4(k-1)$. Thus for $m_0 = t - 4(k-1)$
II : $2^{k+1} - 4 - 2^k + 4(k-1) = 2^k + 4(k-2)$,
III : $2^{k+2} - 12 - 2^k + 4(k-1) = 3 \times 2^k + 4(k-4)$.

The upper bounds are obtained for $m_0 = 0$. □

For $k = 3$, the intervals reduce to the numbers $12 = 2 \times 6$ (II) and $20 = 4 \times 5$ (III).

Example 3.1 Non classical SVD in the sedenions A_4

For $k = 4$, let $c = (c_1, c_2) \in \mathcal{D}_4$ with $c_1, c_2 \in \Im m\mathbb{G}$, $||c|| = 1$. Then $m_1 = 4$ or 12 is nonzero.

1) When c_1 and c_2 are independent, $G = [c_1, -, c_2] \neq 0$ and $m_1 = 4$. The eigenvalues of $-L_c^2$ different from 1 are $1 \pm ||G||$, thus $d = 3$. Moreover,

   $||G|| = 1 \iff ||c_1|| = ||c_2|| = 1/\sqrt{2}$ and $< c_1, c_2 >= 0 \iff c$ is a zerodivisor $\iff m_0 = 4$ (see [3, Theorem 8.9]).

   The values for D and T are listed below:

	m_0	III	II	I
D	4	12	6	3
	0	13	7	
T	4	48	24	16
	0	$52 = 4 \times 13$	$28 = 2 \times 14$	

2) When c_1 and c_2 are dependent, c is alternative thus $m_1 = 12$, $d = 1$ and $m_0 = 0$. The corresponding values of D and T are listed below:

	III	II	I
D	5	3	1
T	52	28	16

△

Example 3.2 $c = (b, \tilde{b})$ in $\mathcal{D}_k$, with b alternative in $\mathcal{D}_{k-1}$

We suppose $k \geq 4$, see [3, Section 10.3 and 11.1]. We suppose that b is alternative in $\mathcal{D}_{k-1}$, $k-1 \geq 3$, with $||b||^2 = \frac{1}{2}$ so that $||c|| = 1$. Then $\sigma_c = \{0, 1, 2\}$ with $m_1 = 4$, $m_0 = 2^{k-1} - 4 \geq 4$, thus $d = 3$. The values of D and T are listed below, with $\frac{t}{2} = 2^{k-1} \geq 8$.

	III	II	I
D	12	6	3
T	$7 \times \frac{t}{2} - 8$	$3 \times \frac{t}{2}$	t

$\triangle$

3.4 Characteristic lines and points in $\mathbb{C}_{\tilde{1}}$ for a in A_k, $k \geq 3$

Let be given a in A_k, $k \geq 3$, split into $a = \alpha + \beta\tilde{1} + c$, $||c|| = 1$. We consider a in reference with the 3D-frame $F_3(c) = lin(1, \tilde{1}, c)$ in A_k, where α, β vary in $\mathbb{R}$, but c is fixed in $\mathcal{D}_k$. The 3 normalized vectors $1, \tilde{1}, c$ form an orthogonal trihedron isomorphic to $\mathbb{R}^3$, depending on c: $F_3(c) = \mathbb{C}_{\tilde{1}} \oplus \{c\}$.

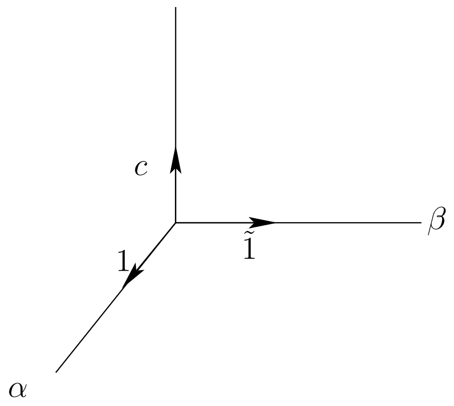

Figure 3.2: The reference frame $F_3(c)$ for a

Definition 3.2 *To the vector a in A_k and its d distinct tail eigenvalues λ in $\hat{\sigma}_c$, are associated 2 families of characteristic lines and points in $\mathbb{C}_{\tilde{1}}$ as follows.*

1) The characteristic lines are either hyperbolas defined by $\beta^2 - \alpha^2 = \lambda > 0$, or the two bisectors $\beta = \pm\alpha$ for $\lambda = 0 (k \geq 4)$.

2) The characteristic points have the abscissa $\pm\sqrt{\lambda}$ on the imaginary axis.

See Figure 3.3

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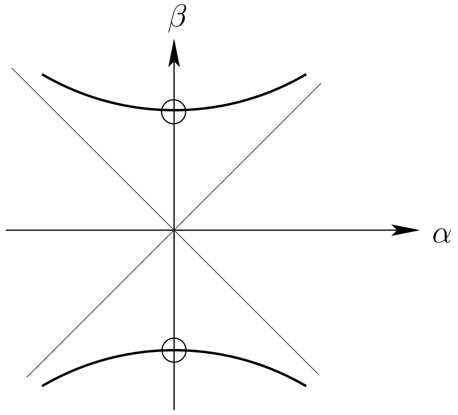


Figure 3.3: Characteristic hyperbola ▬ and points (o) for a, $0 < \sqrt{\lambda} \neq 1^4$ in $\mathbb{C}_{\tilde{1}} \simeq \mathbb{R}^2$

The hyperbolas have the asymptotes $\beta = \pm\alpha$, and they pass though the characteristic points. When $\lambda = 0$, the corresponding hyperbola is degenerate and reduces to $\beta = \pm\alpha$: the origin is a characteristic point iff c is a zerodivisor for $k \geq 4$.

When the head $h = \alpha + \beta\tilde{1}$ of the vector a lies on a characteristic line, or exactly at a characteristic point, we get the following interpretations for $k \geq 3$.

	III	II
h on hyperbola	a is a split zerodivisor	—
h is a characteristic point	—	$d = \beta\tilde{1} + c$ is a split zerodivisor

Table 3.4:

And d is a split zerodivisor in $\Im m\ A_k$ iff $(c, \beta\tilde{1})$ an $(\beta\tilde{1}, c)$ are zerodivisors in $\mathcal{D}_{k+1}$.

4 The evolution of measurement by SVD for a vector a in A_k as a function of $k \in \mathbb{N}$

4.1 $k = 0$ and 1

In $\mathbb{R}$ (resp. $\mathbb{C}$) the measure of a real (resp. complex) number is its absolute value (resp. modulus) which satisfies

$$0 \leq \big||x| - |y|\big| \leq |x+y| \leq |x| + |y|, \quad \text{for} \quad x, y \in \mathbb{R} \text{ (resp.}\mathbb{C}).$$

In the complex plane, the 2D-vectors representing the complex numbers can be orthogonal:

$$< x, y >= 0 \quad \text{iff} \quad |x+y| = \sqrt{|x|^2 + |y|^2}.$$

This is Pythagoras' theorem in its simplest form. In particular, for $0 \neq z = a + ib \in \mathbb{C}$, $|z| = \sqrt{a^2+b^2}$, $a, b \in \mathbb{R}$, that is $z\overline{z} = a^2 + b^2 > 0$.

4.2 $k = 2$

In the associative algebra $\mathbb{H}$ of quaternions, the measure of a 4D-quaternion a is given by its euclidean norm $||a||$, by virtue of Pythagoras' theorem. SVD on the split vector $a = \alpha + \beta\tilde{1} + c$, $c \in \mathcal{D}_2$ gives the same answer $||a||$ because $\hat{\sigma}_c = \emptyset$.

Diversity begins with $k = 3$. The measure of a vector a by SVD is a *multivalued* function in a *nonassociative* Dickson algebra (see Table 3.2).

4.3 Measuring a vector a in A_k, $k \geq 3$

We consider the splitting $a = \alpha + \beta\tilde{1} + c$, $c \in \mathcal{D}_k$ for any given a in A_k according to $1, \tilde{1}$ and c. We assume that $||c|| = 1$.

Definition 4.1 *The measure of $a = \alpha + \beta\tilde{1} + c$ is the map $a \mapsto \xi(a)$ which associates to the vector a the set $SV(a)$ of the singular values for L_a derived from those for L_c. The set $SV(a)$ includes the classical and nonclassical singular values for $k \geq 3$.*

As before, $\sigma_c = \sigma(-L_c^2)$ and $\hat{\sigma}_c = \sigma_c \backslash \{1^4\} \neq \emptyset$ for $k \geq 3$.

4.4 Classical measures for $a = h + c$, with $\lambda \in \sigma_c$, $h \neq 0$

First, the choice $\lambda = 1^4$ yields the euclidean norm $||a|| = \sqrt{N_1} = \sqrt{1 + \alpha^2 + \beta^2}$. It corresponds to Pythagoras' theorem applied to $a = c + h$, thus

$$N(a) = a\overline{a} = N(c) + N(h) = 1 + N(h) = N_1.$$

For $\lambda \in \hat{\sigma}_c$, the other classical measures (different from $||a||$) are given by $\sqrt{N_\lambda} = \sqrt{\lambda + \alpha^2 + \beta^2}$ for $\lambda \neq 1^4$. This amounts to the Pythagoras rule on $a = h + c$ where the measure $\xi(c)$ for c is one of the singular values $\sqrt{\lambda}$, $\lambda \in \hat{\sigma}_c$.

We use the notation for $a = h + c$:

$$\xi_P(h + c) = \sqrt{||h||^2 + \xi^2(c)} = \xi_P(a) = \sqrt{N_\lambda}.$$

When a is written as $\alpha + \beta\tilde{1} + c$, we also have the formula:

$\boldsymbol{A}$: $\xi_P(a) = \xi_P(\alpha + d)$ with $\xi(d) = \xi_P(\beta\tilde{1} + c)$.

Speaking informally, we shall say that formula $\boldsymbol{A}$ is obtained by the "composition" $\xi_P \circ \xi_P$. Formula $\boldsymbol{A}$, which agrees with mathematics and geometry, is called *conservative.*

4.5 Nonclassical meaures for $a = \alpha + \beta\tilde{1} + c$ with $\lambda \in \hat{\sigma}_c$.

In the nonclassical SVD derivation, the associativity of addition is *not* preserved in the splitting of a into *three* vectors. Observe that it is preserved in the Pythagorean rule above, which is an *irrational* formula. This irrationality allows the 3D-reference frame $F_3(c)$ for a to remain rigid under classical measurement, in conformity with the viewpoint of euclidean geometry.

In contradistinction, nonclassical measures *modify the geometry* perceived around c. They transform $F_3(c)$ into the plane $P_2(c) = lin(1, c)$: loosely speaking, $\tilde{1}$ in $F_3(c)$ is "absorbed" by the plane $P_2(c) = lin(1, c)$. We consider c given in $\mathcal{D}_k$, $\|c\| = 1$. Therefore $a = \alpha + \beta\tilde{1} + c \in \mathbb{H}_c$ represents an eigenstate for $-L_c^2$ associated with $N(c) = 1^4$. In the quaternionic framework $\mathbb{H}_c$, a is represented by the vector $(\alpha, \beta, 1, 0)$ with a *zero* component on $\tilde{c}$. The greatest modification is realized by means of the *rational* rule for $a = \beta\tilde{1} + e$, with $e = c + \alpha$ of measure $\xi(e)$:

$$\xi_r(\beta\tilde{1} + e) = \{|\beta - \xi(e)|, |\beta + \xi(e)|\} \in (\mathbb{R}^+)^2$$

In the rational rule, the vector $\beta\tilde{1} + e$ is assigned *two* rational measures, computed as if the 3 vectors $\tilde{1}$, e and $\overline{e}$ were colinear in $P_2(c)$. Observe that one of the measures is zero when $|\beta| = \xi(e)$, thus a is a split zerodivisor.

A milder modification corresponds to $\tilde{1}$ and c perceived as colinear ($c = \pm\tilde{1}$).

Proposition 4.1 *For a split in 3 with $\alpha\beta \neq 0$, there are 2 additional types of measures, for $\lambda \neq 1^4$, given by the formulae below.*

$\boldsymbol{B}$: $\sqrt{\alpha^2 + (\beta \pm \sqrt{\lambda})^2} = \xi_P(\alpha + d)$ *with* $\xi(d) = \xi_r(\beta\tilde{1} + c)$ *in which* $\tilde{1}$ *is perceived as colinear with c:* $a = \alpha + (\beta\tilde{1} + c)$ *and* $c = \pm\tilde{1}$.

$\boldsymbol{C}$: $|\beta \pm \sqrt{\lambda + \alpha^2}| = \xi_r(\beta\tilde{1} + e)$ *with* $\xi(e) = \xi_P(\alpha + c)$. *The complex unit* $\tilde{1}$ *is perceived as colinear with* e *and* $\overline{e}$*:* $a = \beta\tilde{1} + (\alpha + c)$ *and* $\pm\tilde{1} = \frac{1}{||e||}\{e, \overline{e}\}$.

Proof. Clear. The second formula corresponds to the nonclassical singular value computed directly or via $e = c + \alpha$. ☐

Formulae $\boldsymbol{B}$ and $\boldsymbol{C}$ are called *inventive* because they modify the geometry around c: The 3D-frame $F_3(c)$ "looses" one dimension to become the 2D-frame $P_2(c)$ in a way to be described. We call apparent vector any vector constructed from the data $\pm c, \pm\alpha, \pm\beta$ whose measures are given by $\boldsymbol{B}$ or $\boldsymbol{C}$, for $\lambda \in \hat{\sigma}_c$. Let us assume first that $0 < \lambda \neq 1^4$.

Given $\alpha\beta \neq 0$, in case $\boldsymbol{B}$ the vector a in $F_3(c)$ corresponds to 8 antiparallel vectors on 4 independent axes in the plane $P_2(c)$ with common origin O (Figure 4.1 with $0 < \lambda < 1$). $\tilde{1}$ is "absorbed" by $c \in \mathcal{D}_k$.

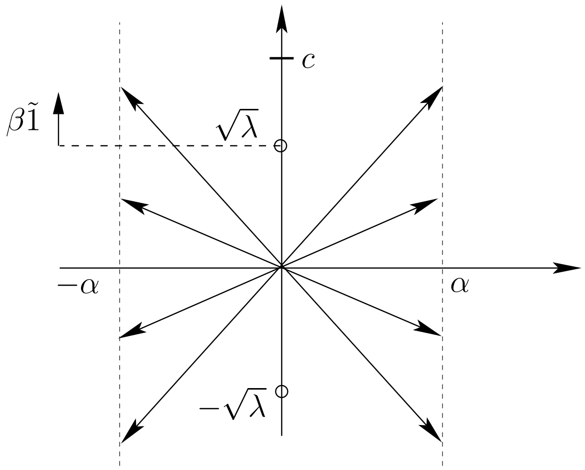


Figure 4.1: Formula $\boldsymbol{B}$: $\tilde{1} = \pm c,\ 0 < \lambda < 1,\ \alpha\beta \neq 0$

In case $\boldsymbol{C}$, $\tilde{1}$ is "absorbed" by the unit vectors $\frac{e}{\|e\|}$ and $\frac{\overline{e}}{\|e\|}$, with $e = c + \alpha$, $\overline{e} = -c + \alpha$, $\Re e(e) = \alpha \neq 0$. The resulting vectors in $P_2(c)$ can be written as $e_\lambda \pm \beta\tilde{1}$, where e_λ is any of the 4 vectors on the two axes defined by e and $\overline{e}$, such that $||e_\lambda|| = \sqrt{\lambda + \alpha^2} = \xi_P(\alpha + c)$.

There are 8 apparent vectors antiparallel and colinear with e and $\overline{e}$, with common origin O (Figure 4.2 with $\lambda > 1$). Four vectors are reduced to 0 iff $|\beta| = \sqrt{\lambda + \alpha^2}$ (a is a split zerodivisor).

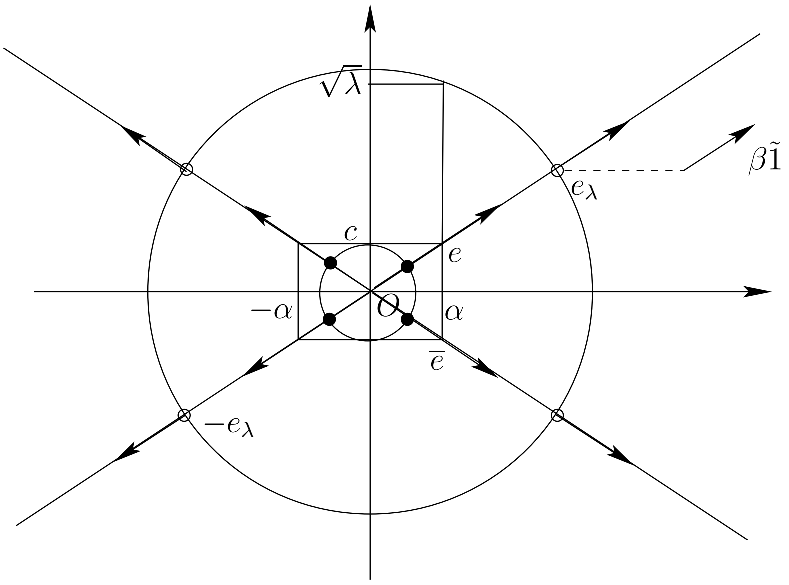


Figure 4.2: Formula $\boldsymbol{C}$: $\tilde{1} = \pm\frac{1}{\|e\|}\{e, \overline{e}\}$, $\lambda > 1$, $\alpha\beta \neq 0$

When $\alpha\beta = 0$, the previous two cases coalesce in 2 different ways.

II: $\alpha = 0$, $\beta \neq 0$, the plane $P_2(c)$ becomes the line $L_1(c)$ generated by c. The 4 or 2 axes associated with $\boldsymbol{B}$ or $\boldsymbol{C}$ become the unique axis $\{c\}$ identified with the imaginary axis $\alpha = 0$. The 4 vectors on c correspond to the measures $|\beta \pm \sqrt{\lambda}| = \xi_r(d) \geq 0$, $d = \beta\tilde{1} + c$.

I:$\alpha \neq 0$, $\beta = 0$, the 8 vectors coalesce into 4 in $P_2(c)$ with unique measure $\sqrt{\lambda + \alpha^2} = \xi_P(e) \geq |\alpha|$.

When $\lambda = 0$, the corresponding number of apparent vectors is listed below:

	III	II	I
$\boldsymbol{B}$	4	4	2 .
$\boldsymbol{C}$	8	4	4

A more complete picture would be obtained by considering the 4D-space $\mathbb{H}_c = \{1, c, \tilde{1}, \tilde{c}\}$, rather than $F_3(c)$. Indeed c and $\tilde{c}$ share the same singular values, and a, $\tilde{a} \in \mathbb{H}_c$. We shall go back to this question in Section 10.

4.6 The diversity of measures for $\lambda \neq 1^4$, $k \geq 3$

The set of measures for a in A_k are listed below, by means of the 2 maps ξ_P and ξ_r, depending on $\lambda \neq 1^4$, and on the location of h in $\mathbb{C}_{\tilde{1}}$, $k \geq 3$. Recall that $\xi(c) = \sqrt{\lambda}$,

$h = \alpha + \beta\tilde{1}$, $d = \beta\tilde{1} + c$, and $e = \alpha + c$. This set, denoted $SV(a)$, is defined by the 3 formulae $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$, where $\boldsymbol{A}$ is conservative, and $\boldsymbol{B}$, $\boldsymbol{C}$ are inventive.

	III	II	I
Classical	$\sqrt{N_\lambda} = \xi_P(h+c)$	$\xi_P(\beta\tilde{1}+c)$	$\xi_P(\alpha+c)$
Non Classical	$\xi_P \ (\alpha+d)$ with $\xi(d) = \xi_r(\beta\tilde{1}+c)$	$\xi_r \ (\beta\tilde{1}+c)$	
	$\xi_r \ (\beta\tilde{1}+e)$ with $\xi(e) = \xi_P(\alpha+c)$		

We observe that ξ_P: $A_k \to \mathbb{R}^{*+}$ whereas ξ_r: $A_k \to (\mathbb{R}^+)^2$. Thus ξ maps A_k onto $\mathbb{R}^{*+}$ or $(\mathbb{R}^+)^2$: it is given, generically, by any one of the 3 "compositions" $\xi_P \circ \xi_P$, $\xi_P \circ \xi_r$, $\xi_r \circ \xi_P$. The "composition" $\xi_r \circ \xi_r$ does not appear.

4.7 d is a split zerodivisor in $\Im m A_k$

When d is a split zerodivisor, then $(c, \pm\sqrt{\lambda}\tilde{1})$ are zerodivisors in $\mathcal{D}_{k+1}$ [3, Theorem 11.5]. Because $Zer(c) \supset Zer(\tilde{1}) = \{0\}$, the Schur rule [3, Definition 11.1 and Lemma 11.1] imposes a shift to *complex* arithmetic.

From a *linear* algebra point of view, the identification $\mathbb{C} \equiv \mathbb{C}_{\tilde{1}}$ is required to interpret ix as $\tilde{x}$ [3, Remark 11.1]. But from the inherently nonlinear viewpoint of A_k, $k \geq 3$, computation can go on in real arithmetic *without* interpretation. See more in Section 5.2.

5 The split vector $a = \alpha + \beta\tilde{1} + c$ is a source of induction from A_k into A_{k+1}, $k \geq 2$

5.1 Induction into A_{k+1} by a in $A_k = \mathbb{C}_{\tilde{1}} + \mathcal{D}_k$, $k \geq 2$

We suppose that $\alpha\beta \neq 0$: the head $h = \alpha + \beta\tilde{1}$, belongs to the domain III in $\mathbb{C}_{\tilde{1}}$.

Given $a = \alpha + \beta\tilde{1} + c$ in A_k, $k \geq 2$, we introduce the notations in $\mathcal{D}_{k+1}$:

$$\begin{array}{lll} \phi_0 = (c + \beta\tilde{1}, 0) = (d, 0) & , & \phi_2 = (0, d) = \phi_0^R \quad , \\ \phi_1 = (c, \beta\tilde{1}) & , & \phi_3 = (\beta\tilde{1}, c) = \phi_1^R \quad , \end{array}$$

summarized as ϕ_i, $i = 0$ to 3. These are the 4 vectors induced by $d = c + \beta\tilde{1}$ into $\mathcal{D}_{k+1}$. They form the 2 couples $C(\phi_{i'}) = \{\phi_{i'}, \phi_{i'}^R\}$ for $i' \equiv i (\text{mod } 2)$, $i' = 0, 1$.

We also set $u_0 = e_0 = 1 = 1_{k+1}$, $u_1 = e_{2^k} = \tilde{1} = \tilde{1}_{k+1}$ denoted u_j, $j = 0, 1$: they are the real ($j = 0$) and imaginary ($j = 1$) units in A_{k+1}.

Lemma 5.1 *The 8 vectors φ_l, $l = 0$ to 7 induced into A_{k+1} by a in A_k, $k \geq 2$, can be written as*

$$\varphi_l = \phi_i + u_j \ , \quad i = 0 \text{ to } 3, \ j = 0, 1$$

according to the correspondence table

l	0	1	2	3	4	5	6	7
i	0	1	1	0	2	3	3	2
j	1	0	1	0	0	1	0	1
i'	0	1	1	0	0	1	1	0

Proof. Clear from Table 1.1 and from the expressions:

$$\begin{array}{lll} \varphi_0 = \phi_0 + \alpha u_1 & , & \varphi_3 = \phi_0 + \alpha u_0 \quad , \\ \varphi_1 = \phi_1 + \alpha u_0 & , & \varphi_2 = \phi_1 + \alpha u_1 \quad , \\ \varphi_4 = \phi_2 + \alpha u_0 & , & \varphi_7 = \phi_2 + \alpha u_1 \quad , \\ \varphi_5 = \phi_3 + \alpha u_1 & , & \varphi_6 = \phi_3 + \alpha u_0 . \end{array}$$

□

Set generically $a = \alpha + A$, $b = \beta + B$ in A_k, with A, B in $\Im m \ A_k$. Then $\varphi = (a, b) = \alpha + \beta\tilde{1}_{k+1} + \phi$ with $\phi = (A, B) \in \mathcal{D}_{k+1}$. We use Proposition 2.2 to get $L_\varphi^T L_\varphi = (\alpha^2 + \beta^2) I_{2^{k+1}} - L_\phi^2$ in A_{k+1}. In what follows, we specialize ϕ to be ϕ_i, $i = 0$ to 3.

Given L_c^2, L_d^2, $M_a = L_a^T L_a$ and $G = [c, -, \tilde{1}] \not\equiv 0$ for $k \geq 3$, we introduce the notation in A_{k+1}:

$$\begin{array}{lll} P_c & = & diag\,(-L_c^2, -L_c^2) = \left(\begin{array}{c|c} -L_c^2 & 0 \\ \hline 0 & -L_c^2 \end{array} \right), \\ P_d & = & diag\,(-L_d^2, -L_d^2)\,, \\ P_a & = & diag\,(M_a, M_a) = (\alpha^2 + \beta^2) I_{2^{k+1}} + P_c, \\ H & = & \left(\begin{array}{c|c} 0 & G \\ \hline -G & 0 \end{array} \right). \end{array}$$

Finally we recall that $i' \equiv i$ (mod 2).

Lemma 5.2 *For $i = 0$ to 3, the maps $-L^2_{\phi_i}$ take one of the two forms:*

1) $i' = 0 : -L^2_{\phi_0} = -L^2_{\phi_2} = \beta^2 I_{2^{k+1}} + P_c = P_d$,

2) $i' = 1 : -L^2_{\phi_1} - L^2_{\phi_3} = P_d + \beta H$.

Proof. 1) Use Proposition 2.2 for $d = c + \beta\tilde{1}$, $-L^2_d = \beta^2 I_{2^k} - L^2_c$. Thus $-L^2_{\phi_0} = diag\,(-L^2_d, -L^2_d)$ by direct calculation.

2) Direct calculation of

$$-L^2_{\phi_1}(x, y) = -(c, \beta\tilde{1}) \times ((c, \beta\tilde{1}) \times (x, y)).$$

$\square$

Below, $\phi_{i'}$ denotes any of the 2 vectors in $C(\phi_{i'}) = \{\phi_{i'}, \phi^R_{i'}\}$ for $i' = 0, 1$, $i' \equiv i$ (mod 2).

5.2 The exact eigenvalues of $-L^2_{\phi_{i'}}$, $i' = 0, 1$ for $\beta \neq 0$

We denote by μ any eigenvalue of $-L^2_\phi$, for $\phi \in \mathcal{D}_{k+1}$. We assume that $\beta \neq 0$: h is not real in $\mathbb{C}_{\tilde{1}}$. We recall that for $\lambda \in \sigma_c$, the condition $\lambda \neq 1^4$, equivalent to $\hat{\sigma}_c \neq \emptyset$, requires that $k \geq 3$.

Proposition 5.3 *The exact eigenvalues of $-L^2_{\phi_{i'}}$ are given in terms of $\lambda \in \sigma_c$ by:*

1) $i' = 0 : for\ \lambda \in \sigma_c,\ \mu = \beta^2 + \lambda$,

2) $i' = 1 : for\ \lambda = 1^4,\ \mu = \beta^2 + 1$,
$for\ \lambda \neq 1^4,\ \mu_\pm = (\beta \pm \sqrt{\lambda})^2 \geq 0\ (k \geq 3)$.

Proof. 1) $i' = 0 : -L^2_d$ is block-diagonal in the eigenbasis for $-L^2_c$. The eigenvalue of each block of order 4 is $\mu = \beta^2 + \lambda$, $\lambda \in \sigma_c$.

2) $i' = 1$. We define $\mathbb{G}_c = \mathbb{H}_c \oplus \mathbb{H}_c \times \tilde{1}$, $F_\lambda = E_\lambda \oplus E_\lambda \times \tilde{1}$ for $\lambda \neq 1^4$, so that $A_{k+1} = \mathbb{G}_c \underset{\lambda \neq 1^4}{\oplus} F_\lambda$.

The map $-L^2_{\phi_1} = -L^2_{\phi_3}$ is block-diagonal in the corresponding basis, with blocks of order 8, (see [3, Section 11.2]). For $\lambda = 1^4$ (resp. 0) the 8×8 block is the diagonal $(1 + \beta^2)I_8$ (resp. $\beta^2 I_8$). For $0 < \lambda \neq 1^4$, we recall that for $x \in E_\lambda$, $Gx = 2\widetilde{c \times x}$. Let us fix $x \in E_\lambda$, $||x|| = 1$. We consider the 4D-building block V_x for E_λ, with the orthonormal basis $\left\{x, \dfrac{c \times x}{\sqrt{\lambda}}, \dfrac{\tilde{c} \times x}{\sqrt{\lambda}}, \tilde{x}\right\}$.

Then $G_{\restriction V_x} = 2\sqrt{\lambda}\left(\begin{array}{c|c} \mathrm{O}_2 & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \\ \hline \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} & \mathrm{O}_2 \end{array}\right)$ and $GG^T = 4\lambda I_4$. The eigenvalues of $H_{\restriction F_\lambda}$ are $\pm 2\beta\sqrt{\lambda}$ and these for $-L^2_{\phi_1 \restriction F_\lambda}$ are $\mu_\pm = \beta^2 + \lambda \pm 2\beta\sqrt{\lambda} = (\beta \pm \sqrt{\lambda})^2$. $\square$

Corollary 5.4 *The multiplicities of the eigenvalues of $-L^2_{\phi_{i'}}$ are multiples of 8 (resp. 8 or 4) for $i' = 0$ (resp. $i' = 1$ and $k \geq 3$).*

Proof. 1) $i' = 0$: clear for all $\lambda \in \sigma_c$.

2) $i' = 1$: the multiplicity of $\beta^2 + 1$ ($\lambda = 1^4$) is exactly 8, the multiplicity of β^2 ($\lambda = 0$) is a multiple of 8. For $0 < \lambda \neq 1^4$, the multiplicities are multiples of 4 for $\beta \neq 0$ ($k \geq 3$). $\square$

We revisit the Schur rule for $\sigma\left(-L^2_{\phi_1}\right)$ [3, Section 11].

Theorem 5.5 *When $\beta\lambda \neq 0$, complex arithmetic in A_k underlies the property*

$$\sigma\left(-L^2_{\phi_1}\right) \supset \{(\beta \pm \sqrt{\lambda})^2, \lambda \in \hat{\sigma}_c\}.$$

Proof. We apply [3, Lemma 11.1 and Remark 11.1] to $-L^2_{\phi_1} - \mu I_{2^{k+1}}$, with $\mu \in \sigma\left(-L^2_{\phi_1}\right)$: $-L^2_{\phi_1} - \mu I_{2^{k+1}} = \left(\begin{array}{c|c} -L^2_d - \mu I_{2^k} & \beta G \\ \hline -\beta G & -L^2_d - \mu I_{2^k} \end{array}\right)$.

To compute $det(-L^2_{\phi_1} - \mu I_{2^{k+1}})$, we can apply the Schur complement formula when the eigenvalue μ for $-L^2_{\phi_1}$ is not an eigenvalue of $-L^2_d$.

By Proposition 5.3, this condition writes $\mu_\pm = \left(\beta \pm \sqrt{\lambda}\right)^2 \neq \lambda + \beta^2$, that is $\beta\sqrt{\lambda} \neq 0$. When $\beta\lambda \neq 0$, $\mu_\pm$ is a pair of eigenvalues for $-L^2_{\phi_1}$ and $-L^2_{\phi_3}$ iff the 4 linear maps $\beta G \pm i\left(L^2_d + \mu_\pm I_{2^k}\right)$ are not invertible. Observe that $\beta\lambda \neq 0$ iff $\boldsymbol{A} \not\equiv \boldsymbol{B}$. $\square$

It is possible to interpret i in A_{k-1} by setting $i \equiv \tilde{1}_k$ [3, Section 11]

Corollary 5.6 *When $\mathbb{C} \equiv \mathbb{C}_{\tilde{1}}$, and c is not a zerodivisor, the exact singular values of L_{ϕ_1} and L_{ϕ_3} are explained by the complex structure of $A_k = A_{k-1} \star \mathbb{C}_{\tilde{1}} \equiv A_{k-1}(\mathbb{C})$, $k \geq 3$.*

Proof. Clear by the Schur rule applied at any eigenvalue μ for $-L^2_{\phi_1}$ with $\lambda > 0$. With the notation of [4], $A_k = A_{k-1} \star \mathbb{C}_{\tilde{1}}$ can be identified with $A_{k-1}(\mathbb{C})$ when $\mathbb{C}_{\tilde{1}} \equiv \mathbb{C}$, that is $\tilde{1}_k \equiv i$. $\square$

For example, for $k = 3$ (resp. 4) and $\mathbb{C} \equiv \mathbb{C}_{\tilde{1}}$, the octonions (resp. sedenions) have the complex structure $\mathbb{H}(\mathbb{C})$ (resp. $\mathbb{G}(\mathbb{C})$). The complex representation of $\mathbb{G}$

with coefficients in $\mathbb{H}$ is significant or phenomenological. It becomes explicit by derivation and is not invariant by the automorphisms of $\mathbb{G}$ (with $Aut(\mathbb{G}) \simeq Z_4$, the set of zerodivisors in A_4).

By comparison, for $k \geq 4$, the complex representation of A_k, which uses coefficients in A_{k-1}, $k-1 \geq 3$, is fundamental in nature [4].

When $\beta \neq 0$, $\phi_{i'}$ cannot be a zerodivisor for $i' = 0$ since $\mu = \beta^2 + \lambda \geq \beta^2 > 0$. This need not be true for $i' = 1$, and $k \geq 3$.

Lemma 5.7 *For $k \geq 3$, ϕ_1 and ϕ_3 are zerodivisors in $\mathcal{D}_{k+1}$ iff $\beta^2 = \lambda$, $\lambda \neq 1^4$.*

Proof. $\phi_1 = (c, \pm\sqrt{\lambda}\tilde{1})$ and $\phi_3 = (\pm\sqrt{\lambda}\tilde{1}, c)$ are zerodivisors for $k \geq 3$ [3, Theorem 11.5]. For $k = 3$, $\hat{\sigma}_c \neq \emptyset$. Observe that ϕ_1 and ϕ_3 satisfy the characterization of zerodivisors in $\mathcal{D}_4$. □

This lemma "explains" for $k \geq 3$ the existence of the imaginary split zerodivisor $d = c + \beta\tilde{1} \in \Im m A_k$. Observe that d in $\Im m\mathbb{H}$ cannot be a split zerodivisor ($k = 2$, and $\hat{\sigma}_c = \emptyset$).

5.3 Classical singular values for L_{φ_l}, $l = 0$ to 7 when $\alpha\beta \neq 0$ $k \geq 2$

For $k \geq 2$, the classical derivation of the singular values for L_{φ_l} in A_{k+1}, from that for $L_{\phi_{i'}}$ in $\mathcal{D}_{k+1}$ is based on the two representations:

$$\begin{aligned} 1)\ i' = 0 :\ P_a &= \alpha^2 I_{2^{k+1}} - L^2_{\phi_0} \\ &= -L^2_{\varphi} = L^T_{\varphi_3} L_{\varphi_3} = L^T_{\varphi_4} L_{\varphi_4} = -L^2_{\varphi_7} \\ 2)\ i' = 1 :\ & P_a + \beta H = \alpha^2 I_{2^{k+1}} - L^2_{\phi_1} \\ &= L^T_{\varphi_1} L_{\varphi_1} = -L^2_{\varphi_2} = -L^2_{\varphi_5} = L^T_{\varphi_6} L_{\varphi_6}. \end{aligned}$$

Theorem 5.8 *Let $k \geq 2$. The classical singular values for the 8 multiplication maps L_{φ_l}, $l = 0$ to 7 when $\alpha\beta \neq 0$, can be of two types.*

1) *For $l \equiv 0, 3$ (mod 4), they are equal to the classical singular values $\sqrt{N_\lambda}$ for L_a, with $N_\lambda = \alpha^2 + \beta^2 + \lambda$, $\lambda \in \sigma_c$.*

2) *For $l \equiv 1, 2$ (mod 4), they are equal to the nonclassical singular values for L_a computed via $d = c + \beta\tilde{1}$. The values are $||a|| = ||\varphi_l|| = \sqrt{N_1}$ for $\lambda = 1^4$, and $\sqrt{\alpha^2 + (\beta \pm \sqrt{\lambda})^2} \geq |\alpha|$ for $\lambda \neq 1^4$, which differ from $\sqrt{N_\lambda}$ for $\beta \neq 0$, and $k \geq 3$.*

Proof. Clear by Table 3.1. The case for $l = 0, 3, 4, 7$ is derived from $i' = 0$, and the one for $l = 1, 2, 5, 6$ from $i' = 1$. □

5.4 The case $\alpha\beta = 0$.

The condition $\alpha\beta = 0$ covers the two axes $\alpha = 0$, $\beta \neq 0$ (II) and $\beta = 0$ (I).

1) When the head is pure imaginary and nonzero (II), there are 4 different vectors induced into $\mathcal{D}_{k+1}$ by $d = c + \beta\tilde{1} = a$:

$$\begin{array}{ccc} \varphi_1 = \varphi_2 = \phi_1 & , & \varphi_0 = \varphi_3 = \phi_0, \\ \varphi_5 = \varphi_6 = \phi_3 & , & \varphi_4 = \varphi_7 = \phi_2. \end{array}$$

   There are 2 types of singular values given by Proposition 5.3 (with $\alpha = 0$), which cover the classical *and* nonclassical singular values for L_d, $d = c+\beta\tilde{1} = a$.

2) When the head is real, $\neq 0$ ($\mathrm{I}\backslash\{0\}$), the 4 different vectors induced by $e = c+\alpha$ are:

$$\begin{array}{ccc} \varphi_0 = \varphi_2 = (c, \alpha) & , & \varphi_1 = \varphi_3 = (c+\alpha, 0), \\ \varphi_4 = \varphi_6 = (\alpha, c) & , & \varphi_5 = \varphi_7 = (0, c+\alpha). \end{array}$$

   The role of $d = c + \beta\tilde{1}$ (which can be split when $\beta \neq 0$) with respect to $a = d + \alpha$, is now played by c (unsplittable) with respect to $e = c + \alpha$. The singular values are derived from $-L_c^2$.

Proposition 5.9 *For $k \geq 3$ and $\alpha\beta = 0$, the classical singular values for L_{φ_l}, $l = 0$ to 7 are listed below for $\lambda \neq 1^4$.*

II		I	
$l = 0, 3, 4, 7$	$\sqrt{\lambda + \beta^2}$	l *odd*	$\sqrt{\lambda + \alpha^2}$
$l = 1, 2, 5, 6$	$\lvert\sqrt{\lambda} \pm \beta\rvert \geq 0$	l *even*	$\lvert\sqrt{\lambda} \pm \alpha\rvert \geq 0$

Proof. By Theorem 5.8 with α or $\beta = 0$. ☐

For $\lambda = 1^4$ (resp. 0) the two values coalesce respectively into $\sqrt{1+\beta^2}$ (II) and $\sqrt{1+\alpha^2}$ (I) (resp. $|\beta|$ (II) and $|\alpha|$ (I)). We observe that the roles of α and β are interchanged when I and II are exchanged.

It is remarkable that the *classical* SVD for L_{φ_l} can provide an interpretation of *half* of the nonclassical singular values for L_a for $k \geq 3$: the ones computed via d. This special role of d in the explanation by induction is strengthened by Lemma 5.7.

What about the other half of the nonclassical singular values for L_a? Since the inventive formula $\boldsymbol{B}$ can be interpreted mathematically in A_{k+1}, let us see what the *nonclassical* SVD derivation for L_{φ_l} in A_{k+1} has to say about $\boldsymbol{C}$.

6 Nonclassical singular values for L_{φ_l} when $\varphi_l \in \Im m A_{k+1}$, $j = 1, k \geq 3$

When $j = 0$, that is $l = 1, 3, 4, 6$, then $\varphi_l = \phi_i + \alpha$, $i = 0$ to 3. For these 4 vectors without component on $\tilde{1}_{k+1}$, the classical and nonclassical singular values agree by Theorem 3.1, and yield the values given in Theorem 5.8.

6.1 Nonclassical singular values for $L\varphi_l$, $j = 1, \alpha\beta \neq 0$

Below, we suppose that $j = 1$, that is $l = 0, 2, 5, 7$. The corresponding vectors $\varphi_l = \phi_i + \alpha\tilde{1}$ are imaginary, $\tilde{1} = \tilde{1}_{k+1}$. The nonclassical singular values for L_{φ_l} *differ* for $\alpha \neq 0$ and $\lambda \neq 1^4$. Therefore we assume that $k \geq 3$.

Theorem 6.1 *For $k \geq 3$ and $\alpha\beta \neq 0$, the nonclassical singular values for L_{φ_l}, $l = 0, 2, 5, 7$, $\varphi_l \in \Im A_{k+1}$ are listed below for $\lambda \in \sigma_c$, with their multiplicities mult.*

	$l = 0, 7$	$l = 2, 5$	$mult$
$\lambda = 1^4$	$\sqrt{N_1} = \sqrt{1 + \alpha^2 + \beta^2}$	$\sqrt{N_1} = \Vert\varphi_l\Vert$	8
$\lambda \neq 1^4$	$\lvert\sqrt{\lambda + \beta^2} \pm \alpha\rvert \geq 0$	$\Big\lvert \lvert\beta \pm \sqrt{\lambda}\rvert \pm \alpha \Big\rvert \geq 0$	$mult(\lambda)$

Table 6.1: $\alpha\beta \neq 0$

Proof. 1)For $i' = 0$, $l = 0$ and 7, and $\mu = \lambda + \beta^2$. Thus $\mu = 0$ iff $\lambda = \beta = 0$, and $\mu \neq (N(\phi_0)$ with eigenspace $\mathbb{G}_c)$ iff $\lambda \neq 1^4$.

2) For $i' = 1$, $l = 2$ and 5, $\varphi_2 = \phi_1 + \alpha\tilde{1}_{k+1}$ and $\varphi_5 = \phi_3 + \alpha\tilde{1}_{k+1}$. There exists a unique route from ϕ_1 to φ_2 and from ϕ_3 to φ_5. This yields the 4 values $\left|\sqrt{\mu_\pm} \pm \alpha\right| = \left||\beta \pm \sqrt{\lambda}| \pm \alpha\right|$, because $\mu_\pm = \left(\beta \pm \sqrt{\lambda}\right)^2$ for $\lambda \neq 1^4$ by Proposition 5.3.

Observe that $\mu_\pm = 0$ iff $\beta^2 = \lambda$ for $\lambda \neq 1^4$ is equivalent to $d = c + \beta\tilde{1}$ is a split zerodivisor in A_k. □

The values listed in Table 6.1 for $\lambda \neq 1^4$ differ from the values given by Theorem 5.8 in III.

It is remarkable that the values for $l = 0, 7$, $\lambda \neq 1^4$ are the nonclassical singular values for L_{a^M} with $a^M = \beta + \alpha\tilde{1} + c$ that are computed directly, or via $e^M = c + \beta$ (M for mirror).

For $k \geq 3$, the vectors a and a^M share the *same* tail $c \in \mathcal{D}_k$ with respective heads $h = \alpha + \beta\tilde{1}$ and $h^M = \beta + \alpha\tilde{1}$. The two heads are *symmetrical* in $\mathbb{C}_{\tilde{1}}$ with respect to the *first* bisector: they are mirror images of each other.

Remark 6.1 Because of the nature of the formula $|\sqrt{\lambda + \beta^2} \pm \alpha|$, there are 4 (resp. 2) different vectors $a_j = \pm|\beta| \pm |\alpha|\tilde{1} + c$, $j = 1$ to 4, for $\alpha\beta \neq 0$ (resp. $\alpha\beta = 0$) which share the same measure when $a^M \neq c$.

They all differ from a iff $|\alpha| \neq |\beta|$. When $|\alpha| = |\beta| \neq 0$, one of the a_j coalesces with a: the resulting 3 vectors are either colinear with, or orthogonal to, a. This can happen for $\alpha\beta \neq 0$ only. Observe that $\tilde{a} = -\beta + \alpha\tilde{1} + \tilde{c} = -\overline{h^M} + \tilde{c}$.

In the event, classical and nonclassical SVD for L_{φ_l} provide an interpretation for the nonclassical singular values for L_a computed via $d = c + \beta\tilde{1}$, and the nonclassical singular values for L_{a^M} computed directly, or via $e^M = c + \beta$.

The role of a^M is twofold : i) for the computed values, α and β are exchanged and ii) for the computational route, $d = c + \beta\tilde{1}$ is replaced by $e^M = c + \beta$: 1 and $\tilde{1}$ are exchanged.

During the inductive SVD computation, four *new* values have appeared for $l = 2, 5$: They are the nonclassical singular values for L_{φ_l}, $l = 2, 5$, equal to $\Big||\beta \pm \sqrt{\lambda} \pm \alpha|\Big|$. For $\alpha\beta\ \lambda \neq 0$, they have no other interpretation at this stage. When $\alpha\beta\ \lambda = 0$, the 3 possibilities:

$\alpha = 0$	$\beta = 0$	$\lambda = 0$
$\|\beta \pm \sqrt{\lambda}\|$	$\|\alpha \pm \sqrt{\lambda}\|$	$\|\alpha \pm \beta\|$

can all receive an interpretation in terms of a or a^M.

The results when $\alpha\beta \neq 0$ are summarized in Figure 6.1 which is commented below.

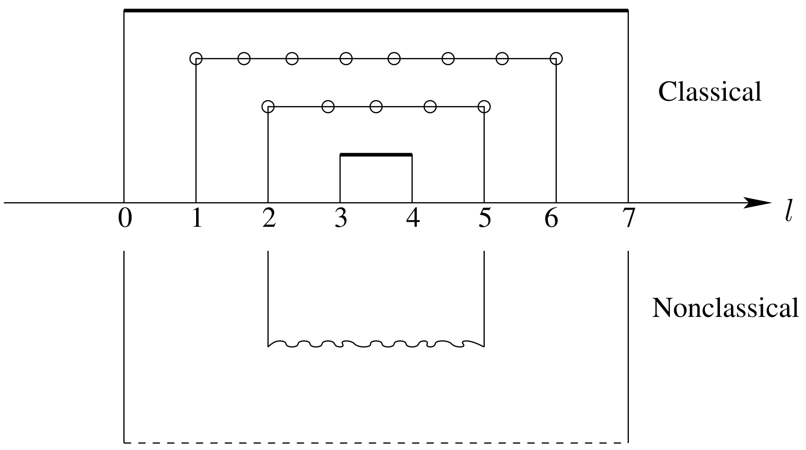


Figure 6.1: $\alpha\beta \neq 0$, III

The figure displays the 3 types of interpretation which can be given to the classical (upper part of the figure) and nonclassical (lower part) SVD derivation for the 8 maps L_{φ_l}, $l = 0$ to 7, according to the legend:

━━━━━ classical SVD for L_a

o—o—o—o—o nonclassical SVD for L_a (via $d = c + \beta\tilde{1}$)

– – – – – nonclassical SVD for L_{a^M} (direct or via $e^M = c + \beta$)

∼∼∼∼∼ not interpreted for $\lambda \neq 0$.

6.2 The case $\alpha\beta = 0$.

For $l = 1, 3, 4, 6$ the classical and nonclassical singular values, which agree because $\varphi_l = \phi_i + \alpha$ has no component on $\tilde{1}$, are given by Theorem 5.8.

For $l = 0, 2, 5, 7$, $\varphi_l = \phi_i + \alpha\tilde{1}$ and for $\alpha \neq 0$ one should add, to the classical singular values, the nonclassical ones given for $\lambda \neq 1^4$ by Theorem 6.1.

6.3 Summary for $k \geq 3$

The eigenvalues for $L^T_{\varphi_l} L_{\varphi_l}$, $l = 0$ to 7, obtained by classical *and* nonclassical SVD derivation in A_{k+1} are listed below, depending on $\lambda \neq 1^4$ and on $h \neq 0$ in $\mathbb{C}_{\tilde{1}_k}$. For $\lambda = 1^4$, there is a unique value $N_1 = 1 + N(h) = N(\varphi_l)$ in all cases.

III l	$3, 4$	$1, 6$	$0, 7$	$2, 5$
$0 < \lambda \neq 1^4$	$N_\lambda = \lambda + \alpha^2 + \beta^2$	$\alpha^2 + (\beta \pm \sqrt{\lambda})^2$	N_λ, $(\sqrt{\lambda + \beta^2} \pm \alpha)^2$	$\alpha^2 + (\beta \pm \sqrt{\lambda})^2$, $(\lvert\beta \pm \sqrt{\lambda}\rvert \pm \alpha)^2$
$\lambda = 0$	$\alpha^2 + \beta^2$		$(\beta \pm \alpha)^2$	

II l	$0 \quad 3 \quad 4 \quad 7$	$1 \quad 2 \quad 5 \quad 6$
$0 < \lambda \neq 1^4$	$\lambda + \beta^2$	$(\beta \pm \sqrt{\lambda})^2$
$\lambda = 0$	β^2	

I l	$1 \quad 3 \quad 4 \quad 6$	$0 \quad 2 \quad 5 \quad 7$
$0 < \lambda \neq 1^4$	$\lambda + \alpha^2$	$\lambda + \alpha^2, (\sqrt{\lambda} \pm \alpha)^2$
$\lambda = 0$	α^2	

Table 6.2: $0 \neq h \in \mathbb{C}_{\tilde{1}_k}$, $\lambda \in \hat{\sigma}_c$

For $\lambda = 1^4$ (resp. $\lambda = 0$ for $k \geq 4$) there can be 1 (resp. 1 or 2) eigenvalues. In the generic case $0 < \lambda \neq 1^4$ for $k \geq 3$, there can be 1,2,3, or 6 eigenvalues for the multiplication map defined by φ_l. The 8 indices l are grouped into 4 pairs (III) or 2 quadruples (I,II).

We observe that the eigenvalues depend continuously the components α and β for h in $\mathbb{C}_{\tilde{1}}$. The groupings into 2 quadruples are different on the real and imaginary axes. They can be transformed continuously into each other through the region III.

6.4 Split zerodivisors in $\Im m A_{k+1}$, $k \geq 3$

Lemma 6.2 *The source vector* $a = \alpha + \beta\tilde{1} + c$ *in* A_k, $k \geq 3$ *can induce split zerodivisors* φ_l *into* $\Im m A_{k+1}$ *for* $\lambda \neq 1^4$ *according to the following rule:*

	l	necessary and sufficient condition
III	$0, 7$	$\alpha^2 = \lambda + \beta^2$
	$2, 5$	$\alpha^2 = (\beta \pm \sqrt{\lambda})^2$
II	$1, 2, 5, 6$	$\beta^2 = \lambda$
I	$0, 2, 5, 7$	$\alpha^2 = \lambda$

Table 6.3:

Proof. Use Table 6.2 and check that $\varphi_l \in \Im m A_{k+1}$. $\varphi_0 = \phi_0 + \alpha\tilde{1}$, $\varphi_7 = \phi_2 + \alpha\tilde{1}$, $\varphi_2 = \phi_1 + \alpha\tilde{1}$, $\varphi_5 = \phi_3 + \alpha\tilde{1}$, belong to $\Im m A_{k+1}$. For $\alpha = 0$ (II), $\varphi_1 = \varphi_2 = \phi_1$ and $\varphi_5 = \varphi_6 = \phi_3$ belong to $\mathcal{D}_{k+1}$. □

By Lemma 5.7, ϕ_1 and ϕ_3 are zerodivisors iff $d = c + \beta\tilde{1}$ is a split zerodivisor in $\Im m A_k$ ($\beta^2 = \lambda$). A similar property holds for $d^M = c + \alpha\tilde{1}$.

Corollary 6.3 *The following statements are equivalent for* $\alpha^2 = \lambda$*:*

i) $d^M = \alpha\tilde{1} + c$ *is a split zerodivisor in* $\Im m A_k$, $k \geq 3$,

ii) $\phi_1^M = (c, \alpha\tilde{1})$ *and* $\phi_3^M = (\alpha\tilde{1}, c)$ *are zerodivisors in* $\mathcal{D}_{k+1}$.

Proof. Use $\alpha^2 = \lambda$. $\phi_3^M = (\phi_1^M)^R$. □

For $\alpha \neq 0$ and $i' = 0, 1$ we consider for $\phi_{i'} \in C(\phi_{i'})$ the vectors $\psi_{i'} = (\phi_{i'}, \alpha\tilde{1})$ and $\psi_{i'}^R = (\alpha\tilde{1}, \phi_{i'})$ induced into $\mathcal{D}_{k+2}$. Below, $\psi_{i'}$ denotes any vector in $C(\psi_{i'}) = \{\psi_{i'}, \psi_{i'}^R)\}$: there are 8 such vectors in $\mathcal{D}_{k+2}$.

Corollary 6.4 *For* $\alpha \neq 0$, *a necessary and sufficient condition for* $\psi_{i'}$, *to be a zerodivisor in* $\mathcal{D}_{k+2}$, $k \geq 3$ *is*

1) $\alpha^2 = \beta^2 + \lambda$ *for* $i' = 0$

2) $\alpha^2 = (\beta \pm \sqrt{\lambda})^2$ *for* $i' = 1$

with $\lambda \neq 1$[4].

Proof. Clear by Lemma 6.2. □

It is important to observe that when $\alpha = 0$, $a = d = \beta\tilde{1} + c \in \Im m A_k$, $k \geq 3$, can induce zerodivisors into $\mathcal{D}_{k+1}$ which are 1 level ahead. Whereas when $\alpha \neq 0$, zerodivisors can be induced (by $a = \alpha + \beta\tilde{1} + c$) into $\mathcal{D}_{k+2}$ only: they are 2 levels ahead.

A striking consequence follows about zerodivisors in $\mathcal{D}_4$. They can be explained by split zerodivisors in $\Im m\mathbb{G}$, but they *cannot* be related to a source vector a in $\mathbb{H}$: for $k = 2$, $\hat{\sigma}_c = \emptyset$.

Lemma 6.5 *$a^M = \beta + \alpha\tilde{1} + c \in A_k$, $k \geq 3$ with $\alpha \neq 0$ is a split zerodivisor iff $\alpha^2 = \beta^2 + \lambda$, $\lambda \neq 1^4$.*

Proof. See [3, Corollary 9.12] and Theorem 3.1. We have to exchange the roles of α and β, and consider $\alpha \neq 0$. ☐

This Section 6.4 can be summarized as follows. Given $a = \alpha + \beta\tilde{1} + c$ in A_k, $k \geq 3$, and given σ_c, inductive SVD derivation yields computational artifacts, through which 4 of the induced vectors $\varphi_l (l = 0, 7, 2, 5)$, can appear as split zerodivisors in $\Im m A_{k+1}$, under characteristic relations between $0 \neq \alpha, \beta$ and $\lambda \neq 1^4$. Moreover, when $\alpha = 0$, $\varphi_1 = \varphi_2$ and $\varphi_5 = \varphi_6$ can appear as zerodivisors in $\mathcal{D}_{k+1}$.

6.5 Characteristic lines and points for φ_l in A_{k+1}, $k \geq 3$

We introduce the analogue of Definition 3.2 for the vectors φ_l induced by a into A_{k+1}.

Definition 6.1 *For φ_l, $l = 0$ to 7, there are 2 types of characteristic lines and points defined in $\mathbb{C}_{\tilde{1}_k}$ as follows.*

1) *The characteristic lines consist of the hyperbolas*

$$\alpha^2 - \beta^2 = \lambda, \ for \ \lambda > 0,$$

and of the straight lines

$$|\alpha| = |\beta \pm \sqrt{\lambda}|, \ for \ \ \lambda \geq 0.$$

2) *The characteristic points have the coordinates $\pm\sqrt{\lambda}$ on the real $(\lambda > 0)$ and imaginary axes of $\mathbb{C}_{\tilde{1}} \simeq \mathbb{R}^2$*

See Figure 6.2 for a given λ, $0 < \lambda \neq 1^4$.

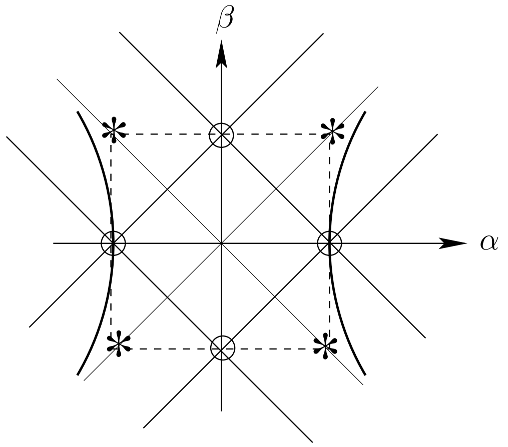


Figure 6.2: Characteristic lines ▬ and points ($\circ$) for φ_l, $0 < \sqrt{\lambda} \neq 1^4$, in $\mathbb{C}_{\tilde{1}}$.

The hyperbolas have the asymptotes $\beta = \pm\alpha$, and they pass through the real characteristic points.

The orthogonal array of straight lines passes through the real and imaginary characteristic points.

When $\lambda = 0$, the corresponding hyperbola $\alpha^2 = \beta^2$ is degenerate and coalesces with the two straight lines $\beta = \pm\alpha$: the origin is an imaginary characteristic point when c is a zerodivisor for $k \geq 4$. Figure 6.2 should be contrasted with Figure 3.3. The dotted square, centered at O with side length $2\sqrt{\lambda}$ will be commented in Section 7.4.

The various interpretations which can be attributed when the head $h = \alpha + \beta\tilde{1}$ of the source $a = \alpha + \beta\tilde{1} + c$ is either a characteristic point treated discretely, or lies on a characteristic line are summarized below for $k \geq 3$.

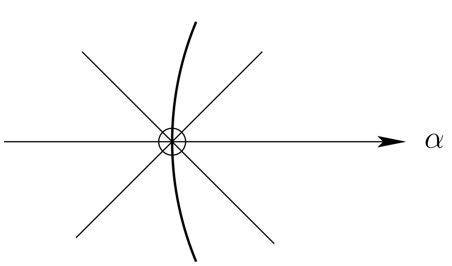
Lines	A_k	$\Im m A_{k+1}$	$\mathcal{D}_{k+2}$
Hyperbola	1 split zerodivisor $a^M = \beta + \alpha\tilde{1} + c$	2 split zerodivisors φ_0 , φ_7	**4 zerodivisors** corresponding to $\boldsymbol{\psi_0}$
Straight lines	——	2 split zerodivisors φ_2 , φ_5	**4 zerodivisors** corresponding to $\boldsymbol{\psi_1}$

Table 6.4:

Points	$\Im m A_k$	$\Im m A_{k+1}$	$\mathcal{D}_{k+2}$
Real $\beta = 0$	1 split zerodivisor $d^M = \alpha\tilde{1} + c$	4 split zerodivisors $\varphi_l,\ l = 0, 2, 5, 7$	**8 zerodivisors** corresponding to $\boldsymbol{\psi_0}$ **and** $\boldsymbol{\psi_1}$
		2 zerodivisors $\boldsymbol{\varphi_0^M}$**,** $\boldsymbol{\varphi_2^M}$ in $\boldsymbol{\mathcal{D}_{k+1}}$	——
Imaginary $\alpha = 0$	1 split zerodivisor $d = \beta\tilde{1} + c$	**2 zerodivisors** $\boldsymbol{\varphi_1 = \varphi_2 = \phi_1}$, $\boldsymbol{\varphi_5 = \varphi_6 = \phi_3}$	——

Table 6.5: Characteristic points treated in isolation

When a zerodivisor is finally found (highlighted in boldface) then the computed singular value is explained. The role of induction appears clearly when one contrasts these 2 tables with Table 3.4. A more complete picture is obtained when the characteristic points are treated in context, as points which belong to characteristic lines in $\mathbb{C}_{\tilde{1}}$, according to the Figure 6.3.



$\beta = 0,\ \alpha = \sqrt{\lambda} > 0$ $\qquad$ $\alpha = 0,\ \beta = \pm\sqrt{\lambda},\ \lambda \geq 0$

Figure 6.3: Real and imaginary characteristic points in context.

Convergence to a *real* characteristic point for $\lambda > 0$ can be realized in *two* ways. The limits $\alpha \to \varepsilon\sqrt{\lambda}$ for $\varepsilon = \pm 1$ and $\beta \to 0$ occur together under any of the two constraints

i) $|\alpha - \varepsilon\sqrt{\lambda}| = |\beta|$, or

ii) $\alpha^2 - \lambda = \beta^2 = (\alpha - \sqrt{\lambda})(\alpha + \sqrt{\lambda})$.

The second constraint ii) is the product of the two different cases of i). It imposes the conditions $\alpha > \sqrt{\lambda} > c$ (resp. $\alpha < -\sqrt{\lambda} < 0$) for $\varepsilon = +1$ (resp. $\varepsilon = -1$) when $\beta \neq 0$. Moreover let v_+(resp v_-) denote the rate of convergence to $(\sqrt{\lambda}, 0)$ (resp $(-\sqrt{\lambda}, 0)$) on the straightlines. The rate of convergence to the real characteristic points $(\pm\sqrt{\lambda}, 0)$ on the hyperbola is the *geometric mean* $\sqrt{v_+ v_-}$ of the two rates v_+ and v_-.

Convergence to an *imaginary* characteristic point for $\lambda \geq 0$ can be realized in *one* way. The limits $\beta \to \varepsilon\sqrt{\lambda}$, $\varepsilon = \pm 1$ and $\alpha \to 0$ occur together under the unique constraint $|\beta - \varepsilon\sqrt{\lambda}| = |\alpha|$, with two independent rates of convergence v_+ and v_-.

6.6 The diversity of measures for φ_l, $l = 0$ to 7, in A_{k+1}

Proposition 6.6 *There are 4 types of measures for φ_l, given according to l by the following formulae, for $\lambda \in \hat{\sigma}_c$:*

label	l	*formula*
$\boldsymbol{A}$	$0, 3, 4, 7$	$\sqrt{N_\lambda} = \xi_P(\varphi_l) = \xi_P(\alpha + \phi_0)$ *with* $\xi(\phi_0) = \xi_P(d)$.
$\boldsymbol{B}$	$1, 2, 5, 6$	$\sqrt{\alpha^2 + (\beta \pm \sqrt{\lambda})^2} = \xi_P(\alpha + \phi_1)$ *with* $\xi(\phi_1) = \xi_r(d)$.
$\boldsymbol{C^M}$	$0, 7$	$\lvert\sqrt{\lambda + \beta^2} \pm \alpha\rvert = \xi_r(\alpha\tilde{1}_{k+1} + \phi_0)$ *with* $\xi(\phi_0) = \xi_P(d)$.
$\boldsymbol{D}$	$2, 5$	$\left\lvert\alpha \pm \lvert\beta \pm \sqrt{\lambda}\rvert\right\rvert = \xi_r(\phi_1 + \alpha\tilde{1}_{k+1})$ *with* $\xi(\phi_1) = \xi_r(d)$.

Proof. Clear by Table 6.2. □

Corollary 6.7 *In Proposition 6.6 the vectors $\phi_{i'}$, $i' = 0, 1$, in $\mathcal{D}_{k+1}$ have a unique type of measure according to i':*

i) $i' = 0$, $\xi(\phi_0) = \xi_P(d) = \sqrt{\lambda + \beta^2}$ *in* $\boldsymbol{A}$, $\boldsymbol{C^M}$,

ii) $i' = 1$, $\xi(\phi_1) = \xi_r(d) = |\beta \pm \sqrt{\lambda}|$ *in* $\boldsymbol{B}$, $\boldsymbol{D}$.

Proof. Clear. Observe that in the induction $d = c + \beta\tilde{1} \mapsto \phi_{i'}$, the case $i' = 0$ (resp. $i' = 1$) corresponds to d treated as a whole (resp. d split), that is $d \mapsto (d, 0)$ or $(0, d)$ in $\Im m A_{k+1}$ (resp. $d \mapsto (c, \beta\tilde{1})$ or $(\beta\tilde{1}, c)$ in $\mathcal{D}_{k+1}$). □

Observe that 4^{th} composition $\xi_r \circ \xi_r$ appears in Proposition 6.6 for $l = 2, 5$, as a result of induction (formula $\boldsymbol{D}$). The set of measures corresponding to the induced vectors φ_l, $l = 0$ to 7, is denoted $IM(a) = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C^M}, \boldsymbol{D}\}$

Proposition 6.6 is summarized in Table 6.6 below, which also gives for each l from 0 to 7 the number n_l of distinct measures

$n_{\bar{l}} = n_l$	1	2	3	6	
l	3	1	0	2	$\left.\right\}\ l+\bar{l}=7$
$\bar{l}$	4	6	7	5	
formulae	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{A}, \boldsymbol{C^M}$	$\boldsymbol{B}, \boldsymbol{D}$	

Table 6.6: The set $IM(a)$

6.7 Inductive computation and complexification of the arithmetic.

Formulae $\boldsymbol{C^M}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ give the exact eigenvalues for $-L^2_{\psi_0}$, $-L^2_{\psi_0^M}$ and $-L^2_{\psi_1}$ which differ from $N_1 = \alpha^2 + \beta^2 + 1 = N(\psi_0) = N(\psi_1)$, by induction from those of $-L^2_{\phi_0} + \alpha^2 I_{2^{k+1}}$, $-L^2_{\phi_0^M} + \beta^2 I_{2^{k+1}}$ and $-L^2_{\phi_1} + \alpha^2 I_{2^{k+1}}$. The latter ones are given by $\boldsymbol{A}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. By the Schur rule, a shift to complex arithmetic in A_{k+1} is necessary to understand the computation when $\boldsymbol{A} \not\equiv \left\{\boldsymbol{C^M}, \boldsymbol{C}\right\}$ or $\boldsymbol{B} \not\equiv \boldsymbol{D}$. The result is summarized in the

Proposition 6.8 *For $\lambda \in \hat{\sigma}_c$, the eigenvalues of $-L^2_\psi$ for $\psi \in \left\{\psi_0, \psi_0^M, \psi_1\right\}$ are generically explained by complex arithmetic in A_{k+1}. The suficient condition are respectively:*

- *i) for ψ_0: $\boldsymbol{A} \not\equiv \boldsymbol{C^M} \Longleftrightarrow \alpha\sqrt{\lambda + \beta^2} \neq 0$,*
- *ii) for ψ_0^M: $\boldsymbol{A} \not\equiv \boldsymbol{C} \Longleftrightarrow \beta\sqrt{\lambda + \alpha^2} \neq 0$,*
- *iii) for ψ_1: $\boldsymbol{B} \not\equiv \boldsymbol{D} \Longleftrightarrow \alpha(\beta \pm \sqrt{\lambda}) \neq 0$.*

When $l = 0, 7$ (resp. 2,5) and $\boldsymbol{A} \not\equiv \boldsymbol{C^M}$ (resp. $\boldsymbol{B} \not\equiv \boldsymbol{D}$), interpretation of the induction $\Im m A_{k+1} \to \mathcal{D}_{k+2}$ requires the identification $1 \equiv \tilde{1}_k$ and $i \equiv \tilde{1}_{k+1}$, thus $A_{k+1} \equiv A_k(\mathbb{C})$.

When we consider the set $IM(a)$ for $\boldsymbol{A} \not\equiv \boldsymbol{B} \not\equiv \boldsymbol{C^M} \not\equiv \boldsymbol{D}$, we are facing *two* levels of induction. Each one requires a change of the basis field from $\mathbb{R}$ to $\mathbb{C}$. One can make sense of the two simultaneous identifications $i \equiv \tilde{1}_k$ and $i \equiv \tilde{1}_{k+1}$ in the following way.

Theorem 6.9 *The two-level inductive computation realized to get the set $IM(a)$ can be interpreted with the identifications $i = (i, 0) \equiv q = (\tilde{1}_k, 0)$ and $j \equiv \tilde{1}_{k+1}$ in A_{k+1}. Therefore $\mathbb{H} \equiv \mathbb{H}_q$ and $A_{k-1}(\mathbb{H}) \equiv A_{k+1}$.*

Proof. Clear by $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \times j$. □

7 The measurement loop for the source vector a

7.1 The measurement loop $ML(a)$

Definition 7.1 *The measurement loop for a consists of the set of singular values for L_a and L_{φ_l}, $l = 0$ to 7. It is denoted $ML(a) = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{C^M}, \boldsymbol{D}\}$.*

Among the singular values for L_{φ_l}, $l = 0$ to 7 (Tables 6.2 and 6.6), one recognizes some of the singular values for the vector a, the source of induction which yields the φ_l. Formulae $\boldsymbol{C^M}$ and $\boldsymbol{D}$ which do *not* belong to $SV(a)$ are called *correlative.*

7.2 Comparison between $SV(a)$ and $IM(a)$.

Formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical for L_{φ_l} and L_a: $SV(a) \cap IM(a) = \{\boldsymbol{A}, \boldsymbol{B}\}$. The inventive formula $\boldsymbol{B}$ is interpreted by induction. $\boldsymbol{C}$ is absent from $IM(a)$. Can we further interpret $SV(a)$ by means of $IM(a)$?

Formula $\boldsymbol{D}$ does not relate readily with $SV(a)$. As for $\boldsymbol{C^M}$, it is related to $\boldsymbol{C}$ in the

Lemma 7.1 $\boldsymbol{C} \equiv \boldsymbol{C^M} \iff \lambda(\alpha^2 - \beta^2) = 0$.

Proof. Clear from the condition $|\alpha|\sqrt{\lambda + \beta^2} = |\beta|\sqrt{\lambda + \alpha^2}$.

We already know that $\boldsymbol{C^M}$ yields the eigenvalues ($\neq N_1$) for $-L^2_{\psi_0}$. Correspondingly $\boldsymbol{C}$ yields the eigenvalues ($\neq N_1$) for $-L^2_{\psi_0^M}$, where $\psi_0^M = (\phi_0^M, \beta\tilde{1}_{k+1})$ and $\phi_0^M = (c + \alpha\tilde{1}, 0)$. □

The inventive formula $\boldsymbol{C}$ has no direct interpretation unless $\boldsymbol{C} \equiv \boldsymbol{C^M}$. The interpretation is exact when $a = a^M$ ($\alpha = \beta$). For $a \neq a^M$, $\boldsymbol{C}$ can receive a *pseudo* interpretation by considering, instead of a, any of the a_j, $j = 1$ to 4 (Remark 6.1). In the neighborhood of the bisectors $|\beta| = |\alpha|$ the interpretation of $\boldsymbol{C}$ using a_j remains *approximatively* valid. See more in Sections 7.5 and 7.6.

However, the interpretation of $\xi(\varphi_l)$ by means of $\xi(a)$ or $\xi(a^M)$ is **incomplete**. The formula $\boldsymbol{D}$ for $l = 2, 5$ remains totally uninterpreted by SVD in A_k when $\beta^2 \neq \lambda$. When $\beta^2 = \lambda$, L_a has the singular values $|\alpha|$ and $\sqrt{\alpha^2 + 4\lambda}$ corresponding to $\boldsymbol{B}$. Whereas $\boldsymbol{D}$ yields $|\alpha|$ and $|\alpha \pm 2\sqrt{\lambda}|$. In that case only 2 values remain uninterpreted.

It is conceivable to go beyond SVD in A_k, and to accept the formula $\boldsymbol{D}$ given by $\left|\alpha \pm |\beta \pm \sqrt{\lambda}|\right|$ as a measure for a, of a *different kind* than SVD, but such that $\xi(a) = \xi(\varphi_l)$ for $l = 2, 5$.

The consequence of this **hypothesis** are studied below (§7.3).

We first analyze in A_{k+1} the result

$$\xi(\varphi_l) = \left|\alpha \pm |\beta \pm \sqrt{\lambda}|\right| = \xi_r(\phi_1 + \alpha\tilde{1}_{k+1}), \text{ with } \ \xi(\phi_1) = \xi_r(d) \text{ for } \ l = 2, 5,$$

given by Table 6.2 and Proposition 6.6.

This is formula $\boldsymbol{D}$ which corresponds to $\xi_r \circ \xi_r$. The vectors $\varphi_2 = \phi_1 + \alpha\tilde{1}_{k+1}$ and $\varphi_5 = \phi_1^R + \alpha\tilde{1}_{k+1}$ are measured as twice split: first externally with $\alpha\tilde{1}_{k+1}$, then internally because ϕ_1 is the induction of $d = c + \beta\tilde{1}_k$ split into $(c, \beta\tilde{1}_k)$.

7.3 Modification of the geometry by $\boldsymbol{D}$

When we add the values $\left|\alpha \pm |\beta \pm \sqrt{\lambda}|\right|$ to $SV(a)$ these new measures modify even more the perceived geometry around c. Now both $F_3(c)$ and $P_2(c)$, of dimension 3 and 2, are interpreted as the 1D-line $L_1(c) = \{c\}$. The 3 orthonormal vectors $1, \tilde{1}$ and c appear, under this assumption, as colinear along $L_1(c)$. Even though they appear colinear, the 3 vectors remain *qualitatively different* (like 3 in 1). They yield the 4-fold measure:

$$\xi_c(a) = \left\{\left|\alpha + |\beta + \sqrt{\lambda}|\right|, \left|\alpha + |\beta - \sqrt{\lambda}|\right|, \left|\alpha - |\beta - \sqrt{\lambda}|\right|, \left|\alpha - |\beta + \sqrt{\lambda}|\right|\right\} \in (\mathbb{R}^+)^4,$$

generically for $\alpha\beta\lambda \neq 0$ and $\xi(c) = \sqrt{\lambda} \neq 1^4$.

The subscript "c" for $\xi(a)$ stands the reference vector c. From its own perspective, the doubly pure vector c, $||c|| = 1$ is seen as the support for the real unit 1_k, as well as for $\tilde{1}_k$. The 4 measures are taken along $L_1(c)$, with the 3 vectors $1, \tilde{1}$ and c playing the role of 3 different measuring rods.

The subalgebra $\mathbb{C}_{\tilde{1}}$ has been ultimately "absorbed" into $\{c\}$ in the measurement process originating in $c \in \mathcal{D}_k$.

7.4 The measurement loop for a at step $k \geq 3$

For $0 < \lambda \neq 1^4$, there are generically 5 types of measures for L_a obtained by the measurement loop $ML(a) = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{C^M}, \boldsymbol{D}\}$.

The 5 formulae express 3 different aspects of the act of producing a measure. Formula $\boldsymbol{A}$ is conservative because it extends straightforwardly the theorem of Pythagoras to all the different singular values for c, $k \geq 4$. Formula $\boldsymbol{A}$ agrees with euclidean geometry, and with mathematical logic. This is not the case for the inventive formulae $\boldsymbol{B}$ and $\boldsymbol{C}$ which modify significantly the geometry around c. Finally, formulae, $\boldsymbol{C^M}$ and $\boldsymbol{D}$ are correlative to a: they are induced by a, but do not, in general, belong to $SV(a)$.

One can view $ML(a)$ as an *information kit* for a where only $\boldsymbol{A}$ has the traditional meaning associated with a euclidean norm in classical geometry. The inventive formulae $\boldsymbol{B}$ and $\boldsymbol{C}$ are nonclassical singular values for L_a. They are computed in A_k, but can receive an interpretation in A_{k+1} only. The given interpretation is general for $\boldsymbol{B}$ ($l = 1, 6$ and $2, 5$), but it is restricted by the condition $|\alpha| = |\beta|$ for $\boldsymbol{C}$ ($l = 0, 7$). Finally, the correlative formulae are computed in A_{k+1} as singular values for L_{φ_l}, and *not* for L_a. Each of the 5 formulae in $ML(a)$ plays a specific and essential role in providing information to a about itself in A_k and about itself in the larger algebra A_{k+1}. Let us take a close look at the formulae for $0 < \lambda \neq 1^4$:

$\boldsymbol{A}$ $\quad N_\lambda = \lambda + \alpha^2 + \beta^2$ (arithmetic mean)

$\boldsymbol{B}$ $\quad N_\lambda \pm 2\beta\sqrt{\lambda} = \alpha^2 + (\beta \pm \sqrt{\lambda})^2 \geq \alpha^2$

$\boldsymbol{C}$ $\quad N_\lambda \pm 2\beta\sqrt{\lambda + \alpha^2} = (\beta \pm \sqrt{\lambda + \alpha^2})^2 \geq 0$

$\boldsymbol{C^M}$ $\quad N_\lambda \pm 2\alpha\sqrt{\lambda + \beta^2} = (\alpha \pm \sqrt{\lambda + \beta^2})^2 \geq 0$

$\boldsymbol{D}$ $\quad N_\lambda + 2\beta\varepsilon\sqrt{\lambda} \pm 2\alpha|\beta + \varepsilon\sqrt{\lambda}| = (\alpha \pm |\beta + \varepsilon\sqrt{\lambda}|)^2 \geq 0$ with $\varepsilon = \pm 1$.

Generically for $|\alpha| \neq |\beta|$, $\alpha\beta \neq 0$, there are 11 distinct eigenvalues with common arithmetic mean N_λ.

More precisely we have the

Proposition 7.2 *For $\lambda \in \sigma_c$, the number N of distinct singular values in the measurement loop for L_a is given below for $\alpha\beta \neq 0$ when smaller than 11.*

$0 < \lambda \neq 1^4$		$\lambda = 0$	$\lambda = 1^4$
$(\beta^2 - \alpha^2)\lambda = \alpha^2\beta^2$	9	3	1
$\alpha^2 \neq \beta^2 = \lambda$	9		
$\alpha^2 = \beta^2 \neq \lambda$	8		
$\alpha^2 = \beta^2 = \lambda$	6		

For $\alpha\beta = 0$, $\alpha \neq \beta$ the situation is described below

$0 < \lambda \neq 1^4$		$\lambda = 0$ or 1^4
$\alpha = 0$	3	1
$\beta = 0$	3	

For $\alpha = \beta = 0$, there is one singular value $\sqrt{\lambda}$ for any $\lambda \in \sigma_c$.

Proof. The generic number for $\alpha\beta \neq 0$ is $N = 11$. The smaller values obtained for $\lambda = 0$, or 1^4 are clear. For $0 < \lambda \neq 1^4$, we have the following.

1) $\boldsymbol{B}$ and $\boldsymbol{C^M}$ are identical iff $|\beta|\sqrt{\lambda} = |\alpha|\sqrt{\lambda + \beta^2}$. There are $N = 9 = 11 - 2$ distinct singular values when $(\beta^2 - \alpha^2)\lambda = \alpha^2\beta^2$.

2) For $\alpha^2 \neq \beta^2 = \lambda$, then $\boldsymbol{D}$ (resp. $\boldsymbol{B}$) yields the 3(resp. 2) values α^2 and $(\alpha \pm 2\sqrt{\lambda})^2$ (resp. α^2 and $\alpha^2 + 4\lambda$): $N = 11 - 2 = 9$.

3) For $\alpha^2 = \beta^2 \neq \lambda$, then $\boldsymbol{C}$ and $\boldsymbol{C^M}$ are identical, $\boldsymbol{D}$ yields the 3 values λ and $(2\alpha \pm \sqrt{\lambda})^2$: $N = 11 - 3 = 8$.

4) For $\alpha^2 = \beta^2 = \lambda$, the 6 distinct measures are $\sqrt{m\lambda}$, with $m \in \{3-2\sqrt{2}, 1, 3, 5, 3+2\sqrt{2}, 9\}$ with the correspondence

	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C} \equiv \boldsymbol{C^M}$	$\boldsymbol{D}$
m	3	$1, 5$	$3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$	$1, 9.$

Now, for $\lambda = 0$, the 3 measures are $\alpha^2 + \beta^2$ and $(\alpha \pm \beta)^2$. For $\lambda = 1^4$, the unique measure is $N_1 = \alpha^2 + \beta^2 + 1$.

When $\alpha\beta = 0$, $\alpha \neq \beta$ we distinguish 2 cases for $0 < \lambda \neq 1^4$. When $\beta \neq 0$, $\alpha = 0$ then $\boldsymbol{D}$=$\boldsymbol{B}$=$\boldsymbol{C}$ and $\boldsymbol{C^M}$=$\boldsymbol{A}$: there are $N = 3$ distinct eigenvalues corresponding to $\boldsymbol{A}$ and $\boldsymbol{B}$. When $\beta = 0$, $\alpha \neq 0$ then $\boldsymbol{A}$=$\boldsymbol{B}$=$\boldsymbol{C}$=$\boldsymbol{D}$. There are $N = 3$ distinct eigenvalues corresponding to $\boldsymbol{A}$ and $\boldsymbol{C^M}$.

Altogether the number N of distinct singular values associated with $0 < \lambda \neq 1^4$ can take the 6 values: $N = 1, 3, 6, 8, 9, 11$. □

By Lemma 5.7 and Corollary 6.3, the condition $\alpha^2 = \beta^2 = \lambda$ is equivalent to: $d = d^M = c \pm \sqrt{\lambda}\tilde{1}$ are split zerodivisors in $\Im m A_{k+1} \iff \phi_1$ and ϕ_3 are zerodivisors in $\mathcal{D}_{k+1}$.

For $0 < \lambda \neq 1^4$, the 4 points $|\alpha| = |\beta| = \sqrt{\lambda}$ in $\mathbb{C}_{\tilde{1}_k}$ represent the 4 vertices (marked *) of the dotted square in Figure 6.2.

On Figure 7.1 are plotted for $0 < \lambda \neq 1^4$ the graphs representing the nongeneric points $h = \alpha + \beta\tilde{1}$ in $\mathbb{C}_{\tilde{1}}$ for which $N = N(\lambda) \in \{1, 3, 6, 8, 9\}$.

Each of the sets has the label (N) attached to it, except for 0 with corresponds to $N = 1$. It displays the points and lines of discontinuity in $\mathbb{C}_{\tilde{1}}$, where $N(\lambda) < 11$ for λ such that $0 < \lambda \neq 1^4$.

For $\lambda = 1^4$, the map $(\alpha, \beta) \mapsto N(1^4) = 1$ is everywhere constant. For $\lambda = 0$, the map $(\alpha, \beta) \mapsto N(0)$ is discontinuous on the 2 axes $\alpha\beta = 0$: $N(0)$ drops from 3 to 1.

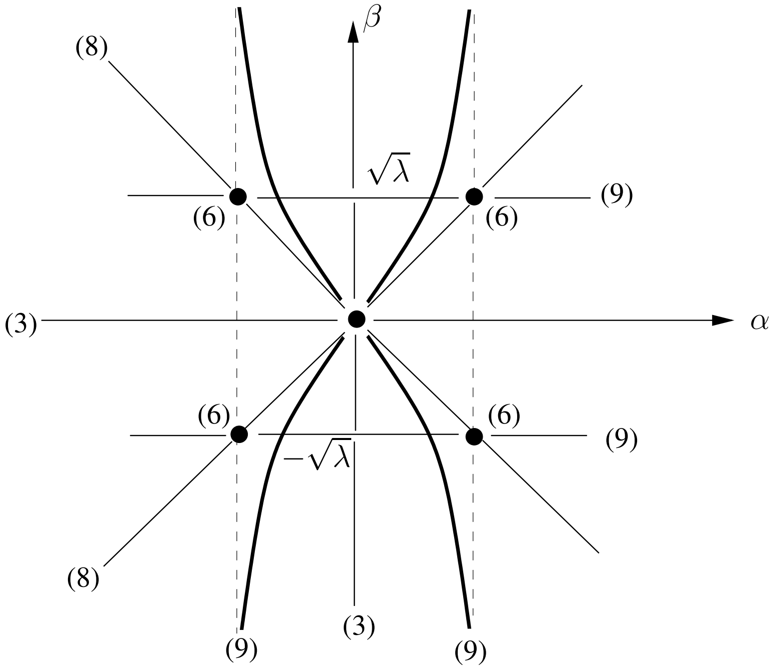


Figure 7.1: Set of non generic points in $\mathbb{C}_{\tilde{1}}$, $N < 11$, $0 < \lambda \neq 1$[4]

The two curves $\dfrac{\beta}{\sqrt{\lambda}} = \pm \dfrac{\alpha}{\sqrt{\lambda}\sqrt{1 - \left(\dfrac{\alpha}{\sqrt{\lambda}}\right)^2}}$ defined for $\alpha^2 < \lambda \Longleftrightarrow \dfrac{|\alpha|}{\sqrt{\lambda}} < 1$ can be parametrized as follows with $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$:

$$\begin{cases} \alpha = \sqrt{\lambda}\sin t \\ \beta = \pm\sqrt{\lambda}\tan t \end{cases}$$

7.5 An homogeneous formulation when $\lambda > 0$

When λ is positive, that 5 formulae can be written in terms of the ratios $u = \dfrac{\alpha}{\sqrt{\lambda}}$, $v = \dfrac{\beta}{\sqrt{\lambda}}$ and $w = \dfrac{\xi}{\sqrt{\lambda}}$, where ξ is the resulting singular value:

$$\begin{array}{l|l} \boldsymbol{A} & w^2 = 1 + u^2 + v^2, \\ \boldsymbol{B} & w^2 = 1 + u^2 + v^2 \pm 2v = u^2 + (v \pm 1)^2, \\ \boldsymbol{C} & w^2 = 1 + u^2 + v^2 \pm 2v\sqrt{1+u^2}, \quad \text{or}, w = \left|v \pm \sqrt{1+u^2}\right|, \\ \boldsymbol{C^M} & w = \left|u \pm \sqrt{1+v^2}\right|, \\ \boldsymbol{D} & w = \big|u \pm |1 \pm v|\big|. \end{array}$$

The formulae represent surfaces in 3D of the type

$$(u,v) \in \mathbb{R}^2 \mapsto w \in \mathbb{R}^+.$$

They create 5 different landscapes as, $u, v \in \mathbb{R}$.

The level curves $w = l$, where l is a nonnegative constant are listed below for $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$:

$\boldsymbol{A}$	the circle $u^2 + v^2 = l^2 - 1$ for $l > 1$, and the point $u = v = 0$ for $l = 1$.
$\boldsymbol{B}$	the two circles $u^2 + (v \pm 1)^2 = l^2$ for $l > 0$, and the points $u = v \pm 1 = 0$ for $l = 0$.
$\boldsymbol{D}$	the four straight lines $v = \pm u \pm (1 + l)$ for $l \geq 1$

For $\boldsymbol{C}$ and $\boldsymbol{C^M}$ the level curves are best interpreted with the change of variables $u' = u^2 \geq 0$, $v' = v^2 \geq 0$, $w' = w^2 \geq 0$, $l' = l^2 \geq 0$, and $r = l' - 1 \geq -1$.

We set $X = u' + v' = \dfrac{\alpha^2 + \beta^2}{\lambda} = \dfrac{N(h)}{\lambda}$ and $Y = v' - u' = \dfrac{\beta^2 - \alpha^2}{\lambda}$, which satisfy $0 \leq |Y| \leq X$.

Lemma 7.3 *With the above change of variables, the level curves* $u^2 + v^2 \pm 2v\sqrt{1+v^2} = r = l' - 1$, *for* $\boldsymbol{C}$ *are given by the 2 relations:*

i) $Y = X$ *for* $0 \leq X \leq X_- = \left(\sqrt{l'} - 1\right)^2$ *and* $X \geq X_+ = \left(\sqrt{l'} + 1\right)^2$,

ii) $Y = \frac{1}{2}(X - l')^2 + \frac{1}{2} - l' = \frac{1}{2}X^2 - l'X + \frac{1}{2}(l' - 1)^2$ *for* $X_- \leq X \leq X_+$.

Proof. By calculation. The relation $\pm 2v\sqrt{1+u^2} = r - u^2 - v^2$ implies that $4v^2(1 + u^2) = (r - u^2)^2 + v^4 - 2v^2(r - u^2)$. This is equivalent to $(u^2 - v^2)^2 - 2r(u^2 + v^2) - 4v^2 + r^2 = 0$, or else $(u' - v'^2 - 2(r + 1)(u' + v') - 2(v' - u') + r^2 = 0$.

Thus $X^2 - 2l'X - 2Y + r^2 = 0$, $Y = \frac{1}{2}X^2 - l'X + \frac{1}{2}(l' - 1)^2 = \frac{1}{2}(X - l')^2 + \frac{1}{2} - l'$.

We now deal with the condition $0 \leq |Y| \leq X$. The minimum value for Y corresponds to the vertex $(l', \frac{1}{2} - l')$. The parabola intersects $Y = X$ at the two points $X_\pm = \Big(\sqrt{l'} \pm 1)^2 = (l \pm 1)^2$ for $l' > 0$, which coalesce into $X_- = X_+ = 1$ for $l' = 0$. In this latter case, the level curves are reduced to the bisector $Y = X$, which corresponds to the axis $u' = 0$, that is the imaginary axis $\alpha = 0$. □

Corollary 7.4 *The level curve for* $\boldsymbol{C^M}$ *is the reflection in the axis* $Y = 0$ *of the level curve for* $\boldsymbol{C}$*. The coincidence* $\boldsymbol{C} \equiv \boldsymbol{C^M}$ *may happen for* $l' \geq \frac{1}{2}$*. It takes place at the following points in* $\mathbb{C}_{\tilde{1}}$*, where* $g_\pm = l' \pm \sqrt{2l' - 1}$ *and* $l' \geq \frac{1}{2}$*:*

1) The 4 points $|\alpha| = |\beta| = \dfrac{\sqrt{\lambda}}{2}$ *for* $l' = \frac{1}{2}$, *2) The 5 points* $\alpha = \beta = 0$ *and* $|\alpha| = |\beta| = \sqrt{\lambda}$ *for* $l' \neq 1$*,*

3) The 8 points $|\alpha| = |\beta| = \sqrt{\frac{\lambda}{2} g_\pm}$ *for* $l' > \frac{1}{2}$ *and* $l' \neq 1$*.*

Proof. The exchange of u' and v' leaves $X = u' + v'$ invariant, whereas $Y = v' - u'$ becomes $-Y = u' - v'$. $\boldsymbol{C} \equiv \boldsymbol{C^M}$ iff $Y = 0$ and $X = g_\pm = l' \pm \sqrt{2l' - 1}$ for $l' \geq \frac{1}{2}$. We check that g_- is nonnegative: $l'^2 > 2l' - 1 \iff (l' - 1)^2 \geq 0$ is always satisfied. Going back to $|\alpha| = |\beta|$, $g_\pm = 2\frac{\alpha_\pm^2}{\lambda}$, where $\alpha_\pm^2 = \frac{\lambda}{2} g_\pm$ are defined for any $l' \geq \frac{1}{2}$ (for $l' = \frac{1}{2}$, $X_- \simeq 0.08 < g_- = g_+ = \frac{1}{2} < X_+ \simeq 2.914$).

For $\frac{1}{2} < l' \neq 1$, one has the ordering: $0 < X_- < g_- < l' < g_+ < X_+$ on the X-axis.

For $l' = 1$, then $X_- = g_- = 0$, hence $a = c$, and $g_+ = 2$, $X_+ = 4$, thus $|\alpha| = |\beta| = \sqrt{\lambda}$. We get the origin $\alpha = \beta = 0$ and the 4 vertices for the dotted square in Figure 6.2, which are marked *. $\square$

See Figure 7.2.

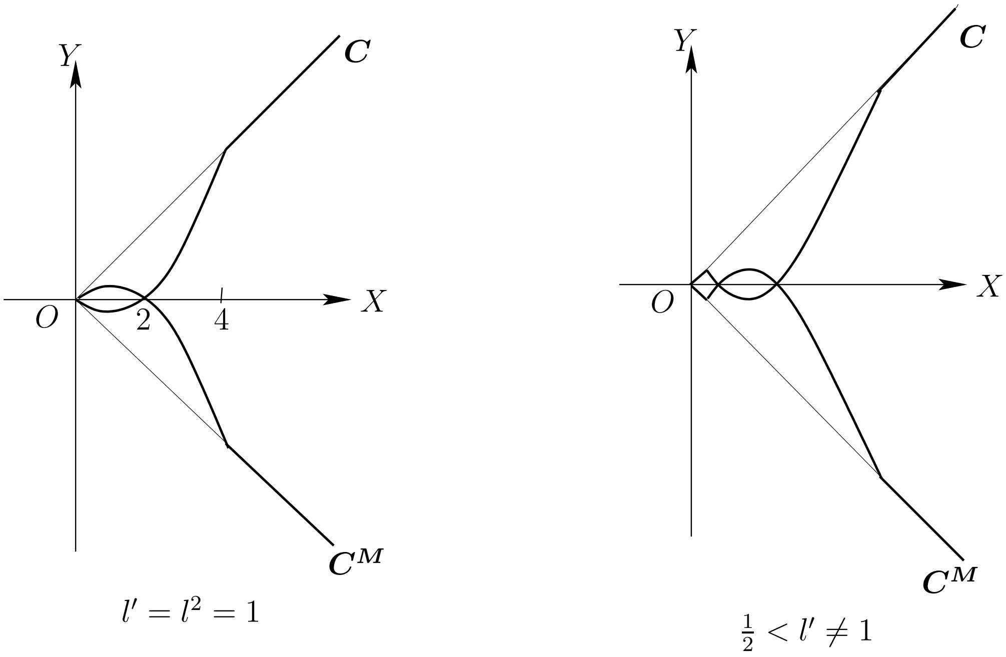


Figure 7.2: The curves of level $l' = l^2$ for $\boldsymbol{C}$ and $\boldsymbol{C^M}$ in the X-, Y-axes

The parameter $l = \sqrt{l'} \geq 0$ represents the level of the ration $\frac{\xi}{\sqrt{\lambda}}$: it indicates how the computed measure ξ compares with the internal singular value $\sqrt{\lambda}$ associated with L_c.

The level curve gives the set of vectors $h = \alpha + \beta\tilde{1}$ in $\mathbb{C}_{\tilde{1}}$ such that ξ agrees at $a = h + c$ with the *internal* measure $l\sqrt{\lambda}$ of lc where $\xi(c) = \sqrt{\lambda}$. The vector lc is a magnified (resp. shrunk) version of c when $l > 1$ (resp. $0 \leq l \leq 1$).

The use of $\boldsymbol{C}$ (resp. $\boldsymbol{C^M}$) yields the imaginary (resp. real) axis in the limit $l \to 0$ is impossible or $l \to \infty$. Moreover the coincidence $\boldsymbol{C} \equiv \boldsymbol{C^M}$ which requires $l' \geq \frac{1}{2}$ is impossible in the limit $l \to 0$.

We mention, for future consideration, that the level curves for $\boldsymbol{C}$ and $\boldsymbol{C^M}$ can be parametrized by $l' = l^2$. This opens the possibility of considering $l = \pm\sqrt{l'}$ positive *and* negative.

Definition 7.2 *Let ξ represent one of the possible measures for a. The condition $\dfrac{\xi}{\sqrt{\lambda}} = 1$ defines the equivalence for a between the external computed measure ξ and the internal singular value $\sqrt{\lambda}$.*

Proposition 7.5 *The equivalence takes place respectively*

i) at the origin $\alpha = \beta = 0$ for $\boldsymbol{A}$,

ii) on the circles of radius 1 and centers $(0, \pm 1)$ for $\boldsymbol{B}$,

iii) on the imaginary axis $\alpha = 0$ for $\beta = 0$ or $|\beta| \geq 2\sqrt{\lambda}$, and for $0 < |\alpha| < \sqrt{\frac{\lambda}{2}}$, $\beta^2 = \lambda - \alpha^2 \pm \sqrt{\lambda(\lambda - 2\alpha^2)}$.

iv) on the 4 lines $\beta = \pm(\alpha \pm 2\sqrt{\lambda})$ for $\boldsymbol{D}$.

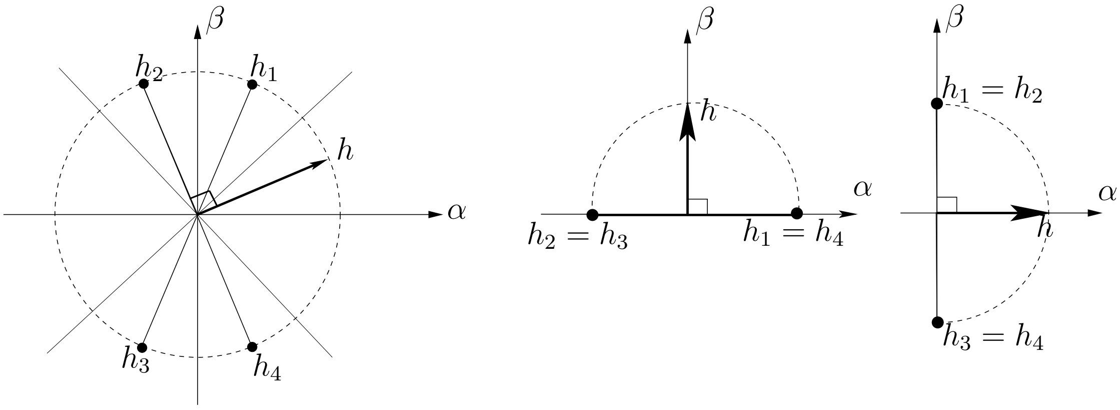
Proof. Clear by leting $l' = l = 1$ in the equations for the level curves. ☐

It is not surprising that the exact result $a = c$ is the only possibility for $\boldsymbol{A}$. We shall go back to this question for $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ in Section 10. The end of Section 7 is now devoted to $\boldsymbol{C^M}$ and to the coincidence $\boldsymbol{C} \equiv \boldsymbol{C^M}$ for $\lambda > 0$.

7.6 The angles $\theta_j = \measuredangle(a, a_j)$ for $j = 1$ to 4

We consider the set $\{a_j\}$ of vectors associated with a in Remark 6.1: we set $a_j = h_j + c$ with $h_1 = -h_3 = \beta + \alpha\tilde{1}$ and $h_2 = -h_4 = -\beta + \alpha\tilde{1}$.

There are 4 different vectors for $\alpha\beta \neq 0$ and 2 for $\alpha\beta = 0$. See Figure 7.3.



$\alpha\beta \neq 0$
III

$\alpha = 0$
II

$\beta = 0$
I

Figure 7.3: $h \neq 0$ in $\mathbb{C}_{\tilde{1}}$ and its images

We set $j' \equiv j \pmod 2$: $j' = 0$ or 1 when j is even or odd. When $\alpha\beta \neq 0$, we observe that for $j' = 0$, h_2 and h_4 are orthogonal with h:

$$h_j = (\exp\tilde{1}(-1)^{\frac{j}{2}+1}\pi/2)h, \quad j = 2, 4$$

To get h_2 or h_4, h is *rotated* about 0 through $\pm\pi/2$. This transformation preserves *chirality* and is intrinsic.

The geometry for $j' = 1$ is different: $h \mapsto h_j^M$, $j = 1$ or 3 is a *reflection* in the first ($j = 1$) or second ($j = 3$) bisector. Such a transformation is *achiral.* The result depends on the choice of axes in $\mathbb{C}_{\tilde{1}}$: it is *not* intrinsic.

This remarkable difference between the 2 possibilities when $\alpha\beta \neq 0$ will have important computational consequences. When $\alpha\beta = 0$, the qualitative difference does not show.

For $h = \alpha + \beta\tilde{1} \neq 0$, we define $q = \dfrac{2|\alpha\beta|}{\alpha^2 + \beta^2} \geq 0$.

The ratio q is defined by α^2 and β^2, as the ratio of the geometric mean $\alpha\beta$ over the arithmetic one $\dfrac{\alpha^2 + \beta^2}{2}$. We recall that $q = 0$ iff $\alpha\beta = 0$, and $q = 1$ iff $|\alpha| = |\beta| \neq 0$ [3, Section 9.5 and 12.2].

We set $h = \alpha + \beta\tilde{1} = ||h||(\cos\gamma + \sin\gamma\tilde{1})$ where the argument $\gamma = \measuredangle(1, h)$ is defined mod 2π. One checks easily that $q = |\sin 2\gamma|$. Set $\eta_j = \measuredangle(h, h_j)$, then

$\cos\eta_j = \dfrac{1}{N(h)} < h, h_j >$.

Lemma 7.6 *For $\alpha\beta \neq 0$, $|\cos\eta_j|$ can take the 2 values 0 for $j' = 0$ and $q \neq 0$ for $j' = 1$. For $\alpha\beta = 0$, $cos\eta_j = 0$ for all j.*

Proof. Clear for $j' = 0$. For $j' = 1$ and $\alpha\beta \neq 0$, use $< h, h_j >= 2\alpha\beta$ ($j = 1$) or $-2\alpha\beta$ ($j = 3$). □

We now consider $\theta_j = \measuredangle(a, a_j)$, $j = 1$ to 4. When $|\alpha| \neq |\beta|$ there are 4 or 2 such angles which are $\neq 0$. When $|\alpha| = |\beta|$, one of the θ_j equal 0. For example, when $N(a) = 2$, that is $N(h) = 1 = N(c)$, one gets geometrically the 4 values $\{0, \pi/2, \pm\pi/3\}$, see Figure 7.3 in $F_3(c)$ when $a = a_1$.

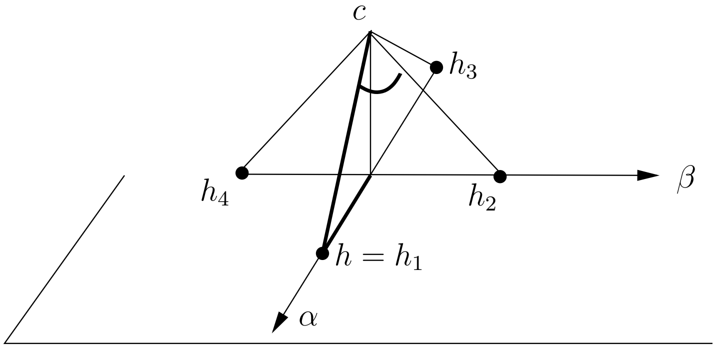


Figure 7.4: $a = \frac{1}{\sqrt{2}}(1 + \tilde{1}) + c = a_1$

To analyze the variation of the 4 angles θ_j as $\alpha, \beta \in \mathbb{R}$, we look at $\cos\theta_j = \frac{1}{N(a)} < a, a_j >$ and $\tau_j = 1 - \cos\theta_j = 2\sin^2\frac{\theta_j}{2}$, for $j = 1$ to 4.

We assume in this section that $|\alpha| \neq |\beta|$, so that a differs from all a_j.

Lemma 7.7 *For j even,* $0 < \tau_2 = \tau_4 = 1 - \frac{1}{N(a)} < 1$. *For j odd,* $\tau_1 = \frac{(\alpha - \beta)^2}{N(a)}$, $\tau_3 = \frac{(\alpha + \beta)^2}{N(a)}$, $\tau_1 + \tau_3 = 2\frac{N(h)}{N(a)}$ *and* $\tau_3 - \tau_1 = 4\frac{\alpha\beta}{N(a)}$.

Proof. Direct calculation for $< a, a_j >$ □

We set $r = N(a) = ||a||^2$.

For $r > 1$ we define $f : r > 1 \mapsto 1 - \frac{1}{r}$. The function f describes the part of the hyperbola define on $\mathbb{R}$ which satisfies $0 < 1 - \frac{1}{r} < 1$.

Proposition 7.8 *When $\alpha\beta \neq 0$, $|\alpha| \neq |\beta|$ the following relations hold for $\tau_j > 0$:*

1) $\tau_2 = \tau_4 = f(r)$

2) $\tau_1 + \tau_3 = 2f(r)$

Proof. Clear by Lemma 7.7. Observe that $\sin^2\frac{\theta_1}{2}+\sin^2\frac{\theta_3}{2}=f(r)$. □

Corollary 7.9 *When* $\alpha\beta=0$, $\tau_1=\tau_2=\tau_3=\tau_4=f(r)$.

Proof. Clear by Lemma 7.7: $\tau_1=\tau_3=f(r)$. □

These computational results indicate a circular invariance for the 2 angles θ_2 and θ_4 when $\alpha\beta\neq 0$, or for the 4 of them when $\alpha\beta=0$.

The circular invariance is expressed by the fact that $\sin^2\frac{\theta}{2}=\dfrac{1}{2}f(r)$ depends on $r=N(a)$ only. Or equivalently on $N(h)=N(a)-1$. All h on a circle of radius $||h||=$ *constant* >0 define the same 2 ($\alpha\beta\neq 0$) or 4 ($\alpha\beta=0$) angles.

We observe that the difference between the two families of images for h (rotated for j even, reflected for j odd) shows clearly when $\alpha\beta\neq 0$, $|\alpha|\neq|\beta|$. For j *even*, the τ_j are individually dependent on $||h||$ only, whereas for j *odd*, this is true for the **sum** $\tau_1+\tau_3$ only.

Another difference concerns the possibility that $|\theta_j|=\pi/2$, that is $\tau_j=1$, or else a is orthogonal to a_j. This can happen for j odd only by Lemma 7.7.

Lemma 7.10 *For* $\alpha\beta\neq 0$, $|\alpha|\neq\beta|$, $<a,a_j>=0$ *is satisfied on the hyperbola* $\beta=-\dfrac{1}{2\alpha}$ *for* $j=1$ *and* $\beta=\dfrac{1}{2\alpha}$ *for* $j=3$, *from which the vertices are deleted.*

Proof. Clear by $<a,a_j>$. For j even, $<a,a_j>=1\neq 0$. □

7.7 Coincidence of a with one of the a_j when $|\alpha|=|\beta|\neq 0$

When $|\alpha|=|\beta|\neq 0$, a coincides with one of the 4 vectors a_j and $\boldsymbol{C}=\boldsymbol{C^M}$. One of the angles $\theta_j=0$, denoted $\theta_{j_\star}=0$. There are several ways to analyze the situation: directly, from the knowledge of the datum a, then $j_\star=1$: $a=a_1=a^M$, or from partial knowledge about which of the 3 angles are nonzero (when $j_\star$ can be any of 4 integers 1,2,3,4).

7.7.1 The direct analysis $a=a^M$

The numbering of a_j is based on the datum a. The only possibility for θ to be zero is that $\alpha=\beta$, thus $a=a^M$ and $j_*=1$. So that $\tau_2=\tau_4=\frac{\tau_3}{2}=f(r)$, $r=1+2\alpha^2$.

The direct analysis, centered on a, can be described as "subjective": the vector a is the actor of its own measurements. In other words, a is the subject who measures.

The situation is illustrated in the

Example 7.1 How can Lemma 7.10 be complemented when $\alpha\beta\neq 0$, $|\alpha|=|\beta|\neq 0$? For $|\alpha|\neq|\beta|$, there are 4 missing vertices on the 2 hyperbolas $\alpha\beta=\pm 1/2$. The *direct* analysis allows us to complete as follows the hyperbola $\beta=\dfrac{1}{2\alpha}$ with real axis the first bisector $\beta=\alpha$. Indeed $\tau_1=0$ and $\tau_2=1$ iff $\tau_2=\tau_4=1/2$ (that is $|\theta_2|=|\theta_4|=\pi/3$). This corresponds to 1 of the 2 vertices on $\beta=\alpha$, depending on the sign of $\alpha=\beta$. Therefore 1 of the 4 missing points has been filled in. △

7.7.2 The backward analysis

Such an analysis can be realized in several ways, as one varies the assumptions that can be made on the *a priori* knowledge about the situation.

We shall consider two possibilities when $j_\star$ is not fixed at 1:

i) either the qualitative difference between $j_\star$ odd or even is maintained,

ii) or the 4 occurrences $j_\star = 1, 2, 3$ or 4 are equally probable.

7.7.3 A qualitative backward analysis

We suppose that the numbering for the images h_j, defined by h, maintains the equivalence: j even $\iff h \perp h_j$ on the bisectors. Therefore $j_\star$ has to be *odd.* We get the computational limit obtained from Proposition 7.8 as $\Big||\alpha| - |\beta|\Big| \to 0$.

Proposition 7.11 *When* $|\alpha| = |\beta|$ *in the limit of* $\Big||\alpha| - |\beta|\Big| \to 0$*, then* $\tau_2 = \tau_4 = f(r)$*. For* j *odd, then either*
$j_\star = 1 \iff \tau_1 = 0$ *and* $\tau_3 = 2f(r) \iff \alpha = \beta$*,*
or $j_\star = 3 \iff \tau_3 = 0$ *and* $\tau_1 = 2f(r) \iff \beta = -\alpha$*.*

Proof. Clear by Lemma 7.7. ☐

When computation is a continuous process near the bisectors, the qualitative difference between j odd or even is maintained.

7.7.4 A geometric backward analysis

In the absence of additional information about the situation, $j_\star$ can be any of the 4 integers 1,2,3,4 with equal probability. This corresponds to the geometric approach in which qualitative internal differences are not considered.

Following the geometric viewpoint, Proposition 7.8 becomes

Proposition 7.12 *When* $|\alpha| = |\beta| \neq 0$*, then for* $j \equiv j_\star \pmod 2$*,* $\tau_{j_\star} = 0$ *and* $\tau_j = 2f(r)$*. For* $j \not\equiv j_\star \pmod 2$*,* $\tau_j = f(r)$*.*

Proof. Direct calculation from trigonometry. ☐

The geometric analysis, which retains no qualitative difference, can be described as “objective”: the 4 values of $j_\star$ have equal probability for an outside observer.

The results of the 3 analyses are summarized below according to the assumption made in $j_\star$

Assumption	Results	Analysis
$j_\star = 1$	$\beta = \alpha \Longleftrightarrow \tau_2 = \tau_4 = \frac{\tau_3}{2} = f(r)$	Subjective
$j_\star$ odd	$j_\star = 1 \Longleftrightarrow \beta = \alpha$ $j_\star = 3 \Longleftrightarrow \beta = -\alpha$	Qualitative
$j_\star \in \{1, 2, 3, 4\}$	$\lvert\beta\rvert = \lvert\alpha\rvert$	Objective

Table 7.1:

Several remarks are in order.

1) The computational limit (Proposition 7.11 with j_* odd) defines the **middle way** between the subjective ($j_\star = 1$) and the objective ($j_\star \in \{1, 2, 3, 4\}$) points of view.

2) Mathematics takes classically the "subjective" point of view, that is the direct analysis from the point of view what is *exact* (here $a = a^M$ hence $\boldsymbol{C}$=$\boldsymbol{C^M}$). It corresponds to the *classical logic* true/false [5].

3) Experimental sciences adopt the "objective" point of view. To assess the validity of a model, scientists perform (most often implicitly) a backward analysis on data which are considered as equally probable. "There cannot exist any qualitative difference between experimental data": this rule underlies the reductionist approach which has been developed by modern Science with great success during the past 4 centuries. The *scientific logic* is based on a purely quantitative approach to backward analysis [6,7].

4) SVD computation without associativity indicates the possibility of a middle path between Mathematics and Experimental Sciences. This middle path maintains a qualitative aspect. It appears much less traveled than the extreme ones, where legions of mathematicians and scientists are busy pushing forward the limits of knowledge, by means of either a direct analysis (mathematics) or of a backward analysis (experimental sciences).

5) The spectacular success of mathematics and other Sciences has blinded us to the necessity to explore the middle path. Computationally, the middle path guarantees the phenomenological *continuity*, a property that the extreme paths cannot provide: they yield either too few, or too many, possible solutions

(see Table 7.3, and compare with Proposition 7.11). We shall go back to this essential aspect in the assessment of SVD computation in Section 10, when we discuss the emergence of an *organic logic* to organize such nonlinear computations.

6) The above distinction is sharp for $k = 3$ because $\lambda = 1^8$, so that $\boldsymbol{C} \equiv \boldsymbol{C^M} \iff |\alpha| = |\beta|$. When k increases above 3, the distinction becomes fuzzy because $\lambda = 0$ is possible, which entails $\boldsymbol{C} \equiv \boldsymbol{C^M}$ for any $\alpha, \beta \in \mathbb{R}$.

8 Autonomous evolution of $\theta = \measuredangle(a, a^M)$ as a function of $r = N(a) = 1 + N(h) > 1$

8.1 A review of the logistic iteration [9, 10, 11]

We consider the quadratic equation

$$x = rx(1 - x) \tag{E}$$

where the unknown x and the parameter r are real. (E) is the simplest form of a non linear difference map used to model a dynamical phenomen. Such a model dates back to Verhulst (1845) at least [10].

(E) has 2 solutions: $x = 0$ defined for all r and $x = 1 - \frac{1}{r}$ defined for $r \neq 0$, which agree for $r = 1$. (E) can be solved by *successive approximations* for r fixed:

$$x_0 \in [0, 1],\ x_{n+1} = rx_n(1 - x_n),\ n \geq 0. \tag{F}$$

Depending on r in $[-2, 4]$, the iterates x_n either converge or remain bounded as $n \to \infty$. For $|r - 1| > 3$, there is divergence at infinity: $|x_n| \to \infty$ as $n \to \infty$ [6, Chapter 2, Section 2.12].

The dynamics of the iteration (F) was extensively studied in the 20 years 1970-1990. Many universal properties were discovered, by Feigenbaum and others. See [10, chapter 3] for a theoretical background and some applications to Physics. See [9,11] for a more computational approach.

There exists an irrational value r_F (due to Feigenbaum), $r_F \simeq 3.5699456\cdots$, such that for $3 < r < r_F$ (resp. $r_F < r \leq 4$) the asymptotic behavior of $\{x_n\}$ is periodic with period 2^k (resp. a mix of periodic and aperiodic behavior, called *deterministic chaos* [9, 10]).

For $r_F < r < r_* = \frac{2}{3}(1 + A + \frac{4}{A})$ (with $A = (19 + 3\sqrt{33})^{1/3}$ [9]) the odd and even iterates x_n do not mix.

$r_* \simeq 3.6785735\cdots$ is the *confluence* parameter value for the iterates: for $r_* \leq r \leq 4$ there is mixing of odd and even iterates and emergence of *odd* periods.

The function $x \mapsto rx(1-x)$ has a smooth maximum for $x_c = 1/2$: x_c is called the *critical* value. A periodic orbit which contains $x_c = 1/2$ is called a supercycle.

The convergence for such supercycles is quadratic rather than linear. This is called *superstability.*

For $r = 4$, the iterates x_n describe $[0,1]$ with invariant density $\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$. This corresponds to the Liapounov exponent ln2 [10, Chapter 3, p.67-68]. By the change of variable $x = \frac{1}{2}(1-\cos 2\pi y)$, $y \in [0,1]$, the iteration (F) for $r = 4$ becomes the Bernoulli shift

y_0, $y_{n+1} = 2y_n \pmod 1$ [10, Chapter 2, p.21-23].

Below are listed some remarkable values for $r \in [0,4]$ which are relevant to the study of (F).

r	Computational phenomenon occuring at r
0	superstability of the solution 0
1	intersection of the 2 solutions
2	superstability of the solution $1 - 1/r$
3	1^{st} bifurcation (period $1 \to 2$)
$1+\sqrt{5}$	supercycle (period 2)
$1+\sqrt{6}$	2^{nd} bifurcation (period $2 \to 4$)
r_F	end of purely periodic behavior (period 2^k)
r_*	confluence
$1+\sqrt{8}$	emergence of period 3
4	$x_n \in [0,1]$

Table 8.1:

When $r > 0$, the iteration (F) can be solved, for the 2 values $r = 2$ and 4, in the *closed* form $x_0,\ x_n = \frac{1}{2}\big(1 - f_r\big[r^n f_r^{-1}(1-2x_0)\big]\big)$, $n \geq 1$.

For $r = 2$, $f_2(t) = e^t$ and $x_n = \frac{1}{2} - \frac{1}{2}\exp\big(2^n \ln|1-2x_0|\big)$.

For $r = 4$, $f_4(t) = \cos t$ for $t \in [0,\pi]$, thus

$$x_n = \frac{1}{2} - \frac{1}{2}\cos\left[2^n \cos^{-1}(1-2x_0)\right] \qquad (C)$$

for $x_0 \in [0,1]$. See [11].

8.2 An emerging law for the evolution of $\theta = \measuredangle(a, a_j)$ for $j = 2, 4$

A specificity of the two angles θ_2 and θ_4 associated with the rotated images h_2 and h_4 is that they satisfy the law

$$\tau_2 = \tau_4 = f(r) \text{ for } r = N(a) = 1 + N(h) > 0.$$

We denote by τ (resp. θ) the quantities τ_j (resp. θ_j) for j *even*. For $r > 1$, $\tau = 2\sin^2\frac{\theta}{2}$ is the positive solution of the quadratic equation

$$\tau = r\tau(1 - \tau)$$

which is of type (E). We assume that for a given $r > 1$, the evolution of τ is computed by the sequential iteration of type (F):

$$\tau_0 \in [0, 1],\ \tau_{n+1} = r\tau_n(1 - \tau_n),\ n \geq 0.$$

The evolution parameter r is chosen to be $r = N(a) = 1 + N(h)$. For $1 < r \leq 4$, or equivalently, $0 < N(h) \leq 3$, the iterates τ_n, for a fixed r, belong to $[0, 1]$. Moreover, $\lim_{n\to\infty} \tau_n = f(r)$ iff $1 < r \leq 3$.

For $3 < r \leq 4$ (that is $2 < N(h) \leq 3$) the iterates remain bounded in $[0, 1]$. The values $\tau = 0$ and $\tau = 1$ are computed by (F) for $r = 4$. In theory, $\tau = 0$ is achieved for $r = 1$ and $\tau = 1$ in the limit $N(a) \to \infty$.

The critical value is $\tau_c = 1/2$, that is $\sin\frac{\theta_c}{2} = 1/2$ and $\theta_c = \frac{\pi}{3}$. At the first bifurcation, $N(a) = 3$, $\tau = \frac{2}{3}$ and $\left|\sin\frac{\theta}{2}\right| = \frac{1}{\sqrt{3}}\tan\frac{\pi}{6}$.

Remark 8.1 The special role played by the angles $\pi/6$ and $\pi/3$ is explained in part by the following trigonometric identities:

- $(1 - \cos\theta)\cos\theta = \cos\theta - \cos\pi/3 - \cos\pi/3\cos 2\theta$,
- $\cos\theta - \cos\pi/3 = -2\sin(\frac{\theta}{2} + \frac{\pi}{6})\sin(\frac{\theta}{2} - \frac{\pi}{6})$,
- $2\cos\pi/3\cos 2\theta = \cos(2\theta + \frac{\pi}{3}) + \cos(2\theta - \frac{\pi}{3})$.

Remark 8.2 In this current application of (E) and (F) to computation, the evolution parameter $r = N(a)$ is *internal* to the measurement process. This differs from classical applications to Physics, where the parameter is *external* to the system [10].

8.3 The evolution of θ_j, $j = 1, 3$

For $j = 1, 3$, the images h_j for h are obtained by *reflection* on each bisector. The mean $\hat{\tau} = \dfrac{\tau_1 + \tau_3}{2} = \sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_3}{2}$ satisfies the same law as τ_2 and τ_4 when $\alpha\beta \neq 0$: $\hat{\tau} = \tau_2 = \tau_4 = f(r)$, $r > 1$. Moreover, when $\alpha\beta = 0$, $\tau_j = f(r)$ for $j = 1$ to 4.

8.4 The limit $N(a) \to \infty$

When $N(a) \to \infty$, $\tau(r) \to 1$. From far away, $\|c\| = 1$ is negligeable by comparison with $N(h)$. In the limit, the 3D-frame $F_3(c)$ becomes indistinguishable from the plane $\mathbb{C}_{\tilde{1}}$.

It is remarkable that this reduction from 3D to 2D occurs for $r = N(a) = 4 < \infty$ when the successive approximation (F) is used.

9 Autonomous evolution of c out of $\mathcal{D}_k$, $k \geq 3$, as a function of σ_c

In Section 8, we considered the evolution of θ while r varies, c being fixed, $\|c\| = 1$.

We shall now consider a different kind of evolution where c itself is modified in a way which depends on its internal state specified by $\lambda \in \sigma_c$.

9.1 The maximum spread of singular values for L_c, $c \in \mathcal{D}_k$

Let $\varphi = (a, b) \in \mathcal{D}_k$, with $a, b \in \Im m A_{k-1}$. With the notations of [3, Section 11]:

$$v = (x, y) \text{ and } \|\varphi \times v\|^2 = Q(x, y) + 2 < x, Gy >= \|L_\varphi v\|^2$$

$$\begin{aligned} \text{with } Q(x,y) &= \; < Nx, x > + < Ny, y > \\ &= \|a \times x\|^2 + \|b \times x\|^2 + \|a \times y\|^2 + \|b \times y\|^2 \end{aligned}$$

(Q was previously denoted $\mathcal{N}$ in [3]).

Lemma 9.1 $0 \leq \|\varphi \times v\|^2 \leq 2Q(x, y)$.

Proof. By definition, $\|\varphi \times v\|^2 \geq 0$, therefore $2| < x, Gy > | \leq Q(x, y)$. When equality is achieved, then, depending on v, $\|L_\varphi v\|$ is either minimum at 0 or maximum. We recall that when $k \geq 5$, $\|L_\varphi v\| = 0$ does not necessarily imply that φ is a zerodivisor [3, Section 12.2].

When $\|a\| = \|b\| = 1$, $Q(x, y) \leq \Big(\lambda_{max}(a) + \lambda_{max}(b)\Big)\|v\|^2$. □

Lemma 9.2 *Let* $\psi = (a, \tilde{a}) \in \mathcal{D}_k$ *with* $a \in \mathcal{D}_{k-1}$, $k \geq 4$. *Then* $\|L_\psi\|/\|\psi\| = \sqrt{2}\|L_a\|/\|a\|$.

Proof. See [3, Proposition 11.3 and Example 11.3]. ψ is a zerodivisor such that $\dim \operatorname{Zer}(\psi) = 2^{k-1} - 4 + \dim \operatorname{Zer}(a) \geq 4$ for $k \geq 4$. Therefore its singular values are spread in $[0, \sqrt{\lambda_{max}(\psi)}\|\psi\|]$ with $\sqrt{\lambda_{max}(\psi)} = \dfrac{\|L_\psi\|}{\|\psi\|} = \sqrt{2\lambda_{max}(a)}$. □

We are interested in the *maximum* spread in $\mathcal{D}_k$:

$$s_k = \max_{\varphi\in\mathcal{D}_k} \sqrt{\lambda_{max}(\varphi)} = \max_{\varphi\in\mathcal{D}_k} \frac{\|L_\varphi\|}{\|\varphi\|}.$$

We know that $s_k = 1$ for $k = 2, 3$ and that $s_4 = \sqrt{2}$ [3, section 8]. For $k \geq 4$ we show that $k \mapsto s_k$ is defined for $k \in \mathbb{N}$. More precisely

Theorem 9.3 *For* $k \geq 4$, $s_k = \left(\sqrt{2}\right)^{k-3} = \sqrt{2^{k-3}}$

Proof. We prove by induction that $s_k = \sqrt{2}s_{k-1}$, $s_3 = 1$.

1) $s_k \geq \sqrt{2}s_{k-1}$ follows from considering $\psi_{max} = (a, \tilde{a})$ where $\lambda_{max}(a) = s_{k-1}$.

2) Conversely, we prove that $s_k \leq \sqrt{2}s_{k-1}$ because s_k is achieved by $\psi_{\max}$. By Lemma 9.1

$$\|L_\varphi\|^2 \leq \left(\lambda_{max}(a) + \lambda_{max}(b)\right)\|\varphi\|^2$$

for $\varphi = (a, b)$, $\|a\| = \|b\| = 1$, $a, b \in \mathcal{D}_{k-1}$. Therefore

$$\max_{\varphi\in\mathcal{D}_k} \left(\frac{\|L_\varphi\|}{\|\varphi\|}\right) \leq 2 \max_{a\in\mathcal{D}_{k-1}} \lambda_{max}(a).$$

The maximum $s_k^2 = 2s_{k-1}^2$ is achieved for ψ_{max}, since $\lambda_{max}(a) = \lambda_{max}(\tilde{a}) = s_{k-1}^2$. □

Let $c \in \mathcal{D}_k$, $\|c\| = 1$. When c varies in $\mathcal{D}_k$, $k \geq 4$, the maximum spread of the singular values for L_c is the interval $[0, \left(\sqrt{2}\right)^{k-3}]$. For $k = 2$ and 3, there is *no* spread: all normalized singular values are 1. For larger k, the singular values can achieve values ≥ 4 for $k \geq 7$.

9.2 The transformation $c \mapsto d_1 = \tilde{1} + c$ by homotopy

We consider the real variable x, $0 \leq x \leq 1$, and set $d_x = x\tilde{1} + c$. Then $d_0 = c$ and $d_1 = \tilde{1} + c$. When x varies in $[0, 1]$, the vector c in $\mathcal{D}_k$ acquires *one* imaginary dimension (corresponding to the complex unit $\tilde{1}$). The evolution parameter $\rho \geq 0$ is related to $\lambda \in \sigma_c \subset [0, \lambda_{max}(c)]$. Various choices are possible (see Section 9.3), each leading to a *finite* number of parameter values. We assume that the autonomous evolution of $c \in \mathcal{D}_k$ into $\Im m A_k$ is ruled by (E) and computed for a given $\rho \in [0, 4]$ by the iteration (F). We observe that d_1 is a split zerodivisor when $m_1 > 0$.

9.3 Three possible evolution parameters

We consider the following 3 parameters ρ_m, $m = 1, 2, 3$, for $\lambda \in \sigma_c$

- $\rho_1 = \sqrt{\lambda} = \xi(c)$ is the measure of c, the initial vector,
- $\rho_2 = \sqrt{1+\lambda} = \xi_P(d_1)$ is the Pythagorean measure of $d_1 = \tilde{1} + c$ (conservative formula $\boldsymbol{A} = \boldsymbol{C^M}$)
- $\rho_2 = \sqrt{1 \pm \lambda} = \xi_r(d_1)$ are the 2 rational measures of the target d_1 (inventive formula $\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{D}$).

  Observe that $\lambda = 1 = \|c\|^2$ for $k = 3$ or for any k when c is alternative. Thus d_1 is a split zerodivisor, and $\rho_1 = 1$, $\rho_2 = \sqrt{2}$, $\rho_3 = 0$ or 2. No evolution is possible for the parameter values 0 and 1.

For $\rho_3 = 2$, (resp. $\rho_2 = \sqrt{2}$), then $x_n \to 1/2$ (resp. $1 - \frac{1}{\sqrt{2}} \simeq 0,3$) and c becomes the imaginary vector $d_{1/2} = \frac{1}{2}\tilde{1} + c$ (resp. $d_1 - \frac{1}{\sqrt{2}}\tilde{1}$).

When c is not alternative then $\rho_1 > 1$ with the choice $\|L_c\| = \sqrt{\lambda_{max}} = \rho_1$. For any c in $\mathcal{D}_k$, $\|c\| = 1$, then $\lambda \in [0, \left(\sqrt{2}\right)^{k-3}]$. So that $\lambda \in [0, 4]$ for $k \leq 7$.

Using Table 8.1, it is straightforward to describe the evolution corresponding to the parameter ρ_m, $m = 1, 2, 3$. Bounded evolution requires that $1 < \rho \leq 4$. For $\rho = 4$, x_n describes $[0, 1]$: c opens into the triangle on Figure 9.1 based on $d_1 = c + \tilde{1}$, with $d_{1/2} = c + \frac{1}{2}\tilde{1}$.

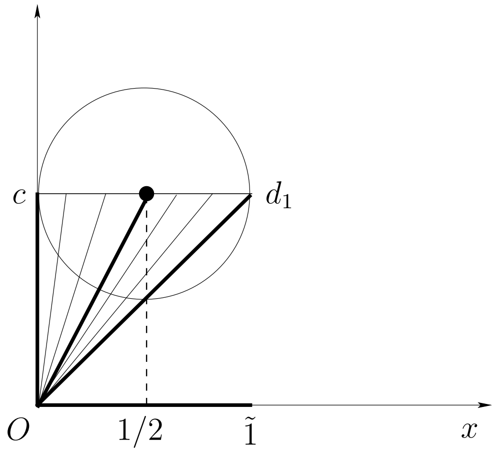


Figure 9.1: Evolution for $\rho = 4$ in $\Im m A_k$

Consider the change of variable $x = \frac{1}{2}(1 - \cos 2\pi y)$ for $y \in [0, 1[$. Then $x_n = \frac{1}{2}(1 - \cos 2\pi 2^n \cos^{-1}(1 - 2x_0))$. This is the closed formula (C).

Equivalently, $d_{x_n} = c + x_n\tilde{1} = d_{1/2} - \frac{1}{2}(\cos 2\pi y_n)\tilde{1}$, for $n \geq 1$. As n varies, the final points of d_{x_n} create a continuous segment. Each of them is the projection, on the axis $c + \mathbb{R} \times \tilde{1}$, of $z_n = e^{2\pi y_n \tilde{1}}$ in $\Im m A_k$. The point z_n belongs to the circle centered at $d_{1/2} = c + \frac{1}{2}\tilde{1}$, of radius $1/2$. The variation of y_n in $[0, 1[$ is such that the two events $y_n \in [0, 1/2[$ and $y_n \in [1/2, 1[$ have equal probability $1/2$ [10].

9.4 The transformation $d \mapsto a_1 = d + 1$ by homotopy

We fix $d = c + \beta\tilde{1}$, with β given in $[0, 1]$, $\|d\| = \sqrt{1 + \beta^2}$. Next we set $a_x = d + x$, $a_0 = d$ and $a_1 = d + 1$. When x varies in $[0, 1]$, the vector d in $\Im m A_k$ acquires *one real* dimension (corresponding to 1). The evolution parameter $\nu(\beta)$ is chosen to be either one of the 2 measures of the initial vector d: $\nu_1(\beta) = \sqrt{\lambda + \beta^2} = \xi_P(d)$ and $\nu_2(\beta) = \left|\beta \pm \sqrt{\lambda}\right| = \xi_r(d)$, or one of the 5 measures of the target $a_1 = 1 + \beta\tilde{1} + c$, given by the formulae $\boldsymbol{A}$ to $\boldsymbol{D}$:

$$\nu_3(\beta) = \sqrt{1 + \lambda + \beta^2},\ \nu_4(\beta) = \sqrt{1 + (\beta \pm \sqrt{\lambda})^2},$$
$$\nu_5(\beta) = \left|\beta \pm \sqrt{1 + \lambda}\right|,\ \nu_6(\beta) = \left|1 \pm \sqrt{\lambda + \beta^2}\right|,$$
$$\nu_7(\beta) = \left|1 \pm |\beta\sqrt{\lambda}|\right|.$$

When $\beta = 0$, the following identifications take place: $\nu_1 = \nu_2 = \rho_1$, $\nu_3 = \nu_4 = \nu_5 = \rho_2$, $\nu_6 = \nu_7 = \rho_3$.

When $\beta = \sqrt{\lambda} = 1$, then $\nu_1 = \sqrt{2}$, $\nu_2 = \{0, 2\}$, $\nu_3 = \sqrt{3}$, $\nu_4 = \{1, \sqrt{5}\}$, $\nu_5 = \nu_6 = \sqrt{2} \pm 1$, $\nu_7 = \{1, 3\}$.

The value $\nu_7 = 1$ (resp. $=3$) corresponds to the shift: no evolution → evolution (resp. a unique limit → 2-cycle).

For $\nu = 4$, we get $a_{x_n} = d + x_n = a_{1/2} - \frac{1}{2}(\cos 2\pi y_n)$ for $n \geq 1$. It corresponds to Figure 9.1 where the plane $lin(\tilde{1}, c)$ is replaced by the plane $lin(1, d)$.

10 Conclusion: The emergence of an organic logic

Computation in genuinely *nonlinear* structures such as non associative Dickson algebras has unexpected but important consequences. The focus of this report has been the SVD of the left (or right) multiplication map by a vector in a nonassociative Dickson algebra A_k, of dimesion 2^k, $k \geq 3$. SVD computation provides a simple example of the logical clash between matrix computation (inherently linear) and the nonlinear algebraic structure in which it takes place. This results in paradoxes related to zerodivisors for $k \geq 4$ [3, 4, 5]. In the conservative view of Hilbert and Turing, paradoxes play a negative role and should be banned [5]. However paradoxes are vital to mathematical creativity, as is testified by the case of the two numbers of 0 and $\sqrt{-1}$ [3,7].

These 2 symbols were, after much debate, accepted as numbers because they allowed to face successfully new computational challenges. Their computing power outweighted the philosophical reservations due to their breaking certains computational laws of the time [3,5,7]. Out of computational necessity, the laws were transformed to accomodate a larger mathematical reality.

But classical logic abhors paradoxes. It is *deductive*, based on a *real* time which orders instants as before or after any given event. It is well suited for classical computation in any of the 3 fields A_k, $k = 0, 1, 2$ where multiplication is associative.

If one wants to compute **beyond associativity**, one is faced with new computational challenges. The paradoxes can be resolved by adapting the old logic to organize the new computational facts. We call *organic logic* this emergent logic which has an *inductive* component necessary to explain the results which carry complexified information [3,4,5]. The organic logic realizes a creative synthesis between the discrete and the continuous. It explains the constructive and creative behaviours exhibited by living organisms, by means of a *complex* time which cannot order [5]. It provides a computational basis for the philosophical vision of Life proposed by Bergson a century ago in “L’évolution créatrice” (1907). It also provides a rational basis for the countless “auto-organized” phenomena which have been discovered in Natural Sciences during the last 3 decades of the 20th Century.

This does not come as a surprise for a practicing mathematician. For example, the lecture on “Mathématiques et réalité” delivered by Pierre Cartier (ENS, Paris) on January 14, 2000, for l’Université de tous les savoirs, is concluded by the following statement:

“... Je considérerais plutôt les mathématiques en termes de physiologie, comme un *organisme*, où il n’y aurait pas de centre mais plutôt un réseau, où diverses parties importantes se répondent, interagissent, cette *unité organique* étant possible parce que les mêmes outils mathématiques peuvent se réemployer dans de nombreuses incarnations.

Là est l’extraordinaire : dans le réemploi des outils mathématiques, dans le dynamisme qui les fait s’engendrer. La meilleure image pour symboliser les mathématiques, c’est la *vie organique*.” (in **Université de tous les savoirs, Les Mathématiques** vol.13, pp 9-26, Poches Odile Jacob, Paris 2002, italics added).

A technical description of this logic will appear in a future work, providing an essential link between three different aspects of computation, all related to non linearity and complexification: the logistic, Dickson algebras and the Homotopic Deviation theory.

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