The dynamics of spectral analysis by Homotopic Deviation.
Part II The evolution field.

Françoise Chaitin-Chatelin *

CERFACS Technical Report TR/PA/08/03

Abstract : This report is Part II of a twofold work devoted to the dynamics of the spectral analysis deriving from the complex coupling of the matrix A of order n, with a deviation matrix E of rank $r \leq n$, resulting in $A(t) = A + tE$. The spectral field $t \mapsto \sigma(A(t)) \in \mathbb{C}^n$ was analyzed in Part I [9]. The study uncoveved the role of the quadratic polynomial (in z)

$$AEA - z(EA + AE) + (z^2 + t)E = \Delta(t, z),$$

where t appears linearly. The map $t \in \hat{\mathbb{C}} \mapsto \{z(t);\ \det \Delta(t, z) = 0\} \in \mathbb{C}^{2n-1}$ defines the evolution field inherent to the coupling (A, E). This derived field is the topic of Part II.

Keywords : Complex coupling, synthesis, spectral filed, evolution field, frontier of analyticity at ∞, core set, spectral potential of a square matrix, quadratic polynomial, pencil.

1 Review of the evolution matrix C_z, $z \in re(A)$

Given A, $E \in \mathbb{C}^{n\times n}$, and $z \in re(A)$, the evolution matrix defined in Part I is

$$C_z = R(0, z)ER(0, z),$$

*Université Toulouse 1 and CERFACS,42 avenue G. Coriolis 31057 Toulouse Cedex 1, France. E-mail: chatelin@cerfacs.fr

with rank $r = \operatorname{rank} E \leq n$.

For $r < n$, the frontier reduction matrix B_ξ, $\xi \in F(A,E)$, is derived from C_z by replacing $E = UV^H$ by $UP_{0\xi}V^H$, where $P_{0\xi}$ is the spectral projection for $0 \in \sigma(M_\xi)$. In general, $B_\xi \neq C_\xi$, however $B_\xi = C_\xi$ when ξ is critical ($P_{0\xi} = I_r$).

1.1 The partial derivative $\frac{\partial}{\partial t} R(0,z)$

Theorem 1.1 *For $z \in re(A)$, the evolution matrix C_z satisfies $-C_z = \frac{\partial}{\partial t} R(0,z)$.*

Proof. For $|t|\rho(M_z) < 1$, $R(t,z) - R(0,z) = -tC_z - tR(0,z)U \sum_{k\geq 1} (tM_z)^k V^H R(0,z)$ is a converging series. Therefore $\lim_{t\to 0} \frac{1}{t}(R(t,z) - R(0,z)) = -C_z$.

The evolution matrix in $z \in re(A)$ is the opposite of the partial derivative of $R(t,z)$ with respect to t at $t = 0$. □

Corollary 1.2 *Let $z \in F_c(A,E)$, be such that $M_z = 0$. Then*

$$C_z = \frac{1}{t}(R(0,z) - R(t,z))$$

for any $t \in \mathbb{C}\backslash\{0\}$.

Proof. Clear by letting $M_z = 0$. We recall that $M_z = 0$ is possible only for $r \leq [n/2]$ (see Part I, Section 3.5). □

1.2 The pencils $C_z + sE$ and $E + tC_z$, for $st = 1$

Let z be fixed in $re(A)$. The coupling of C_z with E by the *reverse intensity* $s = \frac{1}{t}$, $s \in \hat{\mathbb{C}}$, is $C_z(s) = C_z + sE$.

For $s \neq 0$, $\frac{1}{s} C_z(s) = E + tC_z$, with $s \neq 0 \iff |t| \neq \infty$.

The two pencils are proportional for s or t not in $\{0, \infty\}$. They have the same structure, as z varies in $re(A)$.

1.3 The quadratic polynonial $\Delta(z,A,B)$

Definition 1.1 *For $z \in \mathbb{C}$, $A, B \in \mathbb{C}^{n\times n}$, we define the matrix polynomial*

$$\begin{aligned} \Delta(z,A,B) &= ABA - z(AB + BA) + z^2 B \\ &= (A - zI)B(A - zI). \end{aligned}$$

We observe that Δ is quadratic in A and linear in B:

$$\Delta(z, A, \beta B) = \beta\Delta(z, A, B) \text{ for } \beta \in \mathbb{C}.$$

In our HD context, we shall mainly consider $B = E$. When there is no ambiguity, we set $\Delta(z, A, E) = \Delta(z)$.

Proposition 1.3 *$\Delta(z, A, E + tC_z) = \Delta(z, A, E) + tE$ for $z \in re(A)$.*

Proof. For $z \in re(A)$:

$$\begin{aligned} C_z(s) &= C_z + sE = R(0,z)ER(0,z) + sE \\ &= R(0,z)[E + s(A - zI)E(A - zI)]R(0,z) \\ &= R(0,z)[E + s(AEA - z(EA + AE) + z^2E]R(0,z) \\ &= R(0,z)[E + s\Delta(z, A, E]R(0,z). \end{aligned}$$

$$\begin{aligned} \Delta(z, A, E) + tE &= \frac{1}{s}(A - zI)C_z(s)(A - zI) \\ &= \frac{1}{s}\Delta(z, A, C_z(s)) \\ &= \Delta(z, A, E + tC_z). \end{aligned}$$

□

Definition 1.2 $\Delta(t, z) = \Delta(z) + tE = \Delta(z, A, E) + tE$. $\delta(z) = \det \Delta(z)$, $\delta(t, z) = \det \Delta(t, z)$.

Lemma 1.4 *For $\pi(z) \neq 0$, $\det \frac{1}{s}C_z(s) = \frac{1}{\pi^2(z)}\delta(t, z)$ with $st = 1$.*

Proof. $\frac{1}{s}C_z = R(0,z)(\Delta(z) + tE)R(0,z)$. □

For $s \neq 0(|t| \neq \infty)$ and $z \in re(A)$, the structure of the pencil $\frac{1}{s}C_z(s) = E + tC_z$ is determined by the zeros of $\delta(t, z)$.

Definition 1.3 *The map $t \mapsto \{\zeta(t);\ \delta(t, \zeta) = 0\}$ is the evolution field derived from the coupling (A, E).*

Part I of this study [9] was devoted to the spectral field $t \mapsto \sigma(A(t))$, which consists of the roots of $\pi(t, z) = \det(A - zI + tE) = 0$.

In Part II, we shall study the roots of $\delta(t, z) = \det([AEA - z(AE + EA) + z^2E] + tE) = 0$. The pencil $A - zI$ is replaced by the quadratic polynomial $\Delta(z) = \Delta(z, A, E)$. The algebraic modification is remarkably simple. The analysis of the evolution field to be presented follows closely that given in Part I for the spectral field. We shall stress the many differences whenever necessary.

1.4 The limit case $n = r = 1$

When $n = r = 1$, we get only *scalar* relations with $a \neq 0$, e zero or not.

i) $a - z + te = 0$ gives two possibilities:

- if $e = 0$, $z = a$ and t arbitrary in $\mathbb{C}$,
- if $e \neq 0$, $t = \frac{z-a}{e}$

ii) $a^2 e - 2aez + ez^2 + te = 0$ yields also:

- if $e = 0$, t and z arbitrary in $\mathbb{C}$,
- if $e \neq 0$, $t = -(z-a)^2$ which is independant of $e \neq 0$.

The spectral field in i) differs markedly from the evolution field in ii).

Observe that, because $r = n$, there are no frontier points and no secondary sources of induction. For $e \neq 0$, $z \neq a + te$, the resolvent becomes the scalar $\frac{1}{a - z + te}$. And the evolution matrix is the scalar $c_z = \frac{e}{(a-z)^2}$, defined for $z \neq a$.

When z and t belong to the evolution field, we have $t = -(z-a)^2$ for $e \neq 0$, then $c_z = -\frac{e}{t}$ which is independant of a.

1.5 The general case $n \geq 2$, $1 \leq r \leq n$

In reference with notation common in structure analysis [4, 12], we set $K = AEA$, $C = -(EA + AE)$. Thus $\Delta(z) = K + zC + z^2 E = (A - zI)E(A - zI)$ has rank $\leq r$ with the matrix K (resp. C) of rank $\leq r$ (resp. $\leq 2r$).

Lemma 1.5 *The following bounds hold for $t, z \in \mathbb{C}$:*

$$rank\, \{K + tE,\ \Delta(t,z)\} \leq 2r,$$

$$rank\, \{K + zC,\ C + zE\} \leq 3r.$$

Proof. Clear by subadditivity of the rank. □

Notation. We denote by σ_{KE}, σ_{KC}, and σ_{CE} the set of *finite* eigenvalues in the spectra $sp(K,E)$, $sp(K,C)$, and $sp(C,E)$, [4]. For $k < n$ in $\mathbb{N}^*$, $[\frac{n}{k}]$ denotes the largest integer $\leq \frac{n}{k}$.

Corollary 1.6 *The conditions σ_{KE} (resp. σ_{CE} or σ_{KC}) are discrete sets in $\mathbb{C}$ require that $r \geq [n/2]$ (resp. $\geq [n/3]$).*

Proof. Clear by Lemma 1.5. Because $\det K = 0$ for $r < n$, σ_{KE} and σ_{KC} cannot be empty. When $\det C = 0$, σ_{CE} cannot be empty. When C is invertible ($r \geq [\frac{n}{2}]$), then $\sigma_{KC} = \sigma(-KC^{-1})$ and $z \in \sigma_{CE}$ if $0 \neq \frac{1}{z} \in \sigma(-C^{-1}E)$. We observe that σ_{CE} is empty iff $C^{-1}E$ is nilpotent. □

2 The evolution field

2.1 The spectrum $\sigma(A)$ and the information set Z_2

The characteristic polynomial $\pi(z)$ has degree n always, and there are n eigenvalues in $\sigma(A)$.

By comparison the degree d_2 of $\delta(z) = \det \Delta(z)$ can take the 2 values 0 or $2n$. Its root set is $Z_2 = \{\zeta \in \mathbb{C};\ \delta(\zeta) = 0\}$.

Lemma 2.1 *When $r = n$, $Z_2 = (\sigma(A))^2$. And when $1 \leq r < n$, $Z_2 = \mathbb{C}$.*

Proof. $\Delta(z) = (A - zI)E(A - zI)$. Therefore $\delta(z) = \pi^2(z) \det E$.

Definition 2.1 *The set Z_2 is the information set for the coupling (A, E).*

The information set is either discrete for $r = n$ ($Z_2 = (\sigma(A))^2$) or continuous for $r < n$ ($Z_2 = \mathbb{C}$), whereas the spectrum $\sigma(A)$ is always discrete.

2.2 The evolution field

For $t \in \mathbb{C}$, we define

$$Z_2(t) = \{\zeta(t);\ \delta(t, \zeta) = 0\}.$$

The polynomial $\delta(t, z)$, considered as a polynomial in z for a fixed $t \in \mathbb{C}$, has a degree $d_2(t)$ which can vary (with t) between 0 and $2n - 1$. For example, $d_2(0) = 0$ when $r < n$.

First of all, by Lemma 1.3, $\delta(t, z) \equiv 0$ when $r < [n/2]$. Therefore $Z_2(t) = \mathbb{C}$ for any $t \in \mathbb{C}$.

We assume below that $r \geq [n/2]$.

When $d_2(t) \geq 1$, $Z_2(t)$ is a discrete set. When $d_2(t) = 0$, the nature of $Z_2(t)$ depends on the value of $k_t = \det(K + tE)$ which can be $\neq 0$ for $r \geq [n/2]$: $t \notin \sigma_{KE} \neq \mathbb{C}$.

Proposition 2.2 *Let t be given in $\mathbb{C}$. When $r < [n/2]$, $Z_2(t) = \mathbb{C}$ for any $t \in \mathbb{C}$.*

When $r \geq [n/2]$, there are 3 possibilities for $Z_2(t)$:

1) $d_2(t) \geq 1$ and $Z_2(t)$ is discrete, *2) $d_2(t) = 0$, then $Z_2(t) = \mathbb{C}$ (resp. $\emptyset$) for $k_t = 0$ (resp. $\neq 0$).*

Proof. Clear. □

Proposition 2.3 *We assume that $\sigma_{KE} \neq \mathbb{C}$ for $r \geq [n/2]$. Then $Z_2(t) \neq \mathbb{C}$ for all $t \notin \sigma_{KE}$.*

Proof. A necessary condition for $Z_2(t) = \mathbb{C}$ is that $k_t = 0$. When the pencil $K + tE$ is regular (resp. singular) then $\sigma_{KE} \neq \mathbb{C}$ (resp. $\sigma_{KE} = \mathbb{C}$). When $\sigma_{KE} \neq \mathbb{C}$, $Z_2(t) \neq \mathbb{C}$ for all $t \notin \sigma_{KE}$.

When $\sigma_{KE} = \mathbb{C}$, then $Z_2(t) = \mathbb{C}$ for all t is a possibility. □

2.3 The homotopic factorization for $r = n$

When the deviation matrix has rank $r = n$, then $Z_2 = (\sigma(A))^2$. For $z \in re(A) = \mathbb{C} \backslash \sigma(A)$, $\Delta(z) + tE = (I + tE\Delta^{-1}(z))\Delta(z)$ and $T(t,z) = \Delta^{-1}(z)(I + tE\Delta^{-1}(z))^{-1}$. The homotopic deviation for $\Delta(z)$ parallels that for $A - zI$ and $r = n$ in a straightforward way.

We can choose $V = I$ and $U = E$. Then $\Delta^{-1}(z) = R(0,z)E^{-1}R(0,z)$ for $z \in re(A)$ and $N_z = \Delta^{-1}(z)E = R(0,z)E^{-1}R(0,z)E$. We observe that $N_z^{-1} = E^{-1}\Delta(z)$ exists for all $z \in \mathbb{C}$.

This will have an important consequence (Proposition 2.4).

For $z \in re(A)$, $t \mapsto T(t,z) = \Delta^{-1}(z)[I - tE(I + tN_z)^{-1}\Delta^{-1}(z)]$ is analytic around 0 ($|t| < 1/\rho(N_z)$) with $\lim_{t \to 0} T(t,z) = \Delta^{-1}(z)$. It is also analytic around ∞ ($|t| > \rho(N_z^{-1})$) with $\lim_{|t| \to \infty} T(t,z) = 0$.

In particular, the following connection between t and z holds for $r = n$:

Proposition 2.4 *Any $z \in \mathbb{C}$ is an evolution point $\zeta(t)$ for n matrices $\Delta(t,z)$ where t and z are related by $tn_z = -1$, $n_z \in \sigma(N_z)$.*

Proof. $\Delta(\zeta) + tE = E(E^{-1}\Delta(z) + tI)$. Observe that $t \in \mathbb{C}$. □

2.4 A shifted homotopic factorization for $[n/2] \leq r < n$.

When $r < n$, then $Z_2 = \mathbb{C}$ and $\Delta(z)$ is nowhere invertible. No computation by homotopic deviation based on $\Delta(z)$ is possible.

We assume that $\sigma_{KE} \neq \mathbb{C}$. By Proposition 2.3, $Z_2(t) \neq \mathbb{C}$ for all $t \notin \sigma_{KE}$. Recall that $0 \in \sigma_{KE}$ because K is singular. One can choose $\varepsilon \neq 0$ small enough such that $Z_2(\varepsilon) \neq \mathbb{C}$.

Then we consider the shift $t_\varepsilon = t - \varepsilon$.

We write $\Delta(z) + tE = (\Delta(z) + \varepsilon E) + t_\varepsilon E$ where $Z_2(\varepsilon) = \{z;\ \delta(\varepsilon, z) = 0\} \neq \mathbb{C}$ by assumption.

Using $\Delta_\varepsilon(z) = \Delta(z) + \varepsilon E$ which is invertible for $z \notin Z_2(\varepsilon)$, one can develop an homotopic deviation theory with the shifted variable $t_\varepsilon \in \hat{\mathbb{C}}$, where ε can be arbitrarily small. This is the topic of the next Section.

3 Homotopic Deviation for $\Delta_\varepsilon(z) + t_\varepsilon E$, $z \in Z_2(\varepsilon) \neq \mathbb{C}$.

We assume that $[n/2] \leq r < n$ and $\sigma_{KE} \neq \mathbb{C}$. For $\varepsilon \neq 0$ small enough, $Z_2(\varepsilon) \neq \mathbb{C}$. We set $t_\varepsilon = t - \varepsilon \in \hat{\mathbb{C}}$, and $\Delta(\varepsilon, z) = \Delta(z) + \varepsilon E$.

3.1 The communication matrix $N_{\varepsilon z}$

For $z \notin Z_2(\varepsilon)$, we define

$$N_{\varepsilon z} = V^H \Delta^{-1}(\varepsilon, z) U \in \mathbb{C}^{r \times r},$$

and $C_{2\varepsilon}(A, E) = \{z \notin Z_2(\varepsilon);\ \text{rank}\ \ N_{\varepsilon z} < r\}$.

3.2 Characterization of $C_{2\varepsilon}(A, E)$.

We consider the augmented matrix of order $\hat{n} = n + r$ defined by $\begin{pmatrix} \Delta(\varepsilon, z) & -U \\ V^H & 0 \end{pmatrix}$ for $z \in \mathbb{C}$, $\varepsilon \neq 0$.

Its determinant is the polynomial $\hat{\delta}(\varepsilon, z)$, $z \in \mathbb{C}$. Its root set is denoted $\hat{Z}_2(\varepsilon)$.

Proposition 3.1 $C_{2\varepsilon}(A, E) = \hat{Z}_2(\varepsilon) \backslash Z_2(\varepsilon)$

Proof. For $z \notin Z_2(\varepsilon)$, we apply the Schur complement formula to the augmented matrix. This yields $\hat{\delta}(\varepsilon, z) = \delta(\varepsilon, z) \det N_{\varepsilon z}$, for $\varepsilon \neq 0$ small enough. □

3.3 The analyticity of $\Delta^{-1}(t, z)$ with respect to t_ε.

The resolvent $\Delta^{-1}(t, z)$ can be written as

$$T_\varepsilon(t_\varepsilon, z) = \Delta^{-1}(\varepsilon, z)[I_n - tU(I_r + tN_{\varepsilon z})^{-1} V^H \Delta^{-1}(\varepsilon, z)].$$

It is defined for $z \notin Z_2(\varepsilon)$. The map $t_\varepsilon \in \hat{\mathbb{C}} \mapsto T_\varepsilon(t_\varepsilon, z)$ is analytic in t_ε around 0 (resp. ∞) for $z \notin Z_2(\varepsilon)$ (resp. $z \in C_{2\varepsilon}(A, E)$).

The set $C_{2\varepsilon}(A, E)$ represents the frontier of analyticity at ∞ for $z \in \mathbb{C} \backslash Z_2(\varepsilon)$. At such points $T_\varepsilon(\infty, z) = \lim_{|t_\varepsilon| \to \infty} T_\varepsilon(t_\varepsilon, z)$ does not exist. Such points are points of *change* (or modification, alteration).

3.4 The limit set $\mathrm{Lim}\ _2$ for the evolution field when $|t| \to \infty$

When $|t| \to \infty$, some of the evolution points $\zeta(t)$ may converge to a finite limit in $\mathbb{C}$. The set $\mathrm{Lim}\ _2$ denotes the set of these finite limits. Set $\Lambda_{2\varepsilon}(A, E) = \mathrm{Lim}\ _2 \cap Z_2(\varepsilon)$. It is clear that $\Lambda_{2\varepsilon}(A, E) \subset C_{2\varepsilon}(A, E)$.

Also $\Delta(z) + tE = t(E + s\Delta(z))$. The roots of $\det(E + s\Delta(z))$ are denoted $\beta(s)$.

Lemma 3.2 $\zeta(t) \underset{|t|\to\infty}{\to} \xi \Longleftrightarrow |\beta(s) - \xi s| \underset{s\to 0}{\to} 0$

Proof. $\zeta(t) = \dfrac{\beta(s)}{s}$ follows from the relation $\Delta(z) + tE = t(E + s\Delta(z))$. To get $\zeta(t) \to \xi \in \mathbb{C}$ as $|t| \to \infty$, it is necessary and sufficient that $\beta(s) \to 0$ as $s \to 0$ with an order in $s \geq 1$:$\beta(s) = \xi s + o(s)$. □

Corollary 3.3 $E + s\Delta(\beta(s)) = E + sK + O(s^2)$.

Proof. Clear by Lemma 3.2. $\Delta(\beta(s)) = K + \beta(s)C + \beta^2(s)E$. □

Proposition 3.4 *The limit set for $\Delta(z) + tE$ is the limit set of the pencil $K + tE$.*

Proof. The property: $\beta(s)$ is an eigenvalue of the pencil $E + sK$ is equivalent to the property: $\zeta(t) = \dfrac{\beta(s)}{s}$ is an eigenvalue of $K + tE$. □

The following corollary uses the notation introduced in Part I to define the kernel set $\tilde{Z}$ [9].

Corollary 3.5 $\mathrm{Lim}\ _2 \supset \tilde{Z}_2$ *where* $\tilde{Z}_2$ *is the root set of* $\tilde{\delta}(z) = \det\left(P_g K P_{g_{\restriction \mathrm{Ker}\ E}} - z\begin{pmatrix} I_{g_1} & 0 \\ 0 & 0_{g_2}\end{pmatrix}\right)$, *assuming that* $g_1 \geq 1$.

Proof. We apply the theory of Part I to the pencil $K + tE$. The matrix A is replaced by $K = AEA$. And the matrix $P_g K P_{g_{\restriction \mathrm{Ker}\ E}}$ is split into 4 blocks according to the partition $g = g_1 + g_2$, if $g_1 \geq 1$. □

To avoid ambiguity, $\tilde{Z}_2$ is called the *core* set, whereas $\tilde{Z}$ is called the *kernel* set for the coupling (A, E).

3.5 The connection between t_ε and z

For $r < n$, we get an analogue of Proposition 2.4, relating $t_\varepsilon = t - \varepsilon$, $\varepsilon \neq 0$, and z.

Proposition 3.6 *For $\varepsilon \neq 0$ small enough, any $z \notin Z_2(\varepsilon)$ is an evolution point for r matrices $\Delta(t, z)$ where $z \in \mathbb{C}$ and $t_\varepsilon = t - \varepsilon \in \hat{\mathbb{C}}$ are related by $t_\varepsilon n_{\varepsilon z} = -1$, $n_{\varepsilon z} \in \sigma(N_{\varepsilon z})$.*

Proof. Consequence of the factorization, for $\varepsilon \neq 0$,
$\Delta(t, z) = \Delta(\varepsilon, z) + t_\varepsilon E = (I + t_\varepsilon E\Delta^{-1}(\varepsilon, z))\Delta(\varepsilon, z)$. □

4 Generalized eigenvalue problems of order $2n$

4.1 Linearizations of the quadratic eigenproblem $\delta(t,z)=0$

In $\mathbb{C}^{2n\times 2n}$, we consider the strict equivalence for $z\in\mathbb{C}$,

$$\begin{pmatrix} \Delta(z)+tE & 0 \\ 0 & I_n \end{pmatrix} = E(z)(G-zH)F(z),$$

where E and F are matrices depending on z with constant ($\neq 0$) determinant. The matrices G and H may depend on t. The eigenvalues of $\Delta(z)+tE$ and of $G-zH$ concide.

Such a linearization is not unique. We study the *two* standard linearizations into companion form [4, 12].

1) L_1: The *first* companion form is defined by

$$G_1(t)=\begin{pmatrix} 0 & I \\ -(K+tE) & -C \end{pmatrix}, \quad H_1=\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix},$$

$$E_1(z)=\begin{pmatrix} -(C+zE) & -I \\ I & 0 \end{pmatrix}, \quad F_1(z)=\begin{pmatrix} I & 0 \\ zI & I \end{pmatrix}.$$

2) L_2: The *second* companion form is defined by

$$G_2(t)=\begin{pmatrix} -(K+tE) & 0 \\ 0 & I \end{pmatrix}, \quad H_2=\begin{pmatrix} C & E \\ I & 0 \end{pmatrix},$$

$$E_2(z)=\begin{pmatrix} I & zE \\ 0 & I \end{pmatrix}, \quad F_2(z)=F_1(z).$$

4.2 Computation of $\delta(t,z)$ by L_1 and L_2

1) With L_1, $G_1(t)-zH_1=\begin{pmatrix} -zI & I \\ -(K+tE) & -(C+zE) \end{pmatrix}$.

The first (resp. second) diagonal block is nonsingular iff $z\neq 0$ (resp. $z\notin\sigma_{CE}$).

We assume that $\sigma_{CE}\neq\mathbb{C}$. Thus necessarily $r\geq[n/3]$. We may apply the Schur complement formula in two ways to get $\delta(t,z)$.

i) For $z\neq 0$, $\delta(t,z)=z^n\det[(C+zE)+\frac{1}{z}(K+tE)]$.

ii) For $z \in \sigma_{CE} \neq \mathbb{C}$ ($r \geq [n/3]$),

$$\delta(t,z) = \det(C + tE)\det[zI + (K + tE)(C + zE)^{-1}].$$

$z \notin \sigma_{CE} \neq \mathbb{C}$ belongs to $Z_2(t)$ iff $-z$ is an eigenvalue of the matrix $X_{tz} = (K + tE)(C + zE)^{-1}$.

This is a matrix of order n in factored form, where the first (resp. second) factor depends on t (resp. z). It can exist for $r \geq [n/3]$ only.

2) With L_2, $G_2(t) - zH_2 = \begin{pmatrix} -(K + tE + zC) & -zE \\ -zI & I \end{pmatrix}$.

i) $\det I = 1$ unconditionally. This yields $\delta(t,z) = (-1)^n \det(\Delta(z) + tE)$ without restriction.

ii) When $\overline{\omega}(t,z) = \det(K + tE + zC) \neq 0$, then

$$\delta(t,z) = \overline{\omega}(t,z)\det(I + z^2(K + tE + zC)^{-1}E).$$

We observe that t and z appear linearly in $D_{tz} = K + tE + zC$.

Lemma 4.1 *If $r < [n/4]$, $\overline{\omega}(t,z) = \det D_{tz} \equiv 0$ for all $(t,z) \in \mathbb{C}^2$.*

Proof. rank $(K + tE + zC) \leq 4r$ for $(t,z) \in \mathbb{C}^2$. □

For D_{tz} to be invertible, it is necessary that $r \geq [n/4]$.

When $\overline{\omega}(t,z) \neq 0$, we set $W_{tz} = V^H D_{tz}^{-1} U \in \mathbb{C}^{r \times r}$ which satisfies $\sigma(D_{tz}^{-1}E) = \{0\} \cup \sigma(W_{tz})$.

4.3 Semi-implicit spectral connections for r large enough

The linearizations L_1 and L_2 lead to connections between t and z which are expressed through the eigenvalues of X_{tz} (for $r \geq [n/3]$) or W_{tz} (for $r \geq [n/4]$) of order n and r respectively. Because these matrices depend on t and z simultaneously, the connection is *semi-implicit*. The results are stated below. They are straightforward consequences of the factorizations for $\delta(t,z)$ which were obtained in Section 4.2.

Proposition 4.2 *For $r \geq [n/3]$, any observation point z not in $\sigma_{CE} \neq \mathbb{C}$ is an evolution point for any $\Delta(t,z)$ where $t \in \mathbb{C}$ and $z \in \mathbb{C}$ are related by $-z = x_{tz}$ where $x_{tz} \in \sigma(X_{tz})$.*

Now for $r \geq [n/2]$, we can consider $y_{tz} = X_{tz}^{-1}$ defined for $t \notin \sigma_{KE} \neq \mathbb{C}$. We get the alternative

Proposition 4.3 *For $r \geq [n/2]$, any z in $\mathbb{C}$ is an evolution point for $\Delta(t,z)$ with $t \notin \sigma_{KE} \neq \mathbb{C}$ iff $t \in \mathbb{C}$ and $z \in \mathbb{C}$ are related by $zy_{tz} = -1$ where $0 \neq y_{tz} \in \sigma(Y_{tz})$.*

Proof. The condition $z \notin \sigma_{CE}$ guarantees that $y_{tz} \neq 0$, for $t \notin \sigma_{KE}$. □

Proposition 4.4 *For $r \geq [n/4]$, any $z \in \mathbb{C}$ is an evolution point for $\Delta(t,z)$ such that $\overline{\omega}(t,z) \neq 0$ iff t and z are related by $z^2 w_{tz} = -1$ where $0 \neq w_{tz} \in \sigma(W_{tz})$.*

4.4 Comparison between the three ways of computing $\delta(t,z)$

The direct approach of Section 2 yields that $\delta(t,z) \equiv 0$ for $r < [n/2]$. However, this is not the result found by linearization with any of the two companion forms. Computed by linearization, $Z_2(t)$ can be $\neq \mathbb{C}$ for $[n/2] > r \geq [n/3]$ or even $[n/4]$. This yields connections between t and z in $\mathbb{C} \times \mathbb{C}$ which are not found by the direct approach which gives $Z_2(t) = \mathbb{C}$ for all t.

All views are equally correct from a matrix computation point of view. They *coexist*.

Another such example is discussed below.

5 The linear split $\Delta(t,z) = K + zC + \varphi E$ for $r \geq [n/3]$

5.1 An alternative factorization for $\Delta(t,z)$ when $[n/3] \leq r < n$

We assume that $n > r \geq [n/3]$ and $\sigma_{KC} \neq \mathbb{C}$.

We fix z in $\mathbb{C} \backslash \sigma_{KC}$ and set $\varphi = z^2 + t$. Then $\Delta(t,z) = K + zC + \varphi E$ depends on $\varphi \in \mathbb{C}$.

The new variable $\varphi = z^2 + t$ measures the intensity of the coupling in a way which *depends* on z^2. We write $\Delta(t,z) = [I + \varphi E(K+zC)^{-1}](K+zC)$, then, with $V_z = V^H(K + zC)^{-1}U = W_{0z}$, we get $\Delta^{-1}(t,z) = U(\varphi, z) = (K+zC)^{-1}[I_n - \varphi U(I_r + \varphi V_z)^{-1} V^H (K+zC)^{-1}]$,

where z is a parameter in $\mathbb{C} \backslash \sigma_{KC}$. The map $\varphi \mapsto U(\varphi, z)$ is analytic for $|\varphi| < 1/\rho(V_z)$.

We define $C_1(A,E) = \{z \notin \sigma_{KC};\ \text{rank } V_z < r\}$.

The subscript 1 indicates that the notion is related to $K + zC$, the linear split of $\Delta(z)$ in which $z^2 E$ has been deleted. Recall that $\sigma_{KC} \neq \mathbb{C}$ is possible under the condition $r \geq [n/3]$.

We define $\hat{\psi}(z) = \det \begin{pmatrix} K + zC & -U \\ V^H & 0 \end{pmatrix}$.

The root set for $\hat{\psi}$ is $\hat{Z}_1 = \{z \in \mathbb{C};\ \hat{\psi}(z) = 0\}$.

Lemma 5.1 $C_1(A,E) = \hat{Z}_1 \backslash \sigma_{KC}$ *for* $[n/3] \leq r < n$.

Proof. For $z \notin \sigma_{KC}$, $\hat{\psi}(z) = \det(K + zC) \det V_z$. Thus $\det V_z = 0$ implies that $\hat{\psi}(z) = 0$. □

For any $z \notin \hat{Z}_1 \cup \sigma_{KC}$ when $[n/3] \leq r < n$, the map $\varphi \mapsto U(\varphi, z)$ is analytic around ∞ for $|\varphi| > \rho(V_z^{-1})$. And $U(\infty, z) = -(K+zC)^{-1}UV_z^{-1}V^H(K+zC)^{-1} \neq 0$.

5.2 The $z-$dependent intensity $\varphi = z^2 + t$

What are the consequences of the change of measure in the intensity of the coupling, when t is replaced by $\varphi(z) = z^2 + t$?

First of all, for $[n/3] \leq r < [n/2]$, $Z_2(t) = \mathbb{C}$ for all t and there is no analytic dependence expressed with t. For $r \geq [n/2]$, there are *two* possible analytic dependences. The first one, based on $\varphi(z) = z^2 + t$, couples t and z. It introduces a relativity of viewpoint since the measure of intensity φ depends on the observation point z.

The second one is in use when $\sigma_{KE} \neq \mathbb{C}$. It uses also a shift in t, namely $t_\varepsilon = t - \varepsilon$ such that $Z_2(\varepsilon) \neq \mathbb{C}$, $\varepsilon \neq 0$. Here the shift ε is independent of z, and can be chosen arbitrarily small.

Proposition 5.2 *For $[n/2] \leq r < n$, the two following relations coexist:*

1) *For $\varepsilon \neq 0$, $z \notin Z_2(\varepsilon)$ is an evolution point for r matrices $\Delta(t, z)$ where $t_\varepsilon = t - \varepsilon$ and z are related by $t_\varepsilon n_{\varepsilon z} = -1$, $n_{\varepsilon z} \in \sigma(N_{\varepsilon z})$, $t_\varepsilon \in \hat{\mathbb{C}}$.*

2) *Any $z \notin \sigma_{KC}$ is an evolution point for r matrices $\Delta(t, z)$ where $\varphi = z^2 + t$ and z are related by $\varphi v_z = -1$, $v_z \in \sigma(V_z)$, $\varphi \in \hat{\mathbb{C}}$.*

Proof. Clear by Proposition 3.6 and above. □

When $[n/3] \leq r < n < [n/2]$, only the dependence in φ survives. For $n > r \geq [n/2]$, the condition $\varepsilon \neq 0$ is *crucial* since $N_{\varepsilon z}$ does not exist for $\varepsilon = 0$. Therefore $t_\varepsilon \neq t$ when $r < n$, and $t_\varepsilon = t$ with $\varepsilon = 0$ when $r = n$.

5.3 Analytic dependences in φ and $t_\varepsilon \in \hat{\mathbb{C}}$

When they coexist, the two analytic dependences yield two different frontier sets of analyticity at ∞, namely $C_1(A, E)$ for φ, and $C_{2\varepsilon}(A, E)$ for t_ε. They share the same core set $\tilde{Z}_2 \subset \text{Lim}_2$.

We observe that for z outside $\hat{Z}_1 \cup \sigma_{KC}$, the matrix $L_z = z^2 I + V_z^{-1}$ is well-defined. The point 2) in Proposition 5.2 can be modified as follows.

Proposition 5.3 *Any $z \notin \hat{Z}_1 \cup \sigma_{KC}$ is an evolution point for r matrices $\Delta(t, z)$ where t and z are related by $t = -z^2 - \frac{1}{v_z}$, $v_z \in \sigma(V_z)$, $t \in \mathbb{C}$.*

The set $\hat{Z}_1 \cup \sigma_{KC}$ indicates the limit of validity of the linear split $K + zC$ to analyze the evolution field.

Definition 5.1 *The set $\hat{Z}_1 \cup \sigma_{KC}$ is the frontier set for the intelligence of the evolution field by means of the linear pencil $K + zC$ when $[n/3] \le r < n$.*

Remark 5.1 We observe that $K + zC$ can be interpreted as a linear *approximation* of $K + zC + z^2E = \Delta(z)$ when $|z|$ is small enough. Then perturbation techniques apply to relate $K + zC$ and $\Delta(z)$. This is not possible for an arbitrary $z \in \mathbb{C}$. In this case, the global algebraic/analytic theory that we have presented provides us with new insights which are beyond the reach of any perturbation technique.

5.4 The particular case $\det C \neq 0$ for $[n/2] \le r < n$

We have observed that for $[n/2] \le r < n$, the two approaches, the direct one and the one based on $K + zC$, coexist. They coexist also with the three approaches deriving from L_1 or L_2.

When $r \ge [n/2]$ and $\det C \neq 0$, the pencil $K + zC = (KC^{-1} + zI)C$ is regular with $\sigma_{KC} = \sigma(-KC^{-1}) \neq \mathbb{C}$. The pencil has exactly n finite eigenvalues(including 0).

We consider, for $z \in \mathbb{C}$, the factorization

$$\begin{pmatrix} K + zC & -U \\ V^H & 0 \end{pmatrix} = \begin{pmatrix} KC^{-1} + zI_n & -U \\ V^H C^{-1} & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & I_r \end{pmatrix}$$

and the determinant $\hat{\psi}(z)$, with degree $\hat{d}_1$. We suppose that $r < n$.

Lemma 5.4 *$\hat{d}_1 \le n - r$ with equality iff $\det V^H C^{-1} U \neq 0$.*

Proof. Direct consequences of the above factorization. See Part I, and replace A by $-KC^{-1}$ and V^H by $V^H C^{-1}$.

When $\hat{d}_1 = 0$, the $\hat{\psi} \equiv 0$ (resp. $\not\equiv 0$) iff $\hat{\psi}(0) = \det \begin{pmatrix} K & -U \\ V^H & 0 \end{pmatrix} = 0$ (resp. $\neq 0$). □

Corollary 5.5 *Under the assumption $\det C \neq 0$, $\hat{Z}_1$ is either $\mathbb{C}$ or it contains at most $n - r$ points. It is discrete with $n - r$ points iff $\det V^H C^{-1} U \neq 0$.*

Proof. Clear. With $n > r \ge [n/2]$, we get the upper bounds:

- for $n = 2k$, $n - r \le k = [n/2]$,
- for $n = 2k + 1$, $n - r \le k + 1 = [n/2] + 1$. □

6 The deviation matrix E has rank $r = n$

When $r < n$, we have found that the 4 variables $z, t, t_\varepsilon = t - \varepsilon$ ($\varepsilon \neq 0$) and $\varphi = z^2 + t$ play a role in extracting meaning from the evolution field by analytic continuation.

What is the case when $r = n$? The 4 variablaes reduce to 3 since $\varepsilon = 0$ and $t = t_\varepsilon$. What are the other simplifications due to the existence of E^{-1}?

6.1 N_z^{-1} and V_z^{-1} exist for all $z \in \mathbb{C}$

By choosing $V = I$ and $U = E$, we have formally $N_z = \Delta^{-1}(z)E$ and $V_z = (K + zC)^{-1}E$.

Lemma 6.1 *The matrices N_z^{-1} and V_z^{-1} are defined for all $z \in \mathbb{C}$. Moreover, $N_z^{-1} = z^2 I + V_z^{-1} = L_z$.*

Proof. Clear by the algebraic definition of N_z and V_z. $V_z^{-1} = E^{-1}(K + zC)$ and, $N_z^{-1} = E^{-1}\Delta(z) = z^2 I + E^{-1}(K + zC) = z^2 I + V_z^{-1}$. □

Corollary 6.2 *The map*

$$z \mapsto t = -\frac{1}{v_z} - z^2 \tag{1}$$

with $\frac{1}{v_z} \in \sigma(V_z^{-1})$, is well-defined for all $z \in \mathbb{C}$.

Proof. By Lemma 6.1. □

When $r = n$, the connection between t and z expressed by (1) always holds: any $z \in \mathbb{C}$ is an evolution point for $\Delta(t, z)$ with $t = -\frac{1}{v_z} - z^2 \in \mathbb{C}$.

Lemma 6.3 *The sets $\hat{Z}_1$, $\hat{Z}_2$ and $\tilde{Z}_2$ are empty.*

Proof. The augmented matrices $\begin{pmatrix} K + zC & -E \\ I & 0 \end{pmatrix}$ and $\begin{pmatrix} \Delta(z) & -E \\ I & 0 \end{pmatrix}$ have rank $\hat{n} = 2n$. Hence $\hat{Z}_1 = \hat{Z}_2 = \emptyset$.

Moreover, no $\beta(s) \to 0$, hence $\text{Lim}_{\ 2} = \tilde{Z}_2 = \emptyset$. □

6.2 Analyticity with respect to t and φ

Analyticity in t (resp. φ) imposes $z \notin \sigma(A)$ (resp. $z \notin \sigma_{KC}$).

Lemma 6.4 $z \in \sigma_{KC}$ *iff* $z^2 \in \sigma(N_z^{-1})$.

Proof. $K + zC = \Delta(z) - z^2 E = E(N_z^{-1} - z^2 I)$. □

Proposition 6.5 *When $z \in \sigma_{KC}$, there is no analyticity in φ. The connection* (1) *yields $\varphi = z^2 + t = 0$. When $\lambda \in \sigma(A)$, there is no analyticity in t. The connection* (1) *yields $t = 0$.*

Proof. The connection (1) expresses that $-t \in \sigma(N_z^{-1})$. For $z \in \sigma_{KC}$, $z^2 \in \sigma(N_z^{-1})$ and (1) imply $\varphi = 0$. For $\lambda \in \sigma(A)$, $0 \in \sigma(N_z^{-1})$ since $N_z^{-1} = E^{-1}(A-\lambda I)E(A-\lambda I)$. The multilplicity of 0 in $\sigma(N_z^{-1})$ is twice that of λ in $\sigma(A)$. □

7 Summary

In Part I, it was found that the behaviour of the spectral field depends on r, with a binary distinction between $r = n$ and $1 \leq r < n$.

In case of the evolution field, the distinction becomes richer. Four cases are to be distinguished when $r < n$: $1 \leq r < [n/4]$, $[n/4] \leq r < [n/3]$, $[n/3] \leq r < [n/2]$ and $[r/2] \leq r < n$. The larger $r < n$, the more structure is present for the evolution.

For $[r/4] \leq r < n$, up to six matrices define six spectral potentials which can be used to relate z with any of the 3 intensity variables t, $t_\varepsilon = t - \varepsilon$, $\varphi = z^2 + t$. They are listed in Table 7.1.

Matrix	Definition	Condition on $r < n$	Connection
$W_{tz} = V^H(K + zC + tE)^{-1}U$	$\overline{\omega}(t, z) \neq 0$	$[n/4] \leq r$	$t \in \mathbb{C} \quad w_{tz} \neq 0$ $z^2 w_{tz} = -1$
$V_z = V^H(K + zC)^{-1}U = W_{0z}$	$z \notin \sigma_{KC}$	$[n/3] \leq r$	$\varphi = z^2 + t \in \hat{\mathbb{C}}$ $\varphi v_z = -1$
$L_z = z^2 I + V_z^{-1}$	$z \notin \hat{Z}_1 \cup \sigma_{KC}$	$[n/3] \leq r$	$t \in \mathbb{C}$ $-t = z^2 + \frac{1}{v_z}$
$X_{tz} = (K + tE)(C + zE)^{-1}$	$z \notin \sigma_{CE}$	$[n/3] \leq r$	$t \in \mathbb{C},\ -z = x_{tz}$
$Y_{tz} = (C + zE)(K + tE)^{-1}$	$t \notin \sigma_{KE}$	$[n/2] \leq r$	$z \notin \sigma_{CE}$ $z y_{tz} = -1$
$N_{\varepsilon z} = V^H \Delta^{-1}(\varepsilon, z)U,\ \varepsilon \neq 0$	$z \notin Z_2(\varepsilon)$	$[n/2] \leq r$	$t_\varepsilon = t - \varepsilon \in \hat{\mathbb{C}}$ $t_\varepsilon n_{t_\varepsilon z} = -1$

Table 7.1: $[n/4] \leq r < n$

When $r = n$, the number of intensity variables reduces to 2, namely t and $\varphi = z^2 + t$. Rows 3 and 6 in Table 7.1 are modified, as shown by Table 7.2.

Matrix	Definition	$r = n$	Connection
$L_z = z^2 I + V_z^{-1}$ $V_z^{-1} = E^{-1}(K + zC)$	$z \in \mathbb{C}$	$r = n$	$t \in \mathbb{C}$ $-t = z^2 + \frac{1}{v_z}$
$N_z = \Delta^{-1}(z)E$	$z \in re(A)$	$r = n$	$t \in \mathbb{C}$ $t n_z = -1$

Table 7.2: Modification with $r = n$, $\varepsilon = 0$.

The matrices $N_{\varepsilon z}$, N_z, V_z and L_z yield fully explicit connections. The connections provided by W_{tz}, X_{tz}, and Y_{tz} are semi-implicit: z^{-2}, z or z^{-1} are related to the corresponding spectral potentials.

We turn, in the next Section, to fully implicit connections.

8 Fully implicit connections

These connections are obtained by yet other factorizations for $\Delta(t, z)$.

8.1 $z^2 \notin \sigma_{KE}$ for $[n/2] \leq r$

For $z^2 \notin \sigma_{KE} \neq \mathbb{C}$ we write

$$\Delta(t, z) = [I + (tE + zC)(K + z^2E)^{-1}](K + z^2E),$$

and set $\phi_{tz} = (tE + zC)(K + z^2E)^{-1}$.

We also define $q(t, z) = \det(tE + zC)$, which is a polynomial homogeneous in t and z. It is not identically 0 fo $r \geq [n/3]$.

We have the familiar result:

Any z such that $z^2 \notin \sigma_{KC}$ is an evolution point for $\Delta(t, z)$ provided that -1 is an eigenvalues of ϕ_{tz}.

8.2 $q(t, z) \neq 0$ for $[n/3] \leq r$

When $q(t, z) \neq 0$ we can consider as well

$$\phi_{tz}^{-1} = (K + z^2E)(tE + zC)^{-1},$$

and derive a statement analogous to the one above with $-1 \in \sigma(\phi_{tz}^{-1})$.

References

[1] M. Ahmadnasab (2007) *Homotopic Deviation theory: a qualitative study.*, Ph. D. thesis, UT1 and Cerfacs, October 2007. Cerfacs TH/PA/07/120.

[2] M. Ahmadnasab, F. Chaitin-Chatelin (2007) *Parameter analysis of the structure of matrix pencils by Homotopic Deviation theory* in **Proceedings ICIAM07**, Zurich, Wiley, to appear

Also Cerfacs TR/PA/07/108.

[3] M. Ahmadnasab, F. Chaitin-Chatelin, N. Megrez (2005) Homotopic Deviation in the light of Algebra, Cerfacs TR/PA/05/05

[4] F. Chatelin (1993). **Eigenvalues of matrices**. Wiley, Chichester, 1993. Enlarged Translation of the French Publication with Masson.

[5] F. Chaitin- Chatelin (2002) About singularities in Inexact Computing. Cerfacs TR/PA/02/106

[6] F. Chaitin- Chatelin (2003) Computing beyond analyticity: matrix algorithms in inexact and uncertain computing. Cerfacs TR/PA/03/110

[7] F. Chaitin- Chatelin (2005) *The dynamics of matrix coupling with an application to Krylov methods* in **Proceedings NAA 2004 Rousse** (Li Z., et al. (eds), Lect. Notes Comp. Sc. **3401**, pp. 14-24, Springer-Verlag, Berlin

Also Cerfacs TR/PA/04/29

[8] F. Chaitin- Chatelin (2005) On Lidskii's algorithm to quantify the first order terms in the asymptotics of a defective eigenvalue. Part II. Cerfacs TR/PA/05/04

[9] F. Chaitin- Chatelin (2007) The dynamics of spectral analysis by Homotopic Deviation. **Part I** The spectral field. Cerfacs TR/PA/07/118

[10] F. Chaitin-Chatelin and M.B. van Gijzen (2005). *Analysis of parametrized quadratic eigenvalue problems in Computational Acoustics with Homotopic Deviation theory.* NLAAp **13**, 487-512.

Also Technical Report TR/PA/04/05, CERFACS, Toulouse, France, 2004.

[11] F. Gantmacher (1960) **The theory of matrices**, Vol.I and II, Chelsea, New York

[12] F. Tisseur, K. Meerbergen (2001) *The quadratic eigenvalue problem*, SIAM Rev. **43**, 235-286.

All Cerfacs Reports are available from:
`http://www.cerfacs.fr/algor/reports/index.html`