An order reduction method for computing the finite eigenvalues of regular matrix pencils

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AN ORDER REDUCTION METHOD FOR COMPUTING THE FINITE EIGENVALUES OF REGULAR MATRIX PENCILS

MORAD AHMADNASAB*

Abstract. Let $A, B \in \mathbb{C}^{n\times n}$ be two given matrices. Under the assumption that at least one of the matrices A or B is nonsingular, we introduce an order reduction method for computing the generalized eigenvalues of the regular matrix pencil $A - zB$. This method is based on the SVD of the matrix B (resp. A) if the matrix A (resp. B) is nonsingular. When both A and B are nonsingular, the method is based on the SVD of A or B exclusively. The performance of this algorithm is studied and the accuracy of the computed eigenvalues is demonstrated by comparing them with those computed by the QZ method in Matlab.

Key words. Homotopic deviation, regular matrix pencils, generalized eigenvalue problem, singular value decomposition

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1. Introduction. The standard matrix eigenvalue problem for the matrix $A \in \mathbb{C}^{n\times n}$ has the form

$$Ax = \lambda x,$$

where λ is an eigenvalue of A and $x \in \mathbb{C}^n$, $x \neq 0$ is an associated eigenvector. Let A and B be two complex $n \times n$ matrices. The set of all matrices of the form $A - zB$ with $z \in \mathbb{C}$ is said to be a *pencil*. A *generalized eigenvalue problem*, GEVP, consists in finding the eigenvalues of the pencil $A - zB$ defined by

$$sp(A,B) = \{\lambda \in \mathbb{C} \; : \; \det(A - \lambda B) = 0\}, \tag{1.1}$$

where $\det(A - zB)$ is a polynomial of degree n iff the matrix B is *nonsingular*. This means that GEVP has n finite eigenvalues iff the rank of the matrix B satisfies $\text{rank}B = n$. The finite eigenvalues of GEVP are the roots of the scalar polynomial $\det(A - zB)$ defined in (1.1).

The *generalized eigenproblem*, GEP, consists in finding $\lambda \in sp(A,B)$ and $x \in \mathbb{C}^n$, $x \neq 0$ an associated eigenvector such that

$$Ax = \lambda Bx, \qquad x \neq 0. \tag{1.2}$$

The eigenvectors generate a basis $X \in \mathbb{C}^{n\times n}$ iff they are independent. When B is rank deficient, then $sp(A,B)$ may be finite, empty, or infinite [7].

One of the most powerful methods for dense GEP which is an analogue of the QR algorithm is called the QZ algorithm [20]. An important feature of the QZ algorithm is that it functions perfectly well in the presence of infinite eigenvalues [15, 20, 21]. This method is suitable for small to moderate-sized problems because of the requirements of $O(n^3)$ floating point operations and $O(n^2)$ memory locations.

A common approach for a large sparse generalized eigenvalue problem is to reduce the problem (1.2) to a standard eigenvalue problem and then apply an appropriate iterative method for this problem. This technique is called *reduction to standard form.*

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When B is nonsingular, in the approach of *invert* B [5], the following equivalences for the problem (1.2) are proposed:

$$(B^{-1}A)x = \lambda x \quad \text{and} \quad \tilde{y}^H(B^{-1}A) = \lambda \tilde{y}^H, \tag{1.3}$$

where $\tilde{y} = B^H y$. Alternatively, the problem (1.2) is also equivalent to

$$(AB^{-1})\tilde{x} = \lambda \tilde{x} \quad \text{and} \quad y^H(AB^{-1}) = \lambda y^H, \tag{1.4}$$

where $\tilde{x} = Bx$.

The error introduced by this transformation to standard form can be proportional to $\|A\|_2\|B^{-1}\|_2$. If B is ill-conditioned, then the approach is potentially suspect.

When the matrix B is Hermitian positive definite and when the Cholesky decomposition of the matrix B can be computed efficiently a priori, one may consider the approach split-Invert B for the transformation [5, 15, 18]. When both of A and B are singular or B is ill-conditioned, the most common approach is the shift-invert spectral transformation [5, 19].

In this paper, we study the computational properties of an *order reduction method* for computing the finite eigenvalues of the regular matrix pencil $A - zB$. The basic idea of this method was introduced , for the first time, as a result of parameter analysis of the structure of regular matrix pencil $P_z = (A - \xi I) - zB$, for $\xi \in \mathbb{C}$ by means of Homotopic Deviation theory [1, 2, 9]. The method is based on the SVD of the matrix B when the matrix A is nonsingular. The notion of evolving (resp. invariant) eigenvalue ξ in Homotopic Deviation plays an important role in matrix pencil theory [3]. A new interpretation of the index of infinite eigenvalue of matrix pencils is given in [3].

Here we also generalize the idea of the method to the case that B is nonsingular and A can be singular. When both A and B are nonsingular, the method is based on the SVD of A or B exclusively. In this method, we reduce the problem (1.2) to a standard eigenvalue problem of order $r_B = \text{rank}B$ (resp. $r_A = \text{rank}A$) when A (resp. B) is nonsingular and we use the SVD of the matrix B (resp. A).

In section 2, we present a survey of Homotopic Deviation theory. Then in section 3, we present the application of Homotopic Deviation to find the structure of matrix pencils and to introduce the order reduction method for computing the finite eigenvalues of regular matrix pencils. In section 4, a direct transformation approach is used to extract a complete version of this order reduction method. Then the idea is generalized to the case when the matrix B is nonsingular and the matrix A can be singular. The performances of the two versions of this method (which correspond to A nonsingular or B nonsingular) are investigated in section 5. In section 6, accuracy of the computed eigenvalues and execution time of the method are demonstrated by comparing them with those obtained by the QZ method in Matlab.

2. A brief survey of Homotopic Deviation. Given the matrices A and B in $\mathbb{C}^{n\times n}$, the family $A(z) = A - zB$, for $z \in \mathbb{C}$ represents the linear coupling between A and B by the complex parameter z. We denote the *spectrum* of A by $\sigma(A)$ and the *resolvent* set of A by $re(A) = \mathbb{C}\backslash\sigma(A)$. Homotopic Deviation theory [1, 4, 8] studies the singularities in $\mathbb{C}$ of the linear coupling $A(z) = A - zB$, which depend on the parameter z which varies in the completed complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The parameter z defines the intensity of the linear coupling $A - zB$. Let $\lambda_i(z)$, for $i = 1, \ldots, n$ denote the eigenvalues of $A(z)$. When B is full rank, then $|\lambda_i(z)| \to \infty$, for $i = 1, \ldots, n$, as $|z| \to \infty$. But when rank $B = r < n$, it is possible that some eigenvalues of $A(z)$ stay at finite distance, rather than escaping to ∞. A physical exa mple in acoustics is described in [12]. An up-to-date version of Homotopic Deviation is presented in Chapter 7 of [10].

2.1. The communication matrix. Let $B = UV^H$ be the SVD of B, where $U, V \in \mathbb{C}^{n\times r}$ have rank $r \le n$ (an explanation of how to find the matrices U and V from the SVD of the matrix B is given in Section 4). The singularities of $R(z,\xi) = (A(z)-\xi I)^{-1}$ are the eigenvalues $\lambda(z)$ of $A - zB$. The point ξ is the observation point for Homotopic Deviation. When $\xi \in re(A)$, they are easily related to A, U, V by the communication matrix $M_\xi = V^H(A-\xi I)^{-1}U \in \mathbb{C}^{r\times r}$, since we have the following fundamental relation for $\xi \in re(A)$ given in [1, 8, 12]

$$\xi \in \sigma(A - zB) \iff z = 1/\mu_\xi \in \hat{\mathbb{C}}, \text{ where } \mu_\xi \in \sigma(M_\xi). \tag{2.1}$$

2.2. The frontier set. When M_ξ is invertible for $\xi \in re(A)$, then ξ is an eigenvalue of r matrices $A(z_i)$, for $z_i = 1/\mu_{i\xi}$, $i = 1, \ldots, r$. When ξ in $re(A)$ is such that rank$M_\xi < r$, there are *less* than r such matrices. This is possible when $r < n$.

Definition 2.1 ([1, 8]). *The frontier set is the subset of $re(A)$ defined by*

$$F(A,B) = \{\xi \in re(A);\ \text{rank}M_\xi < r\}, \text{ for } r < n.$$

When $\xi \in re(A)\backslash F(A,B)$, the resolvent matrix $R(z,\xi)$ is analytic in z around 0 ($|z| < 1/\rho(M_\xi)$) and around ∞ ($|z| > \rho(M_\xi^{-1})$). When $\xi \in F(A,B)$, the analyticity around ∞ *disappears*.

It is possible that for $\xi \in F(A,B)$ then $\rho(M_\xi) = 0$, that is M_ξ is nilpotent ($M_\xi^\delta = 0$ with $M_\xi^{\delta-1} \neq 0$, $1 \le \delta \le r$). Such a particular frontier point is called *critical* and $R(z,\xi)$ is a polynomial in z of degree less than or equal to r. The critical points form the critical set $F_c(A,B) \subseteq F(A,B) \subseteq re(A)$.

3. The regular matrix pencils. We recall that a matrix pencil $A - zB$ of dimension $n \times n$ is called *regular* if the determinant $\det(A - zB)$ does not vanish identically. Otherwise, the pencil is called *singular*. A criterion for strict equivalence of regular pencils of matrices and a canonical form for such pencils were established by Weierstrass around 1867 on the basis of his theory of elementary divisors. The analogous problems for singular pencils were solved later, in 1890, by the investigations of Kronecker [14].

3.1. The family of regular matrix pencils. The pencil $P_\xi = (A - \xi I) - zB$ for $\xi \in re(A)$ is regular iff $\pi(z,\xi) = \det(A - \xi I - zB) \not\equiv 0$. This is obviously true for $\xi \in re(A)$ since $\pi(0,\xi) \neq 0$. A detailed explanation about how to divide the family of the pencils P_λ for $\lambda \in \sigma(A)$ into regular or singular pencil is given in [3]. When $\xi \in re(A)$, the structure of P_ξ depends on ξ and r (resp. depends on r only) for $\xi \in F(A,B)$ (resp. $\xi \in re(A)\backslash F(A,B)$) [1, 2, 9]. The generic case $\xi \in re(A)\backslash F(A,B)$ corresponds to $g = n - r$ infinite eigenvalues and exactly r finite eigenvalues equal to $1/\mu_{i\xi}$, $i = 1, \ldots, r$. The following proposition summarizes the Propositions 4.3.1 and 4.3.2 of [1].

Proposition 3.1. *Let $P_\xi = (A - \xi I) - zB$ for $\xi \in re(A)$ be a regular matrix pencil for $A, B \in \mathbb{C}^{n\times n}$. The finite eigenvalues of the pencil P_ξ are* *a) the r complex numbers $z = \frac{1}{\mu_{i\xi}}$, for $\mu_{i\xi} \in \sigma(M_\xi)$, $i = 1, \dots, r$ when $\xi \in re(A) \backslash F(A, B)$.*

b) the $r - a_\xi$ complex numbers $z = \frac{1}{\mu_{i\xi}}$, for $0 \neq \mu_{i\xi} \in \sigma(M_\xi)$, when $\xi \in F(A, B)$ and a_ξ is the algebraic multiplicity of $0 \in \sigma(M_\xi)$.

Proof. *a*) When $\xi \in re(A) \backslash F(A, B)$, $z \in sp((A - \xi I) - zB)$ iff $\xi \in \sigma(A - zB)$. Now (2.1) yields $z \in sp((A - \xi I) - zB) \iff z = 1/\mu_{i\xi},\ \mu_{i\xi} \in \sigma(M_\xi),\ i = 1, \dots, r$.

b) When $\xi \in F(A, B)$, then for a_ξ elements of $\{j_1, \dots, j_{a_\xi}\} \subseteq \{1, \dots, r\}$, we have $0 = \mu_{j_k \xi} \in \sigma(M_\xi)$, $k = 1, \dots, a_\xi$. This is equivalent to the existence of a_ξ infinite eigenvalues $|z_j| = 1/|\mu_{j\xi}| = \infty$, $j \in \{j_1, \dots, j_{a_\xi}\}$. Thus the finite eigenvalues of the pencil P_ξ are $z = \frac{1}{\mu_{i\xi}}$, for $\mu_{i\xi} \in \sigma(M_\xi)$, $i \in \{1, \dots, r\} \backslash \{j_1, \dots, j_{a_\xi}\}$. □

An application of (2.1) for the case that $0 = \xi \notin \sigma(A)$ states

$$0 \in \sigma(A - zB) \iff z = 1/\mu_0 \in \hat{\mathbb{C}}, \text{ where } \mu_0 \in \sigma(M_0), \tag{3.1}$$

for $M_0 = V^H A^{-1} U$. The following corollary is obtained directly from the Proposition 3.1 for the case that A is nonsingular and $\xi = 0$.

COROLLARY 3.2. *Let $P_0 = A - zB$ be the regular matrix pencil P_ξ in the Proposition 3.1 for $\xi = 0$, the nonsingular matrix A and the matrix B of rank $r \leq n$. Then for the $r \times r$ matrix M_0 defined in (3.1),*

a) the finite eigenvalues of the pencil $A - zB$ are $z = 1/\mu_{i0}$, $\mu_{i0} \in \sigma(M_0)$, $i = 1, \dots, r$, when $0 \notin \sigma(M_0)$.

b) the finite eigenvalues of the pencil $A - zB$ are $z = 1/\mu_{i0}$, $\mu_{i0} \in \sigma(M_0)$, for $i \in \{1, \dots, r\} \backslash \{j_1, \dots, j_{a_0}\}$ when the algebraic multiplicity of $0 \in \sigma(M_0)$ satisfies $1 \leq a_0 \leq r$.

The Corollary 3.2 suggests an order reduction method for finding the finite eigenvalues of the regular matrix pencil $A - zB$ where the matrix A is nonsingular and the matrix B is singular (or nonsingular but we use the SVD of B). One should remark that the index of the regular matrix pencil P_ξ, that is the size of the largest Jordan block corresponding to the infinite eigenvalue, is larger than 1 when ξ is a frontier point [3]. In this case, the problem is very unstable [6].

In this paper, we assume that $\xi = 0$ is not a frontier point, therefore the index of the pencil $A - zB$ is 0 (resp. 1) when the rank of the matrix B is n (resp. less than n).

In section 4, we first obtain this method (for the nonsingular matrix A and the matrix B of rank $r \leq n$) using the SVD of matrix B and a direct transformation. Then we shall adapt the idea for the case where B is nonsingular and A is singular (or nonsingular but we use the SVD of A).

4. Two versions of the order reduction method for solving generalized eigenvalue problems. In this section, under the assumption that at least one of A or B is invertible, we shall obtain two versions of the order reduction method. One should note that the two versions are applicable when both A and B are invertible.

4.1. When A is nonsingular. When the matrix $A \in \mathbb{C}^{n \times n}$ is nonsingular and $r_B = \text{rank}B \leq n = \text{order}A$, then the process of transforming the problem (1.2) to the associated $r_B \times r_B$ standard eigenvalue problem is as follows: use SVD for the matrix B to get $U_1^H B V_1 = \text{diag}(\sigma_i) = \Sigma$ where U_1 and V_1 of order n are unitary matrices and Σ is a diagonal matrix of order n. The nonzero elements of Σ, which are on the diagonal, are $\sigma_1 \geq \dots \geq \sigma_{r_B} > 0$. This is equivalent to $B = U_1 \Sigma V_1^H$. Now let us denote the first r_B columns of the matrix $U_1 \Sigma$ by U_B. This means that the computation of $U_1 \Sigma$

is implemented just for the r_B columns of the matrix Σ such that for u_j, $j = 1, \ldots, r_B$ as the jth column of the matrix U_1, one has $U_B = [\sigma_1 u_1 \quad \sigma_2 u_2 \quad \cdots \quad \sigma_{r_B} u_{r_B}]$. The first r_B columns of the matrix V_1 is denoted by V_B. Then we get

$$B = U_B V_B^H. \tag{4.1}$$

By multiplying both sides of (1.2) from the left with A^{-1}, we get

$$I_n x = \lambda A^{-1} U_B V_B^H x. \tag{4.2}$$

By multiplying both sides of (4.2) from the left with V_B^H, we get

$$V_B^H x = \lambda V_B^H A^{-1} U_B V_B^H x. \tag{4.3}$$

Now, for $V_B^H x = \hat{x} \in \mathbb{C}^{r_B \times 1}$, we have

$$M\hat{x} = \frac{1}{\lambda}\hat{x}, \tag{4.4}$$

where $M = V_B^H A^{-1} U_B \in \mathbb{C}^{r_B \times r_B}$, the same matrix as what was obtained in (3.1).

For the reduced standard eigenvalue problem (4.4), it is not necessary to evaluate the product $A^{-1}U_B$. One only needs to

(a) solve $AC = U_B$ for C using (say) Gaussian elimination with pivoting (when the order of the problem is not large) or using an iterative linear system solver (when the order of the problem is large and the matrix A is sparse).

(b) use the QR algorithm or an appropriate iterative algorithm (depends on the value of r_B) to compute the eigenvalues of $V_B^H C$.

Then reciprocals of the nonzero eigenvalues computed in (b) are the finite eigenvalues of the matrix pencil $A - zB$.

In this paper, we use Gaussian elimination method with pivoting for (a) and the QR algorithm for (b) and leave the application of iterative algorithms for future work. One reason is that, this method finds its best performance when the order of matrix M (i.e. the rank of matrix B) defined in (4.4) satisfies $r_B \leq n$ or more preferable satisfies $r_B \ll n$, see Sections 5 and 6. Here we consider the problems with small or moderate r_B.

If we want also to compute the eigenvectors of the pencil $A - zB$, we do not use the above order reduction which results in (4.1) for the matrix B. Instead, we continue all the necessary steps from (4.1) by $B = U_n V_n^H$, for $U_n = U_1\Sigma$ and $V_n = V_1$ to get the $n \times n$ matrix $M = V_n^H A^{-1} U_n$. Then the reciprocals of the nonzero eigenvalues of the matrix M are the finite eigenvalues of the matrix pencil $A - zB$. In this case, if the matrix M is nonsingular, then the n eigenvectors associated with its n eigenvalues are the associated eigenvectors of the pencil $A - zB$.

In this paper, we compute the eigenvalues of the pencil $A - zB$ and then we assess their accuracy by comparing them with the results given by the QZ method.

The entire algorithm, may be summed up as follows:

ALGORITHM 4.1. *When A is nonsingular*

1. *Use the SVD of matrix B to write it in the form $B = U_B V_B^H$ as explained before* (4.1)*,*
2. *Solve $AC = U_B$ for C using Gaussian elimination with pivoting,*
3. *Use the QR algorithm to compute the eigenvalues μ_i, $i = 1, \ldots, r_B$ of $V_B^H C = M$,*
4. *$sp(A, B) = \{\lambda_i = 1/\mu_i \ ; \ 0 \neq \mu_i \in \sigma(M)\}$.*

4.2. When B is nonsingular. When the matrix $B \in \mathbb{C}^{n\times n}$ is nonsingular, then the problem (1.2) has n finite eigenvalues. For computing the eigenvalues of the problem (1.2), we use the same idea as those in section 4.1 for transforming the problem (1.2) to the associated standard eigenvalue problem of order $r_A = \text{rank}A$. In this case, we obtain a standard eigenvalue problem whose eigenvalues together with a set of $n - r_A$ zeros construct the n finite eigenvalues of the problem (1.2).

To this end, the same process as what we explained for (4.1) should be used to write the matrix A in the form $A = U_A V_A^H$, for the two $n \times r_A$ matrices U_A and V_A of order r_A: we use SVD for the matrix A to get $U_1 A V_1 = \text{diag}(\sigma_i) = \Sigma$ where U_1 and V_1 of order n are unitary matrices such that $A = U_1 \Sigma V_1^H$ for the diagonal matrix Σ of order n. The nonzero elements of Σ are $\sigma_1 \geq \cdots \geq \sigma_{r_A} > 0$. Now we denote the first r_A columns of the matrix $U_1\Sigma$ by U_A. In other words, the computation of $U_1\Sigma$ is implemented just for the r_A columns of the matrix Σ such that for u_j, $j = 1, \ldots, r_A$ as the jth column of the matrix U_1, we have $U_A = [\sigma_1 u_1 \;\; \sigma_2 u_2 \;\; \cdots \;\; \sigma_{r_A} u_{r_A}]$. The first r_A columns of the matrix V_1 is denoted by V_A. Then we get

$$A = U_A V_A^H. \tag{4.5}$$

Premultiplication of both sides of (1.2) by B^{-1}, gives

$$B^{-1} U_A V_A^H x = \lambda x. \tag{4.6}$$

Premultiplication of both sides of (4.6) by V_A^H, yields

$$V_A^H B^{-1} U_A (V_A^H x) = \lambda (V_A^H x). \tag{4.7}$$

Via $V_A^H x = \hat{x} \in \mathbb{C}^{r_A \times 1}$, we have

$$N\hat{x} = \lambda \hat{x}, \tag{4.8}$$

for the $r_A \times r_A$ matrix $N = V_A^H B^{-1} U_A$.

Again for the reduced standard eigenvalue problem (4.8), it is not necessary to evaluate the product $B^{-1}U_A$. As we are interested in some small to moderate size problems, one only

(a) solve $BC = U_A$ for C using (say) Gaussian elimination with pivoting.

(b) use the QR algorithm to compute the eigenvalues of $V_A^H C$.

Then the eigenvalues of the pencil $A - zB$ (which all are finite) are the union of the set of r_A eignevalues computed in (b) and a set of zeros with the cardinality $n - r_A$. Note that in this case, there is no need to use the reciprocals of the eigenvalues of the matrix N to get finite eigenvalues of the pencil $A - zB$.

If we want also to compute the eigenvectors of the pencil $A - zB$, we do not use the above order reduction which results in (4.5) for the matrix A. Instead, we continue all the necessary steps from (4.5) by $A = U_n V_n^H$, for $U_n = U_1\Sigma$ and $V_n = V_1$ to get the $n \times n$ matrix $N = V_n^H B^{-1} U_n$. In this case, the eigenvalues of the matrix N and their corresponding eigenvectors are the finite eigenvalues and the corresponding eigenvectors of the matrix pencil $A - zB$ respectively.

The entire algorithm is the following:

Algorithm 4.2. *When B is nonsingular*

1. *Use the SVD of the matrix A to write it in the form $A = U_A V_A^H$ as explained before* (4.5)*,*

2. Solve $BC = U_A$ for C using Gaussian elimination with pivoting,
3. Use the QR algorithm to compute the eigenvalues λ_i, $i = 1, \ldots, r_A$ of $V_A^H C = N$. The set of (finite) eigenvalues of the problem (1.2) is the union of the eigenvalues of the matrix N and a set of $n - r_A$ zeros.

5. Performance of the algorithms. We have seen that the process of computing finite eigenvalues of the pencil $A - zB$ by Algorithm 4.1 has four main steps. The necessary operations for each step of this algorithm are the following:

Step 1 involves
- 1-1. $\sim \frac{8}{3}n^3 + nr_B$ flops,
- 1-2. $O(n)$ comparisons for finding r_B,
- 1-3. and memory locations for A,B,U_B,V_B.

Step 2 requires
- 2-1. $\frac{2}{3}n^3$ flops, plus $r_B(n^2 + n)$ flops for solving the multiple right hand side problem $AC = U_B$,
- 2-2. $O(n^3)$ comparisons for computing the complete pivoting factorization of A
- 2-3. and storage for C of order $n \times r_B$.

Step 3 including the QR algorithm, requires
- 3-1. $10r_B^3$ flops, when only the eigenvalues of $V_B^H C$ are desired,
- 3-2. and $O(r_B^2)$ memory locations.

Step 4 needs
- 4-1. at most r_B flops,
- 4-2. r_B comparisons,
- 4-3. and at most r_B memory location.

All together, the number of flops required by Algorithm 4.1 is asymptotic to

$$\frac{10}{3}n^3 + r_B(n^2 + 2n) + 10r_B^3 + r_B.$$

The number of comparisons required by Algorithm 4.1 is at most

$$O(n^3) + r_B.$$

It also needs

$$2n^2 + 3(nr_B) + O(r_B^2) + r_B$$

memory locations.

The same analysis on Algorithm 4.2 shows that:

The total number of flops required by Algorithm 4.2 is asymptotic to

$$\frac{10}{3}n^3 + r_A(n^2 + 2n) + 10r_A^3.$$

The number of comparisons required by Algorithm 4.2 is at most

$$O(n^3).$$

The number of memory locations is

$$2n^2 + 3(nr_A) + O(r_A^2) + r_A.$$

The necessary number of operations for Algorithms 4.1 and 4.2 shows that these algorithms have good relative performances with respect to the QZ algorithm which needs $30n^3$ flops [15]. This fact is supported by the numerical examples in section 6 with $r_B \leq n$ or $r_A \leq n$.

When the matrices A and B are nonsingular, both of Algorithms 4.1 and 4.2 are applicable but Algorithm 4.2 requires a smaller number of operations than Algorithm 4.1: Algorithm 4.2 has 3 steps so it saves n comparisons and at most n flops.

6. Computational results. In this section, we present some computational results to give indications of the performance and accuracy of the order reduction method. The entire processes described in Algorithms 4.1 and 4.2 are implemented in Matlab. The functions ORSGEVP1 and ORSGEVP2 implement Algorithm 4.1 and Algorithm 4.2 in Matlab respectively. From now on, we call them ORSGEVP1 method and ORSGEVP2 method. As a testbed, we used a Pentium 4, 2.8 GHz processor with 512MB RAM.

Let the vector $\hat{s}$ includes the finite eigenvalues of the pencil $A - zB$ computed by ORSGEVP1 or ORSGEVP2 method. We denote by s the vector of finite eigenvalues of the pencil $A - zB$ computed by the `eig` function in Matlab with the option '`qz`'. For simplicity, from now on, we say finite eigenvalues computed by QZ method. We sort the eigenvalues in $\hat{s}$ and s with respect to ascending order of their magnitudes, that is $|\hat{s}(j)| \leq |\hat{s}(j+1)|$, $|s(j)| \leq |s(j+1)|$, for $j = 1, \ldots, \hat{r}_k - 1$, where $\hat{r}_k$ is the number of finite eigenvalues computed by Algorithm 4.1 or Algorithm 4.2. Then the error in the set of finite eigenvalues computed by ORSGEVP1 method or ORSGEVP2 is measured by the following formula

$$D_1 = \|\hat{s} - s\|_2 / \hat{r}_k. \tag{6.1}$$

When the eigenvalues of the pencil $A - zB$ are the complex numbers, and we sort them with respect to ascending order of their magnitudes, there is no guarantee to get a small value for D_1 defined in (6.1) because the positions of the complex conjugate eigenvalues in the sorted vectors $\hat{s}$ and s are not determined by this way of sorting. Therefore, if the application of (6.1) does not obtain reasonably small values for D_1, we directly compare the eigenvalues computed by the ORSGEVP1 (or ORSGEVP2) method with those computed by the QZ method.

Let us recall a definition of normal pencils, given in [13]. An $n \times n$ complex regular pencil $A - zB$ is said to be normal if it has orthogonal right and left eigenvectors, i.e., if it has a decomposition of the form

$$U_l^H (A - zB) U_r = \Lambda_A - z\Lambda_B,$$

where U_l, U_r are unitary and Λ_A, Λ_B are diagonal.

The homogeneous ratio

$$He(A) = \nu(A) / \|A^2\|, \tag{6.2}$$

called Henrici number associated with the matrix A [11] is used to assess *depart from normality* for the matrix A where $\nu(A) = \|AA^H - A^H A\|$. When the matrix B is

Table 6.1
Accuracy achieved by the ORSGEVP1 method for the pencil with the Lehmer matrix A and the diagonal matrices B_k, $k = 1, \ldots, 6$ of order $n = 512$.

k	rankB_k	D_1
1	2	1.6244e−16
2	102	1.1287e−15
3	202	8.5983e−16
4	302	8.0891e−16
5	402	1.3645e−15
6	502	7.8002e−16

invertible, then the pencil $A - zB$ is normal iff both AB^{-1} and $B^{-1}A$ are normal [13]. Therefore the Henrici number (6.2) can be used for both AB^{-1} and $B^{-1}A$ to study the role of departure from normality of the pencil $A - zB$ in the accuracy of the computed eignevalues by our method.

When B is not invertible, then the pencil $A - zB$ can be transformed to an appropriate regular pencil $\widehat{A} - z\widehat{B}$ with invertible matrix $\widehat{B}$ [13]. Then it follows [17] that

$$\widehat{A} - z\widehat{B} \text{ is normal} \Longleftrightarrow A - zB \text{ is normal.}$$

This means that for both cases that B is nonsingular or singular, one can measure depart from normality of the pencil $A - zB$. Nevertheless, we only use (6.2) with the matrix 2-norm $\|\cdot\|_2$ in section 6.2 to show the effect of depart from normality on the accuracy of the eigenvalues computed by the ORSGEVP2 method which is proposed for the case that the matrix B is regular.

In what follows, e_k is the kth canonical vector.

6.1. The matrix A is invertible. In this section, we present some pencils $A - zB$ defined by a fixed invertible matrix A and a family of the singular matrices $B = B_k$ of rank $r_k \leq n = \text{order}A = \text{order}B_k$. The structure of $B = B_k$ is fixed. We use ORSGEVP1 method for computing the finite eigenvalues of these problems.

6.1.1. A is the Lehmer matrix and B is diagonal. Let A be the Lehmer matrix of order $n = 512$. We define some different $n \times n$ diagonal matrices B_k, $k = 1, \ldots, 6$ whose ranks are $r_k = 2 + (k-1)100$ respectively. This means that for instance, the rank of B_1 is $r_1 = 2$ and the rank of B_6 is $r_6 = 502$. The entries of the matrix $B_k \in \mathbb{C}^{n \times n}$, $k = 1, \ldots, 6$ are zeros but some of its entries on the diagonal which are nonzero. The nonzero columns of B_k, $k = 1, \ldots, 6$ are assigned as follows: $B_k(:, j : j) = j\ e_j$, for $j = 1, \ldots, r_k$.

The matrix A and all the matrices B_k, $k = 1, \ldots, 6$ are normal. In Table 6.1, one can see the accuracy achieved by ORSGEVP1 method using the criterion explained for D_1 in (6.1). Figure 6.1 plots the execution times of the two methods QZ and ORSGEVP1. This figure shows that the time taken by ORSGEVP1 method is much less than those in QZ method. Figure 6.2 plots the running times of the two most consuming steps of ORSGEVP1 method for this example which are the steps 1 and 3.

6.1.2. A is the Lehmer matrix and B is symmetric tridiagonal. Let A be the Lehmer matrix of order $n = 512$. We define some different $n \times n$ symmetric tridiagonal matrices B_k, $k = 1, \ldots, 6$ whose ranks are obtained from $r_k = 2+(k-1)100$

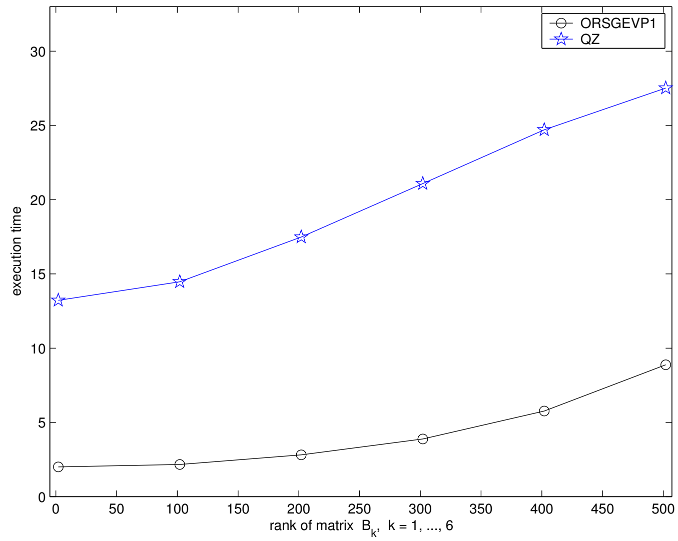

FIG. 6.1. *Comparison of the execution time of the QZ method and the ORSGEVP1 method.*

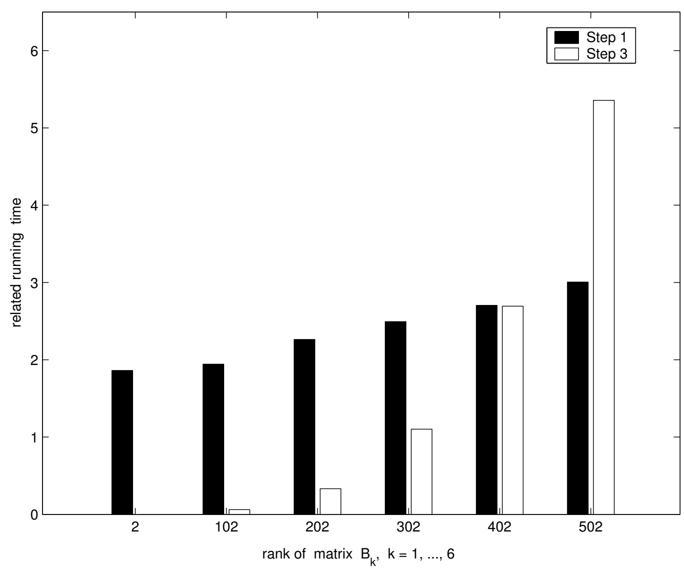

FIG. 6.2. *Running times of the steps 1 and 3 of the ORSGEVP1 method.*

respectively. This means that for instance, the rank of B_1 is $r_1 = 2$ and the rank of B_6 is $r_6 = 502$. The nonzero entries of B_k, for $k = 1, \ldots, 6$ are assigned as follows: $B_k(j,j) = 1$, $B_k(j+1,j) = j$, and $B_k(j,j+1) = j$, for $j = 1, \ldots, r_k - 1$. The matrix A and the matrices B_k, $k = 1, \ldots, 6$ are normal. Table 6.2 displays the accuracy achieved by the ORSGEVP1 method. Figure 6.3 plots the execution times of the two methods QZ and ORSGEVP1. This figure shows that the ORSGEVP1 method is faster than the QZ method when the rank of the matrix $B_k = B$ is less than or equal to 465. Figure 6.4 plots the running times of the steps 1 and 3 of the ORSGEVP1 method which are the two most consuming steps for this example.

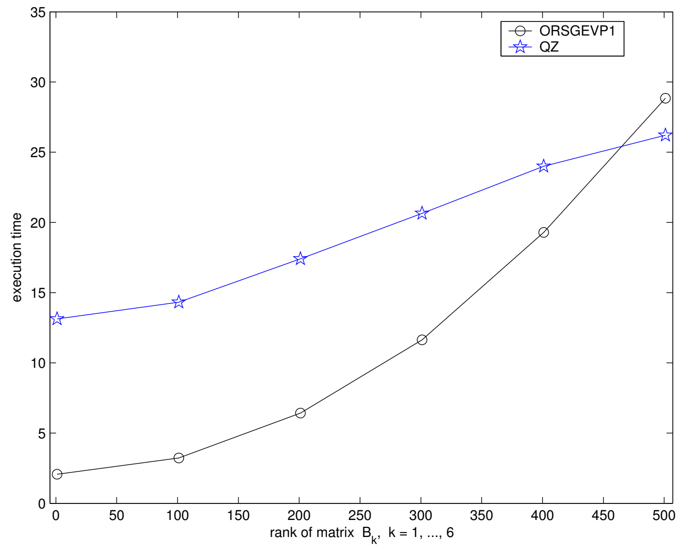

FIG. 6.3. *Comparison of the execution time of the QZ method and the ORSGEVP1 method.*

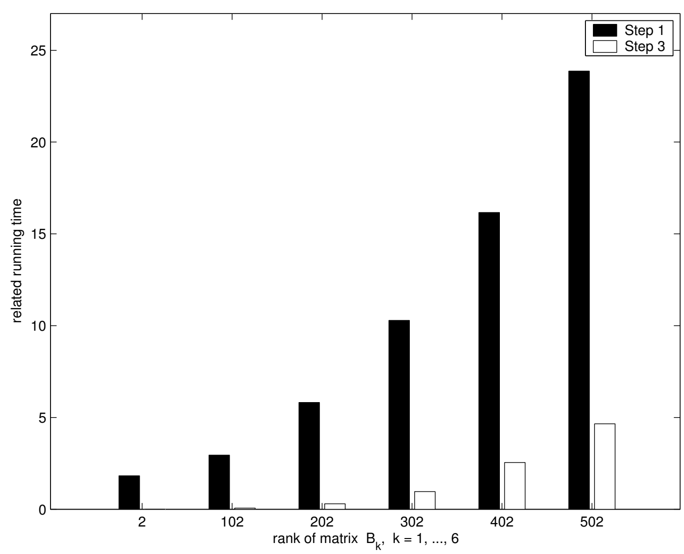

FIG. 6.4. *Running times of the steps 1 and 3 of the ORSGEVP1 method.*

6.1.3. A is a random matrix and B is bidiagonal. Let A be a random matrix of order $n = 1613$ whose entries are chosen from a uniform distribution on

Table 6.2
Accuracy achieved by the ORSGEVP1 method for the pencil with the Lehmer matrix A and the symmetric tridiagonal matrices B_k, $k = 1, \ldots, 6$ of order $n = 512$.

k	rankB_k	D_1
1	2	7.3014e−16
2	102	7.5514e−16
3	202	4.3956e−16
4	302	1.1174e−15
5	402	2.0752e−15
6	502	1.7839e−15

the interval $(0, 1)$. We define 4 different $n \times n$ matrices B_k, $k = 1, \ldots, 4$ whose ranks are $r_k = 111 + (k-1)500$ respectively. This means that for instance, the rank of B_1 is $r_1 = 111$ and the rank of B_4 is $r_4 = 1611$. The nonzero columns of B_k, for $k = 1, \ldots, 4$ are: $B_k(:, j : j) = j\ e_j + (j+1)\ e_{j-1}$, for $j = 2, \ldots, r_k - 1$.

None of the matrices A and B_k, $k = 1, \ldots, 4$ are normal. Table 6.3 shows the accuracy achieved by the method ORSGEVP1. Figure 6.5 plots the execution times of the two methods QZ and ORSGEVP1. This figure shows that the time taken by the ORSGEVP1 method is less than those taken by QZ method when the rank of the matrix $B_k = B$ is less than or equal to 1111.

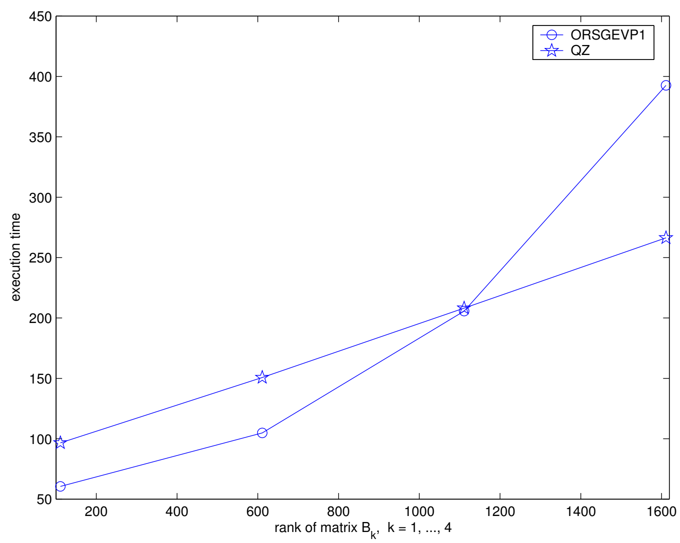

Fig. 6.5. *Comparison of the execution time of the QZ method and the ORSGEVP1 method.*

6.2. The matrix B is invertible. In this section, we present some pencils $A - zB$ defined by a fixed invertible matrix B and a family of the singular matrices $A = A_k$ of rank $r_k \leq n = \text{order}B = \text{order}A_k$. We use the ORSGEVP2 method for computing the finite eigenvalues of these problems.

6.2.1. B is the Hadamard matrix and A is a family of random matrices. Let B be the Hadamard matrix of order $n = 1024$. Let A_{rando} be a random matrix produced by `rando` function in gallery Higham test matrices in Matlab with entries 0

Table 6.3

Accuracy achieved by the ORSGEVP1 method for the pencil with the random matrix A and the bidiagonal matrices B_k, $k = 1, \ldots, 4$ of order $n = 1613$.

k	rankB_k	D_1
1	111	4.8389e−12
2	611	4.3435e−12
3	1111	5.3145e−12
4	1611	1.3585e−14

Table 6.4

Percentage of the nonzeros entries of the matrices A_k for $k = 1, \ldots, 6$.

k	rankA_k	p_k
1	24	98.81
2	224	89.04
3	424	79.29
4	624	69.48
5	824	59.71
6	1024	49.94

or 1. A_k, for $k = 1, \ldots, 6$ are $n \times n$ matrices whose rank satisfy $r_k = 24 + (k-1)200$ respectively. The first r_k columns of the matrix A_k are the first r_k columns of the matrix A_{rando} but its other $n - r_k$ columns are defined as follows: the jth column of the matrix A_k for $j = r_k + 1, \ldots, n$ is equal to the sum of the first r_k columns of the matrix A_k. The matrix B is a dense matrix which has no zero entries. The matrices A_k for $k = 1, \ldots, 6$ are dense too. Let p_k denotes the percentages of the nonzeros entries of the matrix A_k for $k = 1, \ldots, 6$. Table 6.4 shows the p_k associated with the matrices A_k for $k = 1, \ldots, 6$.

The matrix B is normal but the matrices A_k are not. Since the eigenvalues of the pencils $A_k - zB$ are complex number, we directly compare the eigenvalues computed by the ORSGEVP2 method with those computed by the QZ method. Table 6.5 shows the 3 eigenvalues with smallest magnitudes computed by the ORSGEVP2 method and the QZ method. Table 6.6 shows the 3 eigenvalues with largest magnitudes computed by the ORSGEVP2 method and the QZ method. Figure 6.6 plots the execution times of the two methods ORSGEVP2 and QZ.

6.2.2. B **is the Poisson matrix and** A **is tridiagonal.** Let B of order 400 be the Poisson matrix. A_k, for $k = 1, \ldots, 4$ are the tridiagonal singular matrices of order 400 whose rank are 98. Each A_k differs from A_{k-1} by the size of its superdiagonal and underdiagonal nonzero entries which are two times larger than that in the matrix A_{k-1}, for $k = 2, 3, 4$. The nonzero entries of the matrix A_1, are the following: $A_1(j, j+1) = 2$, $A_1(j, j-1) = -1$, $A_1(j, j:j) = 1$ for $j = 2, \ldots, n/4-1$.

The nonzero entries of the matrix A_k, for $k = 2, 3, 4$ are assigned as follows:

$$A_k(j, j+1) = 2A_{k-1}(j, j+1),$$
$$A_k(j, j-1) = 2A_{k-1}(j, j-1),$$
$$A_k(j, j:j) = 1,$$

for $j = 2, \ldots, n/4 - 1$.

The matrix B is the normal matrix but the matrices A_k, for $k = 1, \ldots, 4$ are not. Table 6.7 shows the effect of depart from normality on the accuracy of the eigenvalues computed by the ORSGEVP2 method for the pencils $A_k - zB$, $k = 1, \ldots, 4$.

As it is expected, the experiments done in Sections 6.1 and 6.2 as representative of various examples in their fields, show the performance of the ORSGEVP1 method and

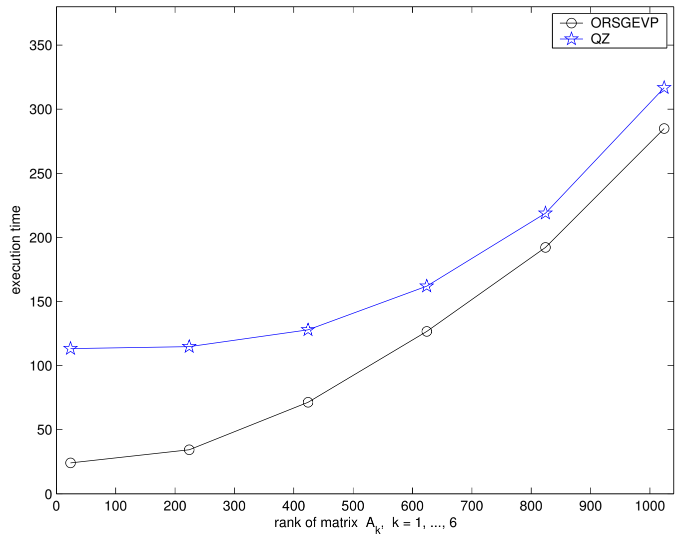

FIG. 6.6. *Comparison of the execution time of the QZ method and the ORSGEVP2 method.*

TABLE 6.5
Comparison of the three eigenvalues of the pencil $A - zB$ with smallest magnitudes computed by ORSGEVP2 method and QZ method.

computed by ORSGEVP2	computed by QZ
-0.00900055846956	-0.00900055846957
$0.02251170159783 - 0.01744975381301i$	$0.02251170159784 - 0.01744975381301i$
$0.02251170159783 + 0.01744975381301i$	$0.02251170159784 + 0.01744975381301i$

the ORSGEVP2 method compared with the QZ method especially for the cases that $r_k \leq n_1$ where $n_1 < n$. Comparison of Fig. 6.2 and Fig. 6.4 shows how the structure of the matrix B affects the running time of SVD method. Also in Section 6.1.3, the matrices B_k, for $k = 1, \ldots, 6$ are bidiagonals whereas the SVD of such matrices can be computed by some new fast methods given in [16, 22], to mention one example. All together support the necessity of extending both of the ORSGEVP1 method and the ORSGEVP1 method to consider some known structured matrix pencils with A or B symmetric (or Hermitian), tridiagonal or bidiagonal, for instance. The first three steps of the ORSGEVP1 method and the ORSGEVP2 method could simply and independently be adapted to consider each one of the mentioned types of matrix pencils.

7. Conclusion. We have studied the computational properties of a novel $O(n^3)$ order reduction method for computing the finite eigenvalues of the regular matrix pencil $A - zB$. When the matrix A is nonsingular, a complete version of this method (called the ORSGEVP1 method) based on the SVD of the matrix B, has been obtained. Generalization of the basic idea has resulted in another version of this method (called the ORSGEVP2 method). This version, for the case when B is nonsingular, is based on SVD of the matrix A. It has been shown that the ORSGEVP1 method (resp. the ORSGEVP2 method) is much cheaper than the QZ method especially when $\text{rank} B \leq n_1$ (resp. $\text{rank} A \leq n_1$) for a specific

Table 6.6

Comparison of the three eigenvalues of the pencil $A - zB$ with largest magnitudes computed by ORSGEVP2 method and QZ method.

computed by ORSGEVP2	computed by QZ
$-0.38534953661660 - 0.51878789785686i$	$-0.38534953661663 - 0.51878789785685i$
$-0.38534953661660 + 0.51878789785686i$	$-0.38534953661663 + 0.51878789785685i$
-0.66778952773254	-0.66778952773259

Table 6.7

Depart from normality and its effect on the accuracy of the eigenvalues computed by the ORSGEVP2 method for the pencil with the Poisson matrix B and the tridiagonal matrices A_k, $k = 1, \ldots, 4$ of order $n = 400$.

k	$He(A)$	$He(B^{-1}A)$	$He(AB^{-1})$	D_1
1	0.5310	0.9312	0.9295	2.4275e−15
2	0.5362	0.9493	0.9451	3.3949e−12
3	0.5368	0.9660	0.9592	1.1490e−9
4	0.5369	0.9777	0.9690	2.8063e−7

natural number $n_1 \leq n = \text{order}A = \text{order}B$ whose value is problem-dependent but usually $n_1 > n/2$. The numerical experiments have confirmed the theoretical results about the performances of the two methods ORSGEVP1 a nd ORSGEVP2. They also show good accuracy of both versions of our method for the cases that A and B are not ill-posed. The influence of "depart from normality" for the pencil $A - zB$ has been studied using an extension of normal matrices [13] and a criterion for depart from normality [11]. An by-product of the experiments, as it is the case for our method, is that the speed of the QZ method implemented in Matlab is affected by rank deficiency in the matrix B. However, the converse of this fact is declared in [15, Section 7.7.7].

The algorithms of both ORSGEVP1 method and ORSGEVP2 method consist in the SVD method, the Gaussian elimination method and the QR method respectively. That is they require computation of the SVD, solving a linear system with multiple right hand side and computing the eigenvalues of a standard eigenvalue problem respectively. These major parts of our methods can be specialized for some special regular matrix pencils. When the order of the matrix pencil $A - zB$ is large and the nonsingular matrix A (resp. B) is sparse, then we can improve the performance of the ORSGEVP1 (resp. ORSGEVP2) method by employing some suitable iterative methods for solving the sparse linear system with multiple rigth hand side at the step 2 of Algorithm 4.1 (resp. 4.2).

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