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ON THE BLOCK TRIANGULAR FORM OF SYMMETRIC MATRICES*

IAIN S. DUFF[†][‡] AND BORA UÇAR[‡]

Abstract. We present some observations on the block triangular form (btf) of symmetric, structurally rank deficient, square, sparse matrices. As the matrix is square and structurally rank deficient, its canonical btf has at least one underdetermined and one overdetermined block. We prove that these blocks are transposes of each other. We further prove that the square block of the canonical btf, if present, has a special fine structure. These findings help us recover symmetry around the anti-diagonal in the block triangular matrix. We also visit the full rank symmetric case.

Key words. sparse matrices, block triangular form, Dulmage-Mendelsohn decomposition, maximum cardinality matchings

AMS subject classifications. 05C50, 05C70, 05D15, 65F50

1. Introduction. We are interested in the block triangular form (btf) of structurally rank deficient, symmetric, sparse matrices. Unless otherwise stated, A is always such a matrix with no all-zero rows or columns. The block triangular form is based on a canonical decomposition of bipartite graphs known as the Dulmage-Mendelsohn decomposition (see [9] for a detailed account). When permuted into the block triangular form, the matrix A assumes the form

$$\begin{array}{c} \\ H_R \\ S_R \\ V_R \end{array} \begin{array}{c} \begin{array}{ccc} H_C & S_C & V_C \end{array} \\ \begin{pmatrix} A_H & * & * \\ O & A_S & * \\ O & O & A_V \end{pmatrix} \end{array} . \tag{1.1}$$

Here, H_R, S_R, and V_R are sets of rows, and H_C, S_C, and V_C are sets of columns. As we shall see, the three diagonal blocks are of special importance. The block A_H, formed by the rows in the set H_R and the columns in the set H_C, is underdetermined; the block A_S, formed by the rows in the set S_R and the columns in the set S_C, is square; the block A_V, formed by the rows in the set V_R and the columns in the set V_C, is overdetermined. As in [9], we will call these three blocks horizontal, square, and vertical, respectively. In (1.1), there are no nonzero entries in the sub-diagonal blocks shown as O.

In the following two subsections, we provide the reader with definitions (mostly standard), and necessary background material from Duff, Erisman, and Reid [3, Chapter 6], Pothen and Fan [9], and Pothen [8, Section 2.7] on the computation and properties of the btf. The computation of the btf is based on maximum cardinality matchings, or just maximum matchings, in bipartite graphs (these are discussed in Sections 1.1 and 1.2). We discuss two transformations on maximum matchings of symmetric matrices in Section 2. One of the transformations is based on [4]; the other is based on the notion of cycles of a permutation, and to the best of our knowledge is discussed and used for the first time in this paper. We use these transformations to show that for a symmetric matrix there is a maximum matching with some special

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properties. In Section 3, we formally state our main theorem on the btf of symmetric matrices. The theorem establishes equivalence relations between the pair of sets H_R and V_C, the pair of sets S_R and S_C, and the pair of sets V_R and H_C. In the same section, we prove our main theorem by exploiting the properties discussed in Section 2.

1.1. Definitions.

As is common, we associate a bipartite graph $G = (R \cup C, E)$ with the $n \times n$ matrix A, where $R = \{r_1, \ldots, r_n\}$ and $C = \{c_1, \ldots, c_n\}$ are the two sets of the vertex bipartition, and E is the set of edges. Here, the vertices in R and C correspond to the rows and the columns of A, respectively, such that $(r_i, c_j) \in E$ if and only if $a_{ij} = 1$. For a given $i \in \{1, \ldots, n\}$, the row r_i and the column c_i are referred to as symmetric counterparts of each other. Similarly, the edges (r_i, c_j) and (r_j, c_i) are called symmetric counterparts of each other. When necessary, we will make it clear whether a vertex is a row or a column vertex. An edge $(r_i, c_j) \in E$ is said to be incident on the vertices r_i and c_j. Two vertices are called adjacent if there is an edge incident on both. The set of vertices that are adjacent to a vertex v are called its neighbours and are indicated by $\mathrm{adj}(v)$. A path is a sequence of edges of the form $\big((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\big)$. A cycle is a sequence of edges of the form $\big((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\big)$ where $v_k = v_0$.

A set of edges $\mathcal{M}$ is a matching if no two edges in $\mathcal{M}$ are incident on the same vertex. In matrix terms, a matching corresponds to a set of nonzero entries no two in a common row or column. A vertex is said to be matched (with respect to a given matching) if there is an edge in the matching incident on the vertex, and to be unmatched otherwise. Given a matching $\mathcal{M}$, an $\mathcal{M}$-alternating path is a path whose edges are alternately in $\mathcal{M}$ and not in $\mathcal{M}$. We use the notation $u \xrightarrow{\mathcal{M}} v$ to denote that vertex u reaches vertex v with an $\mathcal{M}$-alternating path. Note that this is a bidirectional relation in an undirected graph: if $u \xrightarrow{\mathcal{M}} v$, then $v \xrightarrow{\mathcal{M}} u$. An alternating path is called an augmenting path, if it starts and ends at unmatched vertices. The cardinality of a matching is the number of edges in it. A maximum cardinality matching or a maximum matching is a matching of maximum cardinality. Given a bipartite graph G and a matching $\mathcal{M}$, a necessary and sufficient condition for $\mathcal{M}$ to be of maximum cardinality is that there is no $\mathcal{M}$-augmenting path in G (the result is due Berge and summarized in different places, see for example [6, Chapter 1]). We use $\mathrm{mate}(v)$, to denote the vertex matched to the vertex v in a matching $\mathcal{M}$, e.g., if $\mathrm{mate}(r_i) = c_j$, then we also have $\mathrm{mate}(c_j) = r_i$. We use $\langle \cdot, \cdot \rangle$ to differentiate a matching edge from an ordinary edge, e.g., we use $\langle r_i, c_j \rangle$ or $\langle c_j, r_i \rangle$ to denote that the row r_i is matched to the column c_j. We say a vertex set X is completely matched to another one Y, if for all $x \in X$, we have $\mathrm{mate}(x) \in Y$; for clarity we note that $|X| \leq |Y|$, where $|\cdot|$ denotes the cardinality of a set.

Some of the definitions in this paragraph can be found in [7]. Let A be an $n \times n$ matrix, $\mathcal{I}$ and $\mathcal{J}$ be two subsets of $\{1, \ldots, n\}$. The matrix formed by selecting the rows and columns indexed by $\mathcal{I}$ and $\mathcal{J}$, respectively, is called a submatrix of A confined to the rows in $\mathcal{I}$ and the columns in $\mathcal{J}$. The matrix A is said to be partly decomposable if it contains an $s \times (n-s)$ zero submatrix. More explicitly, A is partly decomposable if there exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

with B and D being square. If A contains no $s \times (n-s)$ zero submatrix for $s = 1, \ldots, n-1$, then it is called fully indecomposable, also called irreducible [3]. We note for later use that an $n \times n$ symmetric matrix A, where $n > 2$ and n is odd, $a_{ij} = a_{ji} \neq 0$ for $i = 1, \ldots, n$ and $j \equiv i + 1 \pmod{n}$, and $a_{ij} = a_{ji} = 0$ elsewhere, is irreducible. The bipartite graph of this matrix is a cycle on $2n$ vertices with n row vertices and n column vertices. Any $n \times n$ matrix B whose sparsity structure is a superset of that of A, i.e., $b_{ij} \neq 0$ if $a_{ij} \neq 0$, is also fully indecomposable.

Our last definition is for directed graphs. A vertex v is said to be reachable from another vertex u if there is a directed path from u to v. A strongly connected component of a directed graph $G = (V, E)$ is a maximal set of vertices $U \subseteq V$ such that every pair of vertices in U are reachable from each other.

1.2. Computation and properties of the block triangular form. Given a maximum matching $\mathcal{M}$, the btf of a matrix (of any shape or symmetry) can be computed using the following equations (the equations are rephrased from [8, Section 2.7]):

$$
\begin{aligned}
&U_C = \{c \in C : c \text{ is unmatched}\} && U_R = \{r \in R : r \text{ is unmatched}\} \\
&H_R = \{r \in R : r \xrightarrow{\mathcal{M}} u \text{ for some } u \in U_C\} && V_C = \{c \in C : c \xrightarrow{\mathcal{M}} u \text{ for some } u \in U_R\} \\
&H'_C = \{c \in C : c \xrightarrow{\mathcal{M}} u \text{ for some } u \in U_C\} && V'_R = \{r \in R : r \xrightarrow{\mathcal{M}} u \text{ for some } u \in U_R\} \\
&H_C = U_C \cup H'_C && V_R = U_R \cup V'_R \\
&S_R = R \setminus (H_R \cup V_R) && S_C = C \setminus (H_C \cup V_C)\ .
\end{aligned}
$$

The algorithms to compute the btf of a matrix were discussed some time ago, see for example [5, 9]. We present an essential part of those algorithms in a different form below. Algorithm 1 shows how to find the set of rows H_R and the set of columns H_C of the horizontal block. The algorithm grows the row set H_R and the column set H_C by running a graph search algorithm. At a column vertex c known to be in H_C (whose adjacency is not explored yet), it adds all neighbouring rows to H_R. At a row vertex r known to be in H_R, the algorithm only visits the column $v = \text{mate}(r)$ and adds v to H_C if it is not already there.

Algorithm 1 Algorithm to find horizontal block of the btf

```
Input $\mathcal{M}$: a maximum matching; mate($v$) giving the mate of a vertex $v$
Output $H_R$ and $H_C$: the set of rows and columns of the horizontal block
$H_R \leftarrow H_C \leftarrow \emptyset$
$U \leftarrow$ {unmatched columns}
while $U \neq \emptyset$ do
    Pick a column vertex $c \in U$ and set $U \leftarrow U \setminus \{c\}$
    $H_C \leftarrow H_C \cup \{c\}$
    for each row vertex $r \in \text{adj}(c) \setminus H_R$ do
        $H_R \leftarrow H_R \cup \{r\}$
        $v \leftarrow \text{mate}(r)$    ▷ should exist; otherwise flags an augmenting path
        if column $v \notin U \cup H_C$ then
            $U \leftarrow U \cup \{v\}$
```

A similar algorithm is run to find the rows and columns in the vertical block. In this case, at a column vertex in V_C, only its mate is visited and added to V_R, if necessary; at a row vertex in V_R, the neighbouring columns are added to V_C, whenever necessary. After finding the rows and columns of the horizontal and vertical blocks, the remaining rows and columns are marked to be in the sets S_R and S_C, respectively.

We note the following properties of the block triangular form without proving them. The proofs can be found in [2], [3, Chapter 6] and [8, Section 2.7]. These properties hold for any matrix.

FACT 1.1. *The rows in H_R are completely matched to the columns in H_C. The columns in S_C are completely matched to the rows in S_R and vice versa. The columns in V_C are completely matched to the rows in V_R.*

FACT 1.2. *The block triangular form is unique. In other words, any given maximum matching yields the same sets H_R, H_C, S_R, S_C, V_R, and V_C.*

The previous two properties also imply that all entries of a maximum matching should reside in the diagonal blocks of the btf.

FACT 1.3. *In the block triangular form of a structurally rank deficient, square matrix (not necessarily symmetric), the horizontal and vertical blocks both should be present. The square block may be missing.*

It may be possible to decompose the three diagonal blocks A_H, A_S, and A_V further into smaller submatrices, resulting in a fine decomposition. In the fine decomposition, the horizontal and vertical blocks have block diagonal structure where the individual diagonal blocks are horizontal and vertical, respectively.

The fine decomposition of the square block A_S is obtained by identifying irreducible blocks. Pothen and Fan [9] list the properties of and give an algorithm to compute the fine decomposition of the square block. Here, we summarize some of the properties that we will need in the rest of the paper. Let p be the number of irreducible blocks (all of them square) in the fine decomposition of A_S, and let R_i and C_i be the set of rows and the set of columns in the ith block for $i = 1, \ldots, p$. The rows in R_i are matched to the columns in C_i. The sets R_i and C_i for $i = 1, \ldots, p$ are unique—they are independent of the choice of maximum matching [2]. The blocks cannot be combined to yield another decomposition satisfying the properties given above.

2. Two transformations.

As discussed in the previous section, the btf is unique and can be obtained using any maximum matching. In this section, we discuss two processes that transform any maximum matching in the bipartite graph of a symmetric matrix into another one that has some special properties. Given a matching in the bipartite graph of a square matrix, we define an m-path as a sequence of edges of the form $\big(\langle r_i, c_j\rangle, \langle r_j, \cdot\rangle, \ldots, \langle \cdot, c_k\rangle, \langle r_k, c_l\rangle\big)$, where each edge is in the matching. After a matching edge $\langle r_k, c_l\rangle$, the next edge to be included is either of the form $\langle r_l, \cdot\rangle$ or $\langle \cdot, c_k\rangle$, if any of them exists. Note that an m-path is not necessarily a path in the bipartite graph, as the two consecutive matching edges $\langle \cdot, c_k\rangle$ and $\langle r_k, \cdot\rangle$ are not necessarily connected by the edge (c_k, r_k). An m-path starts at a row vertex and ends at a column vertex. If $l = i$ in the above example, then we have an m-cycle. An m-path is called open if the symmetric counterparts of the start and end vertices of the path are unmatched. An open m-path can be identified with the following process: start from a matched row r_i where the column c_i is unmatched, visit the column $c_j = \mathrm{mate}(r_i)$, and continue from row r_j if it is matched. Note that any matching in a square matrix can be decomposed into m-cycles and open m-paths.

2.1. Automorphic maximum matchings.

We define a matching to be automorphic if it matches a set of rows to the corresponding set of columns. That is, for a matching $\mathcal{M}$ to be automorphic, whenever $\langle r_i, c_j\rangle \in \mathcal{M}$, the column c_i and the row r_j should be matched by $\mathcal{M}$. We restate the following lemma from [4].

LEMMA 2.1 (Property 4.2 of [4]). *Let A be a symmetric matrix and $\mathcal{M}$ be a maximum matching. Let $\mathcal{I}$ and $\mathcal{J}$ be the set of rows and columns matched by $\mathcal{M}$, i.e.,*

$\mathcal{I} = \{r_i : \langle r_i, \cdot\rangle \in \mathcal{M}\}$ *and* $\mathcal{J} = \{c_j : \langle \cdot, c_j\rangle \in \mathcal{M}\}$*. Then, there is an automorphic maximum matching* $\mathcal{M}'$ *that matches the set of rows* $\mathcal{I}$ *to the set of columns that are symmetric counterparts of the rows in* $\mathcal{I}$*. Equivalently, there is an automorphic maximum matching that matches the set of columns* J *to the set of rows that are symmetric counterparts of the columns in* $\mathcal{J}$*.*

We summarize the main points of the proof of the lemma for completeness. Duff and Pralet first note that edges of a maximum matching are either in m-cycles or in open m-paths. Since the m-cycles are already automorphic, they investigate the open m-paths. They note that if an open m-path is formed using the row vertices in the set I and the column vertices in the set J, then $|I\backslash J| = |J\backslash I| = 1$. They show that using the edges symmetric to those of the open m-path (the matrix is symmetric), it is possible to completely match the set of rows in I to the set of columns in $(I \setminus J) \cup (I \cap J) = I$. Notice that since the m-cycles of the original maximum matching are kept intact, and for each open m-path an automorphic matching with the same cardinality is obtained, they end up with an automorphic maximum matching.

2.2. Permutation cycles of an automorphic matching. Let $\mathcal{M}$ be an automorphic matching. Since $\mathcal{M}$ matches a set of rows to the set of corresponding columns, its edges reside in m-cycles, i.e., there is no open m-path. An automorphic matching from $\mathcal{I}$ to $\mathcal{I}$ can be perceived as a permutation of the set $\mathcal{I}$ in an algebraic sense (a one-to-one and onto function). By starting from an element of the set $\mathcal{I}$ and by applying the permutation until the starting element is seen again, we can obtain cycles of the permutation (for more on cycles of a permutation see [1, Section 1.5]). Similarly, by following the matching edges of an automorphic matching $\mathcal{M}$ as $\big(\langle r_i, c_j\rangle, \langle r_j, \cdot\rangle, \ldots, \langle \cdot, c_i\rangle\big)$, we can obtain the cycles of $\mathcal{M}$. Due to this correspondence, we refer to the cycles of an automorphic matching as the permutation cycles.

Note that permutation cycles are also m-cycles and therefore they do not necessarily correspond to ordinary cycles in the underlying bipartite graph. Consider for example the permutation cycle $\big(\langle r_i, c_j\rangle, \langle r_j, c_i\rangle\big)$. If a_{ii} or a_{jj} is zero in A, then we do not have a cycle in the graph; we only have a permutation cycle. The length of a permutation cycle is the number of matching edges in it. The length 2 permutation cycles, also called transpositions [7, p.11], are of the form $\big(\langle r_i, c_j\rangle, \langle r_j, c_i\rangle\big)$ and are of special importance. Figure 2.1 displays permutation cycles of length 1 to 4 in a hypothetical example. An odd permutation cycle is of odd length, and an even permutation cycle is of even length. Note that the edges of an odd permutation cycle (with length greater than one) when put together with their symmetric counterparts form a unique ordinary cycle in the bipartite graph of A. Note also that any pair of a row and a column in an odd permutation cycle are reachable from each other via two alternating paths: one starting and ending with a matching edge, the other starting and ending with an ordinary edge. For an even permutation cycle of length k, adding the symmetric edges partitions the permutation cycle into two ordinary cycles each having $k/2$ matching edges and $k/2$ non-matching edges, where the row vertices in one cycle are symmetric counterparts of the column vertices in the other one; the two cycles may be connected in the bipartite graph due to existence of other edges, but we are not interested in this possibility. Consider, for example, the length 4 permutation cycle $\big(\langle r_i, c_j\rangle, \langle r_j, c_k\rangle, \langle r_k, c_l\rangle, \langle r_l, c_i\rangle\big)$ shown in Fig. 2.1. The permutation cycle is split between two ordinary cycles in the bipartite graph of A: $\big(\langle r_i, c_j\rangle, (c_j, r_k), \langle r_k, c_l\rangle, (c_l, r_i)\big)$ and $\big(\langle r_j, c_k\rangle, (c_k, r_l), \langle r_l, c_i\rangle, (c_i, r_j)\big)$. As seen, each of these cycles contain 2 matching edges and 2 non-matching edges, and the row vertices in one cycle are symmetric counterparts of the column vertices in the other.

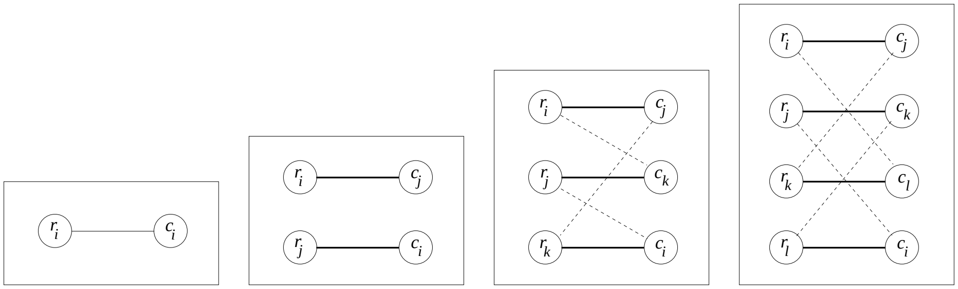

FIG. 2.1. *Permutation cycles of an automorphic matching. The matching edges are shown with bold solid lines. The other edges, shown with dashed lines, are present because of the symmetry of the matrix. Matching edges of the form $\langle r_i, c_i\rangle$ give a permutation cycle of length 1 (first subfigure); two matching edges of the form $(\langle r_i, c_j\rangle, \langle r_j, c_i\rangle)$ give a permutation cycle of length 2 (second subfigure); length 3 (third subfigure) and length 4 (fourth subfigure) permutation cycles are also shown.*

From an automorphic matching $\mathcal{M}$, we construct another automorphic matching $\mathcal{M}'$ which is composed of odd length permutation cycles and length 2 permutation cycles. We proceed as follows. First, all edges of $\mathcal{M}$ that form an odd permutation cycle are copied into $\mathcal{M}'$, e.g., for a length 3 permutation cycle of the form $\big(\langle r_i, c_j\rangle, \langle r_j, c_k\rangle, \langle r_k, c_i\rangle\big)$, these three edges are copied into $\mathcal{M}'$. Then, even length permutation cycles of $\mathcal{M}$ are decomposed into length 2 permutation cycles, and these length 2 permutation cycles are added to $\mathcal{M}'$ such that if $\langle r_i, c_j\rangle \in \mathcal{M}'$, then $\langle r_j, c_i\rangle \in \mathcal{M}'$. As noted above, the even length permutation cycles are split between two ordinary cycles in the bipartite graph when the symmetric edges are considered. By alternating the status of the edges according to the matching in one of the cycles, we can obtain length 2 permutation cycles. The decomposition of an even permutation cycle into length 2 permutation cycles is best seen in matrix terms. Consider the matching shown on the left below

$$\begin{array}{c}
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & & \times & & & & * \\
2 & * & & \times & & & \\
3 & & * & & \times & & \\
4 & & & * & & \times & \\
5 & & & & * & & \times \\
6 & \times & & & & * &
\end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & & \times & & & & * \\
2 & \times & & * & & & \\
3 & & * & & \times & & \\
4 & & & \times & & * & \\
5 & & & & * & & \times \\
6 & * & & & & \times &
\end{array}
\end{array}. \tag{2.1}$$

In the matrix on the left of (2.1), the original matching, with matching entries marked by ×, is automorphic and corresponds to a length 6 permutation cycle. Using the entries symmetric to the matching entries, shown with $*$, we obtain two ordinary cycles: first $\big(\langle r_1, c_2\rangle, (c_2, r_3), \langle r_3, c_4\rangle, (c_4, r_5), \langle r_5, c_6\rangle, (c_6, r_1)\big)$ and second $\big(\langle r_2, c_3\rangle, (c_3, r_4), \langle r_4, c_5\rangle, (c_5, r_6), \langle r_6, c_1\rangle, (c_1, r_2)\big)$ which share the original matching edges evenly. Now by taking the second cycle and alternating the status of the edges according to the matching, we obtain $\big((r_2, c_3), \langle c_3, r_4\rangle, (r_4, c_5), \langle c_5, r_6\rangle, (r_6, c_1), \langle c_1, r_2\rangle\big)$. As seen, the new set of matching edges form three length 2 permutation cycles: $\big(\langle r_1, c_2\rangle, \langle r_2, c_1\rangle\big)$, $\big(\langle r_3, c_4\rangle, \langle r_4, c_3\rangle\big)$, and $\big(\langle r_5, c_6\rangle, \langle r_6, c_5\rangle\big)$. The resulting matching is shown on the right of (2.1). As is clear, this transformation does not change the cardinality of the automorphic matching and hence $\mathcal{M}'$ is of maximum cardinality.

3. The block triangular form of symmetric matrices. As shown in the previous section, any maximum matching in the bipartite graph of a structurally singular, symmetric matrix A can be transformed into an automorphic one consisting of only odd permutation cycles and length 2 permutation cycles. Therefore, we assume that we have a matching $\mathcal{M}$ with those properties. We recall Fact 1.2—the uniqueness of the sets H_R, S_R, V_R, H_C, S_C, and V_C of the block triangular form of a matrix.

Before we prove our main theorem, we start with a series of lemmata.

LEMMA 3.1. *The odd permutation cycles are confined to only one diagonal block of the block triangular form.*

Proof. Permutation cycles of length 1 are trivially confined to only one block. Recall from Section 2.2 that an odd permutation cycle is a part of an ordinary cycle. Since the matching edges within the ordinary cycle are in the diagonal blocks of the btf, having those nonzeros straddle more than one diagonal block can only be possible if the submatrix confined to the rows and columns of the cycle is partly decomposable. However, as noted towards the end of Section 1.1, matrices whose sparsity structure corresponds to a superset of odd length cycles are fully indecomposable. □

We have a result which is stronger than the previous lemma.

LEMMA 3.2. *The odd permutation cycles are confined to the square block.*

Proof. From Lemma 3.1, an odd length permutation cycle is confined to a single block. Take an odd length permutation cycle $\mathcal{C} = \big(\langle r_i, c_j\rangle, \langle r_j, \cdot\rangle, \ldots, \langle \cdot, c_i\rangle\big)$ and suppose, for the sake of contradiction, that $\mathcal{C}$ is in the horizontal block. Since all row vertices in $\mathcal{C}$ are in H_R, each one of these vertices reaches an unmatched column with an alternating path. Suppose row r_i reaches, with an alternating path, an unmatched column c_u, i.e., $r_i \xrightarrow{\mathcal{M}} c_u$ without going through other vertices in the permutation cycle $\mathcal{C}$ (the latter assumption is not weaker but gives a cleaner argument). Let $P = \big((r_i, c_l), \langle c_l, r_{i+1}\rangle, \ldots, \langle \cdot, r_{i+t}\rangle, (r_{i+t}, c_u)\big)$ be that alternating path. Due to the symmetry of the matrix, the same path exist in the reverse direction from row r_u to column c_i. That is we have the path $P^T = \big((r_u, c_{i+t}), (c_{i+t}, \cdot), \ldots, (c_{i+1}, r_l), (r_l, c_i)\big)$. We now show that P^T is an alternating path, i.e., $P^T\!=\!\big((r_u, c_{i+t}),\langle c_{i+t}, \cdot\rangle,\ldots,\langle c_{i+1}, r_l\rangle,(r_l, c_i)\big)$. Note that since r_u and c_u are both unmatched, P^T being an alternating path implies that the path $\big((r_u, c_{i+t}), c_{i+t} \xrightarrow{\mathcal{M}} c_i \xrightarrow{\mathcal{M}} r_i \xrightarrow{\mathcal{M}} r_{i+t}, (r_{i+t}, c_u)\big)$ is an augmenting path, contradicting the assumption that $\mathcal{M}$ is a maximum matching. We first note that $c_i \xrightarrow{\mathcal{M}} r_i$ as c_i and r_i are in an odd permutation cycle. Note that since $\mathcal{M}$ is automorphic, row r_u is not matched. Consider the last row vertex r_{i+t} in P. Column c_{i+t} is the first column vertex in P^T. Since r_u is not matched, c_{i+t} should have a mate (otherwise $\mathcal{M}$ would not be a maximum matching). Therefore, c_{i+t} is in the vertical block (being a matched vertex reaching an unmatched row). Now, since r_{i+t} is in the horizontal block and c_{i+t} is in the vertical one, due to Lemma 3.1 the vertices r_{i+t} and c_{i+t} cannot be in an odd permutation cycle. Therefore, they are in a length 2 permutation cycle. That is, if $\text{mate}(r_{i+t}) = c_x$, then $\text{mate}(c_{i+t}) = r_x$. Consider the next column vertex c_{i+t-1} in P^T. It should have a mate, otherwise an augmenting path $r_u \xrightarrow{\mathcal{M}} c_{i+t-1}$ exists, and should be in the vertical block. With the same reasoning as above, it is matched to the row that corresponds to the mate of row r_{i+t-1} in P. Therefore, the path P^T is an alternating path symmetric to P. Figure 3.1 displays the arguments for a length 3 permutation cycle $\mathcal{C} = \big(\langle r_i, c_j\rangle, \langle r_j, c_k\rangle, \langle r_k, c_i\rangle\big)$ and an alternating path $P = \big((r_i, c_l), \langle c_l, r_m\rangle, (r_m, c_u)\big)$.

With similar arguments, it can be shown that odd permutation cycles cannot be confined to the vertical block. Therefore, the odd permutation cycles are confined to

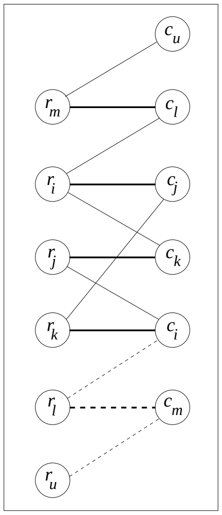

FIG. 3.1. *An example for the proof of Lemma 3.2. The solid bold lines correspond to edges of a maximum matching. A length 3 permutation cycle* $\mathcal{C} = \big(\langle r_i, c_j\rangle, \langle r_j, c_k\rangle, \langle r_k, c_i\rangle\big)$ *is shown; the solid lines represent the edges symmetric to that of the permutation cycle. Column* c_u *is not matched and reachable from row* r_i *with the alternating path* $P = \big((r_i, c_l), \langle c_l, r_m\rangle, (r_m, c_u)\big)$*. The symmetric path* P^T *is shown with dashed lines. Row* r_u *is not matched, as the matching is automorphic. Column* c_m *should have a mate, and since it reaches an unmatched row, it should be in the vertical block. It is shown in the proof that the column* c_m *should have been matched to* r_l*; this matching edge is shown with a dashed bold line. Therefore, the path* $\big((r_u, c_m), \langle c_m, r_l\rangle, (r_l, c_i), \langle c_i, r_k\rangle, (r_k, c_j), \langle c_j, r_i\rangle, (r_i, c_l), \langle c_l, r_m\rangle, (r_m, c_u)\big)$ *is an augmenting path, contradicting the fact that* $\mathcal{M}$ *is a maximum matching.*

the square block. ☐

COROLLARY 3.3. *For each* $\langle r_i, c_j\rangle \in \mathcal{M}$ *in the horizontal block, we have* $\langle r_j, c_i\rangle \in \mathcal{M}$*. Similarly, for each* $\langle r_k, c_l\rangle \in \mathcal{M}$ *in the vertical block, we have* $\langle r_l, c_k\rangle \in \mathcal{M}$*.*

We have a refinement of the previous corollary.

LEMMA 3.4. *The length 2 permutation cycles are not contained entirely in the horizontal or vertical blocks.*

Proof. We prove the lemma for the horizontal block, that is we show that length 2 permutation cycles are not contained in the horizontal block; the vertical block case is similar. Suppose, for the sake of contradiction, $\langle r_i, c_j\rangle \in \mathcal{M}$ and its symmetric counterpart $\langle r_j, c_i\rangle \in \mathcal{M}$ are in the horizontal block. As in the proof of Lemma 3.2, we take an unmatched column c_u that is reachable from row r_i with an alternating path. Again, due to $\mathcal{M}$ being automorphic, row r_u is not matched. However, as in the proof of Lemma 3.2, we have an alternating path from column c_i to unmatched row r_u, contradicting the fact that c_i is in the horizontal block. ☐

We are now ready to state and prove the following theorem regarding the block triangular form of a structurally rank deficient symmetric matrix.

THEOREM 3.5. *Given a structurally rank deficient symmetric matrix* A*, let* H_R, S_R, V_R*,* H_C, S_C, V_C *be the sets in the block triangular form of* A*. Then, the set of horizontal rows* H_R *is equal to the set of vertical columns* V_C*; the set of square rows* S_R *is equal to the square columns* S_C*; the set of vertical rows* V_R *is equal to the set of horizontal columns* H_C*.*

Proof. Since A is square and rank deficient, we know that both horizontal and vertical blocks are present in the block triangular form. The square block may be

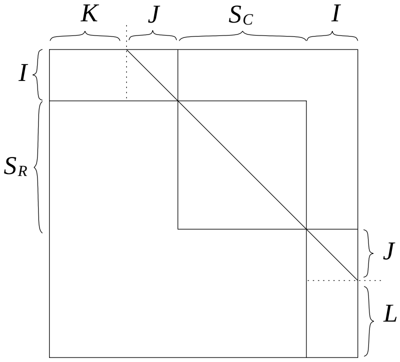

FIG. 3.2. *A matching is shown by the slanted line; the rows in I are matched to columns in J, and the columns in I are matched to rows in J. The borders of the horizontal, square, and vertical blocks are shown with solid lines. The dashed lines divide the set of columns in the horizontal block and the set of rows in the vertical block into two sets. The sets K and L are equal as the set of unmatched columns is equal to the set of unmatched rows.*

missing.

Consider a matching edge $\langle r_i, c_j \rangle$ in the horizontal block. As shown in Lemma 3.2, it is not in an odd permutation cycle and hence, as noted in Corollary 3.3, $\langle r_j, c_i \rangle \in \mathcal{M}$. We know from Lemma 3.4 that $\langle r_j, c_i \rangle \in \mathcal{M}$ is not in the horizontal block. Two cases remain to be investigated: $\langle r_j, c_i \rangle$ is either in the square block or in the vertical block. For the sake of contradiction, suppose $\langle r_j, c_i \rangle$ is in the square block. As in the proof of Lemma 3.2, we take an unmatched column c_u that is reachable from row r_i with an alternating path. Again, due to $\mathcal{M}$ being automorphic, row r_u is not matched. However, as in the proof of Lemma 3.2, we have an alternating path from column c_i to the unmatched row r_u, contradicting the fact that c_i is in the square block. Similar arguments can be used to show that for a $\langle r_k, c_l \rangle$ in the vertical block, $\langle r_l, c_k \rangle$ is in the horizontal block. Therefore, a matching edge $\langle r_i, c_j \rangle$ is in the horizontal block if and only if the matching edge $\langle r_j, c_i \rangle$ is in the vertical block.

We have established two results. First, the set H_R is equal to the set V_C. Second, the set of columns that are matched to the rows in H_R is equal to the set of rows that are matched to the columns in V_C. These equivalence relations are shown in Fig. 3.2. As the matching is automorphic, the set of unmatched columns (K in the figure) is equal to the set of unmatched rows (L in the figure); we have thus established the equivalence between the sets H_C and V_R. Since the matrix is square, the set of remaining rows S_R is equal to the set of remaining columns S_C. □

Once the equivalences between the row and column sets are established, it is easy to recover a structural symmetry in the block triangular form.

COROLLARY 3.6. *The block triangular form of a structurally singular, symmetric matrix can be permuted to be symmetric around the anti-diagonal.*

This can be achieved by fixing a permutation of the rows in H_R and S_R, and the columns in H_C, and then by reorganizing V_R, S_C and V_C such that the reverse order within these later blocks match those of H_C, S_R, and H_R, respectively. It is possible to refine this form by looking at the fine structure of the square block A_S. We first need the following lemma.

LEMMA 3.7. *Let R_i and C_i be the set of rows and the set of columns of the ith irreducible block in the fine decomposition of the square block A_S. Then, either $R_i = C_i$; or $R_i \cap C_i = \emptyset$ and there exist an irreducible block j in the fine decomposition* *of* A_S *with* $R_j = C_i$ *and* $C_j = R_i$.

Proof. Take the ith irreducible block and suppose for the sake of contradiction $R_i \neq C_i$ and $R_i \cap C_i \neq \emptyset$. Define three sets: first $I = R_i \cap C_i$, second $I_1 = R_i \setminus I$, and third $J_2 = C_i \setminus I$. With this partitioning of the rows and columns, we can permute the ith irreducible block into the following form

$$A_i = \begin{array}{ccc} & I & J_2 \\ I & A_{11} & A_{12} \\ I_1 & A_{21} & A_{22} \end{array} . \tag{3.1}$$

First note that $A_{12} \neq O$ and $A_{21} \neq O$, otherwise the block will be reducible. Now consider the larger square submatrix consisting of the set of rows $I \cup I_1 \cup J_2$ and the same set of columns

$$A_L = \begin{array}{cccc} & I & J_2 & I_1 \\ I & A_{11} & A_{12} & A_{21}^T \\ I_1 & A_{21} & A_{22} & * \\ J_2 & A_{12}^T & * & A_{22}^T \end{array} . \tag{3.2}$$

For the columns in the set I_1 and the rows in set J_2 to be in a different block from i, the submatrix of A_L indexed by the row set $I \cup J_2$ and the column set $I \cup I_1$ should be reducible. But the matrix

$$\begin{array}{ccc} & I & I_1 \\ I & A_{11} & A_{21}^T \\ J_2 & A_{12}^T & A_{22}^T \end{array} \tag{3.3}$$

is the transpose of A_i (see (3.1)), as $A_{11} = A_{11}^T$. Since A_i is irreducible, so is its transpose shown in (3.3). Therefore, the rows in row set J_2 and the columns in column set I_1 cannot be in a different block from the one that contains I.

We have established that either $R_i = C_i$ or $R_i \cap C_i = \emptyset$. If $R_i \cap C_i = \emptyset$, all matching nonzeros in this block should be in length 2 permutation cycles, as the odd permutation cycles are irreducible. Therefore, we have another irreducible block j with $R_j = C_i$ and $C_j = R_i$ and the proof is completed. □

Having defined the fine structure of the square block A_S, we refine Corollary 3.6 by using that structure. As before, let R_i and C_i denote the rows and columns of the ith irreducible block of A_S. We will order the rows of A and then apply that order in the reverse direction to the columns. We first order the rows in the horizontal block. Then, we order the rows in square block A_S using the fine structure as follows. Let i be an irreducible block whose rows are yet to be ordered. If the column set C_i is equal to the row set R_i, order R_i. If $C_i \cap R_i = \emptyset$, order R_i and then the rows corresponding to the columns in C_i. After this blockwise ordering of all the rows in the square block A_S, we order the rows in the vertical block. We do not specify the order of the rows in a subblock—it can be arbitrary. Now applying the order obtained to the columns in the reverse direction results in a matrix that is symmetric along the anti-diagonal. A sample matrix ordered with this procedure is shown in Fig. 3.3.

We note that Theorem 3.5, Corollary 3.6, and Lemma 3.7 hold for structurally full rank, symmetric matrices. In particular, the fine decomposition of such a matrix has square irreducible blocks with a row set R_i and a column set C_i where either $R_i = C_i$ or there exists another irreducible block j with $R_j = C_i$ and $C_j = R_i$.

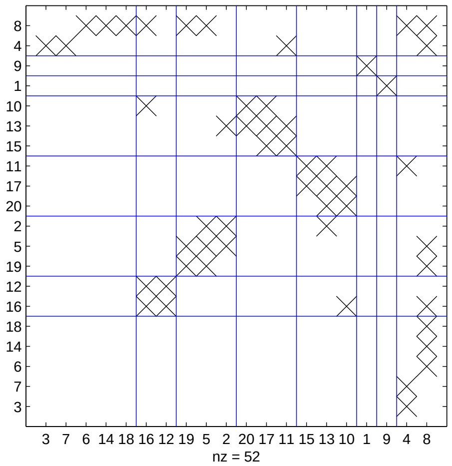

Fig. 3.3. *A* 20×20 *symmetric matrix is shown with the fine decomposition of the square block. Among the six irreducible blocks of* A_S*, two have the same set of rows and columns, i.e.,* $R_i = C_i$*. These appear as the anti-diagonal blocks. The other four come in pairs of two, where the pair is located symmetrically around the anti-diagonal. The permuted matrix is symmetric around the anti-diagonal.*

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