A computational journey into nonlinearity

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The talk to be presented is about the domain of *mathematical computation* which extends *beyond modern calculus* and *classical analysis* when numbers are not restricted to belong to a commutative field. It describes the dynamics of complexification, resulting in an endless remorphing of the computational landscape. Nonlinear computation weaves a colourful tapestry always in a state of becoming. In the process, some meta-principles emerge which guide the autonomous evolution of mathematical computation. These organic principles are essential keys to analyze very large numerical simulations of unstable phenomena: they lie at the heart of the new theory of Qualitative Computing.

What is Qualitative Computing? It is the newly developed branch of mathematical analysis which looks specifically at how the laws of classical analysis (Euler-Cauchy-Riemann) are modified when mathematical computation does not take place in a *commutative field*. Most analysis text books do not consider numbers beyond $\mathbb{R}$ or $\mathbb{C}$, with respective dimension(s) 1 or 2. However, there are important practical domains where such an approach is *too limited*. For example, the quaternions which form a *non*commutative field $\mathbb{H}$ of numbers with 4 real dimensions are the language of Maxwell's electromagnetism, and of special relativity. In the booming field of numerical linear algebra, the basic "numbers" are often taken to be square matrices which belong to a noncommutative associative *algebra* (over $\mathbb{R}$ or $\mathbb{C}$). This is an essential key to the successes of modern numerical software packages like Lapack and Scalapack used worldwide for intensive computer simulations in high tech industries.

The general consensus among mathematicians and physicists at the end of the 19th century was that complex numbers – $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$ – were good enough for every day science. Scientists feared that one could only lose computing power by dropping such properties for multiplication as commutativity or associativity, which were viewed then as essential. F. Klein and Lord Kelvin fiercely attacked Hamilton's quaternions. But theoretical physics has clearly vindicated Hamilton's non commutative field in the 20th century by adopting Clifford algebras C_k, $k \geq 3$, $C_2 = \mathbb{H}$. However, such algebras – heavily used in physics and algebraic geometry – cannot exploit the power of multiplication to its fullest for intrinsic reasons related to their being *associative*.

Therefore one wonders: does a family of multiplicative algebras A_k exist, which does not hinder the computing capabilities of multiplication?

Amazingly enough, the answer is *yes*. It consists of the little-known Dickson algebras A_k of dimension 2^k, $k \geq 0$ (with $A_k = C_k$ for $k \leq 2$), where the multiplication is defined recursively, being *nonassociative* for $k \geq 3$. At the dawn of the 20th century, vectors of dimension 2^k in A_k, $k \geq 2$, have been called *hypercomplex* numbers (Hurwitz, Dickson). And accordingly, computation in A_k, $k \geq 2$, was called hypercomputation.

The talk will show the extent to which hypercomputation in A_k, $k \geq 3$, is *unconventional*, plagued/blessed as it is by computing paradoxes signaling a clash between local (linear) and global (non linear) computation. An important source of paradoxes is found in the act of *measurement*. Let us consider the multiplication map defined by $a \neq 0$, that is $L_a : \ x \mapsto a \times x$, which is a linear map in A_k. For $k \leq 3$, L_a has for unique singular value the euclidean norm $\|a\| > 0$; but for $k \geq 4$, there can exist 2^{k-3} distinct singular values ≥ 0 which differ from $\|a\|$. Moreover the results of the Singular Value Decomposition (if computed inductively) may depend on the computational route, and may even be hypercomplex and uncountable! This is one of the surprises that the Fundamental Theorem of Algebra keeps in store when set in noncommutative algebras. The internal clockwork of hypercomputation is guided in part by such measures which modify the local 3D-geometry defined at a. This results in an expanded logic which provides an arithmetic basis for the emergence of simplexity in life's complex processes, and in highly unstable phenomena.

The computational journey into nonlinearity in the framework of Dickson algebras is *endless*. At every level $k \geq 4$, one gets new vistas, each richer than before. We offer glimpses of the ever changing territory. New computational principles emerge at each level $k \geq 2$ which may supersede some others valid at a lower level $k' < k$. For example, if we drop commutativity in $\mathbb{H}$ ($k = 2$) then the discrete can emerge from the continuous by exponentiation (a generalization of $e^{ni\pi/2} = i^n$). Without associativity ($k \geq 3$), there are several different ways to compute the *multiplicative measures* of vectors which may agree only partially with each other. This creates paradoxes and new options as well. As a rule, the emergence of paradoxes goes hand in hand with an increase in the freedom of choice. This freedom of choice provides a rational basis for the many fuzzy phenomena encountered in experimental sciences at a small scale: they are currently attributed to randomness, as in statistical physics, quantum mechanics, or genetic mutation. However, the proverbial God (i.e. the computing spirit) does not play dice in mathematical computation, but rather offers an ever richer variety of computational options to *choose* from. Hypercomputation supports the old adage: "Variety is the spice of life."

Caveat. The words "hypercomputation", "computability" and "complexity, complexification" are used in their classical mathematical sense. They should not be confused with the same words used in Computer Science. In this specific context, the words applied to programs for Turing machines acquire a meaning which differs greatly from the mathematical one.

The emergence of new mathematical concepts under the evolution pressure of mathematical computing is a recurring phenomenon since Antiquity. For example, irrational numbers, zero and its inverse ∞, negative numbers and complex numbers were finally accepted by our ancestors only after much anguish, inner turmoil and heated debate. Qualitative Computing has been the driving force behind the evolution of mathematical logic from the beginnings, when Pythagoras, and Euclid presented the first known incompleteness result, the proof of the irrationality of $\sqrt{2}$. It is a fact of experience that the classical logic of Aristotle is too limited to capture the dynamics of nonlinear computation. Mathematics provides us with the missing tool, an organic logic (based on $\{\mathbb{R}, \mathbb{C}, \infty\}$) which is tailored on the dynamics of nonlinearity. This organic logic can tame the computing paradoxes stemming from measurements in the absence of associativity; it represents the internal clockwork of computation. It makes full use of the computing potential of rings of numbers with 1,2,4 and 8 dimensions. One salient feature is that the cooperation of results by Fermat, Euler, Riemann and Sierpiński explains the autonomous complex dynamics of the Picard iteration to solve $x = rf(x)$, where $f: \mathbb{R} \to \mathbb{R}$ is continous and r is a real parameter. The necessity to limit the frame of interpretation to 3 dimensions at most brings to light some mechanisms by which computation turns the complex into the simple without reduction.

A detailed technical presentation is provided in the speaker's book *"Qualitative Computing: a computational journey into nonlinearity"*, currently in press at World Scientific, Singapore.

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