Numerical information processing under the global rule expressed by the Euler-Riemann ζ function defined in the complex plane

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When *nonzero*, the ζ function is intimately connected with numerical information processing. Two other functions play a key role, namely $\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s}$ and $\lambda(s) = \sum_{n\geq 0} \frac{1}{(2n+1)^s}$. The paper opens on a survey of some of the seminal work of Euler (1749) and of the amazing theorem by Voronin (1975).

The technical core is an elementary analysis based on the distances of the three complex numbers z, $z/2$ and $2/z$ to 0 and 1. The results applied to ζ, η and λ shed a new epistemological light about a much researched topic (in particular the critical line). We conclude by a cognitive interpretation related to ζ of the Fourier transform for complex signals.

Keywords: information processing, ζ function, η function, Euler's transformation, universality theorem of Voronin, critical line, Fourier transform.

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Life is the dynamical system *par excellence*, which ceaselessly processes information of various kinds in a context–dependent fashion. A look at the historical development of mathematical computation over millennia suggests that numbers can be useful analytic tools to study the polymorphic information processing accomplished by living organisms. In vector form, numbers in $\mathbb{R}^{2^k}$, $k \geq 0$, are endowed with two basic operations: addition and multiplication. The autonomous evolution of nonlinear computation in multiplicative nonassociative ($k \geq 3$) algebras is ruled by an organic logic which extends the classical one.

We present a metric aspect of this broader logic which rules the interpretation of all computations, chaotic or not. It is intimately connected with the ζ function of Euler and Riemann.

I. INTRODUCTION

Calculations with numbers (which can be integral, rational, real, complex or quaternions) play a key role in the scientific attempt to decipher the world. Important aspects of the world have been revealed by the interplays between the discrete and the continous, as well as between the real and the complex, both at work in mathematical computation [1] Chapters 8 and 10. These two interactions are present *simultaneously* in the celebrated function $\zeta : s \mapsto \sum_{n\geq 1} \frac{1}{n^s}$ which was defined for $s \in \mathbb{C}\backslash\{1\}$ by Riemann (1859). An english translation of Riemann's epoch-making paper is provided in the Appendix of [2]. It states the Riemann Hypothesis (RH) that all complex zeros of ζ satisfy $\Re s = \frac{1}{2}$. Because of the arithmetic connection between ζ and prime numbers, the study of the nontrivial zeros of ζ has been by far the primary topic of investigation in all publications about ζ after 1859. In the eyes of the researchers, the "prime obsession" reduces the interest of ζ to its zero set only.

In this paper we propose a less fashionable – albeit broader – look at ζ. We describe the role played in numerical information processing by ζ and by the following two functions derived from ζ: the entire function $s \mapsto \eta(s) = \sum_{n\geq 1} (-1)^{n+1} \frac{1}{n^s}$ and $\lambda = \frac{1}{2}(\zeta + \eta)$ defined on $\mathbb{C}\backslash\{1\}$. This more global view is rooted in the seminal work of the great Euler which spent 42 years (1730 to 1772) thinking about ζ defined at *integral* values $k \in \mathbb{Z}\backslash\{1\}$ [3].

Throughout the text, $s = p + iq$ denotes an arbitrary complex number. Particular cases include $s = p$ real or $s = k \in \mathbb{Z}$.

II. A GLIMPSE OF EULER'S ORIGINAL PERSPECTIVE

While working on ζ, Euler introduced the two auxilliary functions, now known as the Dirichlet functions:

- $\eta(s) = \sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^s}$, called the alternating ζ function.
- $\lambda(s) = \sum\limits_{n\geq 0} \dfrac{1}{(1+2n)^s}$ which defines the arithmetic mean $\frac{1}{2}(\zeta + \eta)$ for $s \neq 1$.

The original notations of Euler [4] are ϕ for η and θ for λ. He only considers $s = k \in \mathbb{Z}$.

A. About ζ and its alternating variant

The series η (resp. ζ) is convergent (resp. absolutely convergent) for $p > 0$ (resp. $p > 1$); it is Abel summable (resp. defined by analytic continuation) for $p \leq 0$ (resp. $p \leq 1$, $s \neq 1$). It follows that η (resp. ζ) is an entire (resp. holomorphic) function defined for $s \in \mathbb{C}$ (resp. $s \in \mathbb{C}\backslash\{1\}$ with 1 as a simple pole). Moreover

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \quad s \in \mathbb{C}. \tag{1}$$

B. Rewriting η by Euler's transformation

Let be given a convergent series A of complex numbers a_n with alternating signs: $A = \sum\limits_{n\geq 1}(-1)^{n+1}a_n$.

For $n \geq 1$ we define the forward difference $\Delta^k a_n$ by the iterative scheme: $\Delta^0 a_n = a_n$, $\Delta^k a_n = \Delta^{k-1}a_n - \Delta^{k-1}a_{n+1} = \sum\limits_{m=0}^{k}(-1)^m C_k^m a_{n+m}$ for $k \geq 1$.

Hence $A = \sum\limits_{j=0}^{k-1} \dfrac{\Delta^j a_1}{2^{j+1}} + \sum\limits_{n\geq 1}(-1)^{n-1}\dfrac{\Delta^k a_n}{2^k}$, $k \geq 1$.

Now the sum of the series $\sum\limits_{n\geq 1}$ converges to 0 as $k \to \infty$ [5]. Therefore A can be rewritten as $A = \sum\limits_{j\geq 0} \dfrac{\Delta^j a_1}{2^{j+1}}$. This rewriting of A is known as Euler's transformation.

If we apply the recursive formula to $\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s}$ (corresponding to $a_1 = 1^{-s}$, $a_m = m^{-s}$) we get:

$$\eta(s) = \sum_{j\geq 0} \frac{\Delta^j 1^{-s}}{2^{j+1}} = \sum_{j\geq 0} \frac{1}{2^{j+1}} \sum_{m=0}^{j} (-1)^m C_j^m (m+1)^{-s}. \tag{2}$$

The formula (2) is a globally convergent formula for $\eta(s)$ [6] which provides readily the analytic continuation for $\zeta(s)$ for all complex $s \neq 1$. It is also easy to prove that for $s = -k < 0$, the series in (2) becomes *finite*: $\eta(-k) = \sum_{j=0}^{k} \frac{\Delta^j 1^k}{2^{j+1}}$ [7].

C. The mean $\lambda(s) = \frac{\zeta(s)+\eta(s)}{2}$

The treatment of $\lambda(s)$ is analogous to that of $\zeta(s)$ in $\mathbb{C}\backslash\{1\}$. In particular,

$$\lambda(s) = (1 - 2^{-s})\zeta(s), \quad s \neq 1. \tag{3}$$

III. THE "UNIVERSALITY" THEOREM OF VORONIN (1975)

A. The theorem

In 1975, the Russian mathematician Voronin proved the following theorem [8]:

> Let be given $\varepsilon > 0$ and any holomorphic function f which is nonzero and continuous on the open disk $\{s;\ |s| < r\}$, $0 < r < \frac{1}{4}$. There exists a real t, depending on f and ε such that $\max_{|s|<r} \left|\zeta(s + \frac{3}{4} + it) - f(s)\right| < \varepsilon$.

It follows that, through a simple translation and rescaling procedure, the nonzero behaviour of *any* analytic function on *any* given disk in $\mathbb{C}$ can be reproduced with *arbitrary* precision by the ζ function acting on an appropriate disk of radius $< 1/4$ in the right half of the critical strip.

B. Significance for information processing

This extraordinary result reveals the universal nature of the information encoded in ζ, and opens radically new perspectives on the role of ζ in information processing. It is perplexing that the message of ζ relayed by Voronin has not yet been heard by most scientists. Among the few listeners, one can cite Woon at Cambridge, UK. The reference[9] applies Voronin's theorem to the holomorphic function ζ itself ($s \neq 1$). This shows that ζ is a fractal which displays self-similarities. It encodes versions of itself at different scales. However ζ cannot be infinitely recursive because it is smooth (infinitely differentiable). This property is shared by η and λ.

This line of thought is developed in [10] wich displays the Julia and Mandelbrot sets associated with the map $F(a, s) = \zeta(s) + a$, for $(a, s) \in \mathbb{C} \times (\mathbb{C}\backslash\{1\})$.

C. The fractal dimension of the set of zeros such that $p = 1/2$

It is known that the complex zeros of ζ live in the critical strip $0 < p < 1$, so that $\zeta = \eta = \lambda = 0$. Because η is an entire function, it has at most countably many zeros. It has been checked numerically that the computed zeros (a total of 10^{13} in 2004) for ζ in the critical strip all lie on the critical line. O. Shanker [11] has found computational evidence that the Riemann zeros on the critical line form a fractal set with the surprisingly large fractal dimension $D = 1.9$. The result should be contrasted with the theoretical value 0 for the topological dimension of a countable set.

IV. THE AUXILIARY TOOLS η AND λ

A. The dependence of η and λ on ζ

The ratios $\frac{\eta}{\zeta}$ and $\frac{\lambda}{\zeta}$ at $s \neq 1$ are given respectively by the functions $1 - 2^{1-s}$ and $1 - 2^{-s}$, cf (2) and (3).

Therefore, the relative differences are $\frac{\zeta-\eta}{\zeta} = 2^{1-s}$ and $\frac{\zeta-\lambda}{\zeta} = 2^{-s}$.

If we set $z = 2^{1-s}$, we recognise the four numbers $z, z/2, 1-z$ and $1-z/2$. In the context of a *group* structure put on complex numbers their moduli express the 4 distances of z and $\frac{z}{2}$ to either the origin 0, i.e. the neutral element for the *additive* group, or the unit 1, i.e.

the identity element for the *multiplicative* group. To further the metric analysis related to ζ, we are lead to the study of the connection between these moduli for $z = 2^{1-s}$. Because of the importance of the functional equation which relates $\zeta(s)$ to $\zeta(1-s)$ for $s \notin \{0,1\}$, we shall also consider the third number $2^s = \frac{2}{z}$. In the next section, we consider more generally $z = \rho e^{i\theta}$, $\rho > 0$, $\theta \in [0, 2\pi[$.

B. The evolution of $|1-z|^2$ when $|z|$ is halved.

We set $z = \rho(\cos\theta + i\sin\theta)$ with $\gamma = \cos\theta$. Thus $1 - z = 1 - \gamma\rho - i\rho\sin\theta$ and $|1-z|^2 = 1 + \rho^2 - 2\gamma\rho = N$. Similarly for $z' = \frac{z}{2}$, $|1-z'|^2 = 1 + \frac{\rho^2}{4} - \gamma\rho = \frac{1}{4}(4 + \rho^2 - 4\gamma\rho) = \frac{D}{4}$.

We define the function

$$(\rho, \gamma) \mapsto \alpha(\rho, \gamma) = \left|\frac{1-z}{1-z/2}\right|^2 = 4\frac{\rho^2 - 2\gamma\rho + 1}{\rho^2 - 4\gamma\rho + 4} = 4\frac{N}{D} \tag{4}$$

which is defined on $(\mathbb{R}^+ \times [-1,1])\backslash\{2\}$, because $D = 0$ iff $z = 2$, that is $\rho = 2$, $\gamma = 1$. The variation of α with $\rho > 0$ and $-1 \le \gamma \le 1$, γ fixed, is described below.

Lemma IV.1 *For any γ in $[-1,1]$, $\alpha(\sqrt{2}, \gamma) = 2$ and $\alpha(0, \gamma) = 1$.*

Proof. Set $\alpha = \alpha(\rho,\gamma)$: $\rho^2 - 2\gamma\rho + 1 = \alpha(\frac{\rho^2}{4} - \gamma\rho + 1)$, thus $(1 - \frac{\alpha}{4})\rho^2 - (2-\alpha)\gamma\rho + 1 - \alpha = 0$. If $\alpha = 2$, the dependence on γ disappears and $\rho^2 = \frac{1}{1-1/2} = 2$. Moreover $\frac{\partial\alpha}{\partial\gamma} = \frac{-8\rho}{D^2}(2-\rho^2) = 0$ for $\rho = 0$ and $\rho = \sqrt{2}$. □

Theorem IV.2 *The function $\rho \mapsto \alpha(\rho,\gamma)$ is such that:*

1) $1 \le \alpha < 4$ for $-1 \le \gamma \le 0$,

2) α belongs to $[\alpha_1(\gamma), \alpha_2(\gamma)]$ for $0 < \gamma < 1$, such that $0 < \alpha_1(\gamma) < 1$ and $4 < \alpha_2(\gamma) < \infty$,

3) $\alpha \in [0, +\infty]$ when $\gamma = 1$.

Proof. Use $\frac{\partial\alpha}{\partial\rho} = \frac{8}{D^2}[(\rho-\gamma)D - (\rho - 2\gamma)N] = \frac{8}{D^2}[-\gamma\rho^2 + 3\rho - 2\gamma] = \frac{8}{D^2}q$. The polynomial q has no positive root for $\gamma \le 0$.

For $\gamma > 0$, the two positive roots are ρ_1 and ρ_2: $0 < \rho_1(\gamma) = \frac{1}{2\gamma}(3 - \sqrt{9 - 8\gamma^2} \le 1 \le 2 \le \rho_2(\gamma) = \frac{1}{2\gamma}(3 + \sqrt{9-8\gamma^2}$, We observe for future reference that the product $\rho_1(\gamma)\rho_2(\gamma) = 2$ for any $1 > \gamma > 0$. And $\lim_{\gamma\to 0}(\rho_1, \rho_2) = (0, \infty)$ and $\lim_{\gamma\to 1}(\rho_1,\rho_2) = (1,2)$.

Accordingly $\lim_{\gamma\to 1}(\alpha_1,\alpha_2)=(0,\infty)$, with $\alpha_1(\gamma)=\alpha(\rho_1,\gamma)\in]0,1[$ and $\alpha_2(\gamma)=\alpha(\rho_2,\gamma)\in\,]4,\infty[$. $\square$

This theorem tells us that, when the complex plane is considered multiplicatively as $\mathbb{C}$ (Euler) and not only additively as $\mathbb{R}^2$ (Descartes), there is a significant metric difference between the positive axis $\mathbb{R}^+$ and the cut plane $\mathbb{C}\backslash\mathbb{R}^+$. If $z=\rho$ is halved, $\alpha(\rho,1)$ describes $[0,+\infty[$.

But when we impose $\gamma\neq 1$ (i.e. $\theta\neq 0$) then $\alpha(\rho,\gamma)$ describes a closed bounded interval $[\alpha_1(\gamma),\alpha_2(\gamma)]\supseteq[1,4]$.

Moreover, the end points α_1 and α_2 are *invariant* at the values 1 and 4 when $-1\leq\gamma\leq 0$, that is $\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$: z belongs to the left half-plane.

C. The hyperbolic maps $\rho>0\mapsto h(\rho)=\frac{2}{\rho}$

For $z''=\frac{2}{z}$, $|1-z''|^2=1+\frac{4}{\rho^2}-4\frac{\gamma}{\rho}=\frac{1}{\rho^2}D$.

We set

$$(\rho,\gamma)\mapsto\beta(\rho,\gamma)=\left|\frac{1-z}{1-2/z}\right|^2=\frac{\rho^2}{4}\alpha(\rho,\gamma)\tag{5}$$

and $\quad(\rho,\gamma)\mapsto r(\rho,\gamma)=\left|\frac{1-z/2}{1-2/z}\right|^2=\frac{\rho^2}{4}$.

The relation $\frac{\beta}{\alpha}=r=\frac{\rho^2}{4}$ holds for any γ, $|\gamma|\leq 1$, yielding for example $r=1$ (resp. $1/2$) iff $\rho=2$ (resp. $\sqrt{2}$).

For $-1\leq\gamma\leq 0$, α increases monotonically with ρ; so does β from 0 to $+\infty$.

We define $\rho\mapsto\alpha_\gamma(\rho)=\alpha(\rho,\gamma)$ for $|\gamma|\leq 1$.

Theorem IV.3 *For any γ, the function α satisfies*

$$\alpha(\rho,\gamma)\alpha(h(\rho),\gamma)=4\tag{6}$$

Equivalently, $\alpha_\gamma\circ h=2h\circ\alpha_\gamma$ for $|\gamma|\leq 1$.

Proof. Let $z\neq 2$ become $\frac{2}{z}$. Then N(resp. D) becomes $\frac{D}{\rho^2}$ (resp. $\frac{4N}{\rho^2}$). $\alpha(1,\gamma)=8\frac{1-\gamma}{5-4\gamma}$ and $\alpha(2,\gamma)=4\frac{1}{\alpha(1,\gamma)}$ for $\gamma\neq 1$. The relation (6) for $z=2$ yields $0\times\infty=4$. For $0<\gamma<1$, $\rho_1(\gamma)\rho_2(\gamma)=2$ hence $\alpha_1(\gamma)\alpha_2(\gamma)=4$.

Observe that $h \circ h = 1$: $\rho \mapsto \frac{2}{2/\rho}$. We derive (6) and $h \circ (\alpha_\gamma \circ h) = \frac{1}{2}\alpha_\gamma$. Similarly $\beta_\gamma \circ h = \frac{1}{2} h \circ \beta_\gamma$, thus $h \circ (\beta_\gamma \circ h) = \beta_\gamma$. $\square$

The fixed-point for h is $\rho = \sqrt{2}$. At this value we get $\alpha(\sqrt{2}, \gamma) = 2$, $\beta(\sqrt{2}, \gamma) = 1$ and $r(\sqrt{2}, \gamma) = \frac{1}{2}$ for any γ.

V. AN APPLICATION TO ζ

A. $z = 2^{1-s}$, $\frac{z}{2} = 2^{-s}$, $\frac{2}{z} = 2^s$

We go back to the functions ζ, η, λ, which correspond to $z = 2^{1-s} = 2^{1-p-iq} = 2^{1-p}e^{-iq\ln 2}$.

Therefore $\rho = 2^{1-p}$, $p = \log_2 \frac{2}{\rho}$, $\theta = -q \ln 2 \pmod{2\pi}$, and $\left|\frac{\eta(s)}{\lambda(s)}\right|^2 = \alpha(\rho, \gamma)$. The correspondence between ρ and $p = \Re s$ is given by the following table

	ρ	0	ρ_1	1	$\sqrt{2}$		2	ρ_2	∞
	p	∞		1	1/2		0		$-\infty$
$-1 \leq \gamma \leq 0$	α	1	$\nearrow$		2		$\nearrow$		4
$0 < \gamma < 1$	α	$1\searrow$	α_1	$\nearrow$	2		$\nearrow$	$\alpha_2\searrow$	4
$\gamma = 1$	α	$1\searrow$		$0\nearrow$	2	$\nearrow$	$\infty\|\|\infty$	$\searrow$	4

Lemma IV.1 reveals that, on the critical line $p = 1/2$, $\alpha(\sqrt{2}, \gamma) = 2$ for any $q = \Im s$. When p decreases from 1 to 0 in the critical strip, the function α is always increasing, passing through the value 2 for $p = \frac{1}{2}$.

The zeros of η and λ for which $\zeta \neq 0$ are defined respectively by

- $1 - 2^{1-s} = 0$, $s \neq 1$, i.e. $s - 1 \in (2i\pi \ln 2)\mathbb{Z}^*$ ($\eta(1) = \ln 2 \neq 0$), $\rho = 1$ and $\gamma = 1$,
- $1 - 2^{-s} = 0 \Longleftrightarrow s \in (2i\pi \ln 2)\mathbb{Z}$, $\rho = 2$ and $\gamma = 1$.

For all $s \neq 1$ such that $\zeta \neq 0$ and $\eta(s) = 0$ (resp. $\lambda(s) = 0$), $\alpha(1, 1) = 0$ (resp. $\alpha(2, 1) = \infty$).

B. Cauchy integrals around the common zeros of η, λ and ζ

Let $u = \sigma + it$ be a zero of ζ: $\zeta(u) = 0$ with either $u = -2k$ or $0 < \Re u < 1$. The Cauchy integral of $\frac{1}{\zeta}$ around u defines the complex number $\mathcal{C}(\zeta, u)$. Accordingly, the Cauchy integral of $\frac{1}{\eta}$ and $\frac{1}{\lambda}$ around u yield the two numbers $\mathcal{C}(\eta, u) = \frac{1}{1-2^{1-u}}\mathcal{C}(\zeta, u)$ and $\mathcal{C}(\lambda, u) = \frac{1}{1-2^{-u}}\mathcal{C}(\zeta, u)$. It follows readily that $\left|\frac{\mathcal{C}(\lambda,u)}{\mathcal{C}(\eta,u)}\right|^2 = \left|\frac{1-2^{1-u}}{1-2^{-u}}\right|^2 = \alpha(g, \cos\psi)$ for $v = 2^{1-u} = ge^{i\psi}$.

For the trivial zeros $u = -2k$, $k \in \mathbb{N}^*$, $v = 2^{1+2k} = g \geq 8$. Hence $\left(\frac{7}{3}\right)^2 = 5.44.. \geq \alpha(g, 1) = 4\left(\frac{g-1}{g-2}\right)^2 > 4$.

By comparison, $\alpha(\sqrt{2}, \gamma)$ remains *invariant* at the value 2 for all zeros lying on the critical line. If RH is true, the *emergence* of new numbers by Cauchy integration around complex zeros corresponds to $\alpha = 2$, hence $\beta = 1$ and $r = 1/2$ idependently of γ.

VI. COMPLEX SIGNALS, FOURIER TRANSORM, AND COGNITION

Since its discovery at the beginning of 19th century, the Fourier transform has played an increasing role in the scientific understanding of the world. Our technological society is dominated by digital filtering. In the mid-20th century, engineers switched from analogue to digital computing machines. A few decades later, all coder-decoder equipments are digital.

This rapid evolution over two centuries suggests to look at the Fourier transform under the light of cognition. The mathematical reference for what follows is [12] . The original Fourier series notion applies to periodic functions. The notion can be extended to non periodic functions by means of the Fourier integral.

A. The Fourier integral

We consider complex functions of the type $f : \mathbb{R} \to \mathbb{C}$ in $L^2 = L^2(\mathbb{R})$, the Hibert space of Lebesgue-measurable functions for which $\int_{\mathbb{R}} |f(t)|^2 dt < \infty$.

The scalar product on L^2 is a sesquilinear form on $\mathbb{C}$ defined by $< f, g >= \int_{\mathbb{R}} f(t)\overline{g(t)}dt$. The non negative quantity $\|f\|_2 =< f, f >^{1/2}$ is the L^2-norm of f. Let $f \in L^2$, its formal continuous Fourier transform is $T_F(f)$ defined by the integral

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-itx}dt, \ x \in \mathbb{R}.$$

It is well-known that the Fourier transform $T_F : f \mapsto \hat{f}$ is a unitary linear operator on L^2 such that $T_F^4 = I$, the identity operator on L^2. If $f \neq 0$ is an eigenfunction such that $T_F(f) = \lambda f$, $\lambda \in \mathbb{C}$ then $\lambda^4 = 1$ and $\lambda \in \{\pm 1, \pm i\}$: these are the 4 eigenvalues of T_F (the point spectrum $P_\sigma(T_F)$). Because T_F^* satisfies $f(t) \mapsto \hat{f}(-x)$, then $f(t) \overset{T_F^*}{\mapsto} \hat{f}(-x) \overset{T_F}{\mapsto} f(t)$ shows that $T_F^* \circ T_F = I$ and T_F is unitary. As a consequence the spectrum of T_F belongs to the unit circle.

B. The spectral analysis of T_F

∂ denotes the derivation operator $f \mapsto f'$, ϖ denotes the product operator $f(t) \mapsto t f(t)$, I denotes the identity operator $f \mapsto f$. We set $E = \partial^2 - \varpi^2$ and consider the linear differential equation

$$E = \mu I, \quad \mu \in \mathbb{C} \tag{7}$$

which is invariant under the Fourier transform.

Classically[12] the equation (7) has a continuous solution iff $\mu = -(2n+1)$, $n \in \mathbb{N}$. The solution ϕ_n corresponding to $\mu = -(2n+1)$ is equal, up to a real constant factor, to $\phi_n(x) = H_n(x) e^{-x^2/2}$, where $H_n(x)$ is the Hermite polynomial with leading term $(2x)^n$. Moreover, $T_F(\phi_n(t)) = (-i)^n \phi_n(x)$.

The Hermite polynomial H_n of degree n satisfies for $n \geq 0$

1) the differential equation

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0, \; H_0(x) = 1, \; H_1(x) = 2x$$

2) the recurrence relation

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0, \quad n \geq 1.$$

Moreover $H_n(0) = 0$ for n odd [Hermite (1864)]. The functions $\phi_n(x) = H_n(x) e^{-x^2/2}$ are called Hermite functions. The set $\{\phi_n, \; n \in \mathbb{N}\}$ is a basis for L^2, that is a complete set of orthogonal functions. ϕ_n satisfies $T_F(\phi_n(t)) = (-i)^n \phi_n(x)$ with $(-i)^n \in \{\pm 1, \pm i\}$. In other words, ϕ_n is an eigenfunction for T_F associated with the n^{th} eigenvalue $(-i)^n$. There are 4 different eigenvalues as $n = 4p + j$, $p \in \mathbb{N}$: $(-i)^n = e^{-in\pi/2} = e^{-ij\pi/2}$, $j = 0$ to 3.

C. The scalar product $<\partial_t f, \varpi_t f> = \sigma(f)$

We assume that f is smooth enough ($f' \in L^2$) and such that

$$\lim_{|t|\to\infty} \sqrt{|t|} f(t) = 0. \tag{8}$$

The condition (8) is satisfied when $|f|$ decreases at ∞ faster than $\frac{1}{|t|^{1/2}}$. This is true if $f(t) = O(\frac{1}{|t|^p})$, $p > 1/2$. $\sigma(f)$ denotes the scalar product $<\partial_t f, \varpi_t f>$ between the two functions obtained by applying ∂ and ϖ to a function f satisfying (8). Such functions are complex signals ruled at ∞ by $\zeta(s)$, with $p > 1/2$.

In general $\sigma(f)$ is a complex number. We assume below that $f \neq 0$ is either real or pure imaginary, so that σ is real-valued. Such functions satisfying (8) form the subset Σ of L^2 with zero Lebesgue measure. Then $\sigma(f) = -\frac{1}{2}\|f\|_2^2 < 0$ (resp. $\frac{1}{2}\|f\|_2^2$) when $f \neq 0$ is real-valued (resp. pure imaginary) in Σ.

Moreover, the scalar product $\sigma(f)$ takes its minimum value $-\|\partial_t f\|_2 \|\varpi_t f\|_2$ for the real function $f(t) = Ce^{-(t/2)^2}$, $C \in \mathbb{R}$. The two cognitive operators ∂ and ϖ are colinear. We denote $\mathcal{N} = \{f(t) = Ce^{-(t/2)^2}, t \in \mathbb{C}\}$ and $\mathcal{N}(A) = \{f \in \mathcal{N}, C \in A \subset \mathbb{C}\}$.

By Cauchy's inequality on $|\sigma(f)| = \frac{1}{2}\|f\|_2^2$, we conclude that $\|\partial_t f\|_2 \|\varpi_t f\|_2 \geq \frac{1}{2}\|f\|_2^2$ for $f \in \Sigma$. One may consider the normalized function $\varphi = \frac{f}{\|f\|_2}$, for $f \neq 0$. The lower bound takes the form $\|\partial_t \varphi\|_2 \|\varpi_t \varphi\|_2 \geq \frac{1}{2}$. It is achieved for $\varphi \in \mathcal{N}(\mathbb{R} \cup i\mathbb{R})$.

In experimental sciences (Quantum Physics, Statistics, signal processing) this bound is often referred to as an *uncertainty principle* which states that the two norms cannot be simultaneously arbitrarily small.

This principle is a simple consequence of the assumption (8) put on $f \in \Sigma$: it is clear that $|\sigma(f)| = \frac{1}{2}\|f\|_2^2$ cannot be 0 unless $f = 0$. If one weakens the constraint (8), it is possible for certains classes of signals associated with $\zeta(s)$, $p \leq 1/2$, to define a *weighted* scalar product $<f, g>_r = \int_{\mathbb{R}} r(t) f(t) \overline{g(t)} dt$, where r is a positive function. With the weight function $r(t) = e^{-t^2}$, the alternative scalar product $<\partial f, \varpi f>_r$ can be zero, meaning that the cognitive operators ∂ and ϖ are orthogonal at f in this new metric. Observe that the weight e^{-t^2} acts as a microscope focused on 0, which dampens what happens to f near $\pm\infty$.

VII. CONCLUSION

We have reviewed some of the reasons which indicate – not too surprisingly – why the importance of $\zeta \neq 0$ in numerical information processing exceeds by far the mystifying connection between its complex zeros and the distribution of prime numbers. The proof of the RH would not exhaust the mystery of the ζ function and its wonderfully subtle connections with numbers of many sorts.

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