# Polymorphic Information Processing in weaving computation: An approach through cloth geometry

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**Abstract :** This report extends the scope of computation with a non standard $\times$ to the more basic case of a non standard $+$, where standard means associative and commutative. Two physically meaningful examples of a nonstandard $+$ are provided by the observation of motion in Special Relativity, from either outside (3D) or inside (2D or more), We revisit the "gyro"-theory of Ungar to present the polymorphic information processing which is created by cloth geometry, a relating computational construct framed in a normed vector space, and based on a non standard $\oplus$ whose commutativity and associativity are ruled (woven) by a relator.

**Keywords** : relator, noncommutativity, nonassociativity, induced addition, organ, metric cloth, weaving Information Processing, cloth geometry, hyperbolic geometry, Special Relativity, liaison, metric entanglement, homotopic link, action at a distance, quantisation.

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# 1 About Relating Computation

## 1.1 Introduction

Hypercomputation, that is nonlinear computation in real multiplicative Dickson algebras $A_k \cong \mathbb{R}^{2^k}$, is developed in (Chatelin 2011 a). For $k \geq 2$ (resp. $k \geq 3$) multiplication is not commutative (resp. associative). However addition remains both associative and commutative.

The situation changes in an essential way when computation is merely linear but there exists a *relator* which rules the way any two numbers are to be *added*. This kind of relating computation will be defined in precise terms in Section 2. It includes the special case of an *explicit metric reference* consisting of a positive *finite* number $\lambda$, $0 < \lambda < \infty$. The classical structure of an abelian additive group is weakened by considering an addition whose commutativity and associativity are controlled by the relator. A physically meaningful example was provided a century ago by 3D-Special Relativity (Einstein) where the role of $\lambda$ as a metric reference is played by $c$, the speed of light, and the relator is a plane rotation.

## 1.2 Special Relativity in the early days

It was soon recognised that hyperbolic geometry underlies Einstein's law of addition for admissible velocities (Varičak 1910, Borel 1914) creating the relativistic effect known today as *Thomas precession* (Silberstein 1914, Thomas 1926).

But, despite Maxwell's valiant efforts (Maxwell 1871), Hamilton's noncommutative $\times$ of 4-vectors was still unacceptable for most scientists at the dawn of the 20th century. Therefore Einstein's noncommutative $+$ of 3-vectors (representing relativistically admissible velocities) was fully inconceivable: Einstein's vision was much ahead of its time! A version of Special Relativity with more appeal to physicists was conceived by Minkowski in 1907, by dressing up as physical concepts the Lorentz transformations in the field of quaternions $\mathbb{H}$ which had been introduced by Poincaré in 1905, see (Walter 1999, Auffray 2005). This is the version adopted until today in physics textbooks.

## 1.3 The mathematical revival in 1988

Einstein's intuition was left dormant for 80 some years until it was brought back to a new mathematical life in the seminal paper (Ungar 1988). During 20 years, Ungar has crafted an algebraic language for hyperbolic geometry which sheds a *natural light* on the physical theories of Special Relativity and Quantum Computation (Ungar 2008). Ungar's geometry, which is expressed in "gyrolanguage", is based on the key concepts of gyrator and gyrovector space. They are mathematical concepts abstracted from Thomas precession, a kinematic effect in 3D-special relativity. The *physical* effect was anticipated in (Borel 1913, 1914). As we shall see, these concepts find an equally natural use beyond physics, in the realm of computation ruled by a relator.

## 1.4 Geometric Information Processing in relating computation

The gyrolanguage is geared towards Hyperbolic Geometry and Physics. In this report, we export some of Ungar's tools developed for mathematical physics into mathematical computation in a *relating* context (Definition 2.1 below). The reward of the shift of focus from physics to computation is to gain insight about the *geometric* ways by which information can be organically processed in the mind when *relation* prevails. This processing exemplifies the computational thesis posited in (Chatelin 2011 a,b) by revealing geometric aspects of organic intelligence.

The change of focus entails some changes in the vocabulary which are signalled by a reference to the original gyroterm defined in (Ungar 2008). The reader will find all the necessary theoretical background for the presentation to follow in Ungar's work, mainly the 2008 book which is a goldmine.

## 1.5 Contents

# 2 Linear organic computation

## 2.1 Preliminaries

A *groupoid* $(S, \oplus)$ is a set $S$ of elements on which a binary operation called *addition* and denoted $\oplus$ is defined : $(a, b) \in S \times S \mapsto a \oplus b \in S$. An element 0 such that $0 \oplus a = a$ (resp. $a \oplus 0 = a$) is called a left (resp. right) *neutral*. An *automorphism* for $(S, \oplus)$ is a bijective endomorphism $\varphi$ which preserves $\oplus$: $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in S$. The set of automorphisms form a group (relative to composition $\circ$) denoted Aut $(S, \oplus)$ with the identity map $I$ as unit element.

The subtraction is denoted $\ominus$ : $a \ominus b = a \oplus (\ominus b)$.

## 2.2 Relators

We suppose that we are given a map:

$$rel \; : \; S \times S \to \text{Aut}\,(S, \oplus)$$
$$(a, b) \mapsto rel(a, b)$$

such that

$$rel(a \oplus b, b) = rel(a, b). \tag{A1}$$

A map *rel* satisfying the *reduction* axiom (A1) is called a *relator*. We set $\boldsymbol{R} = rel(S, S)$ for the range of the relator in Aut $(S, \oplus)$.

## 2.3 Organs for linear relating computation

We suppose that $\oplus$ satisfies the additional axioms:

$$a \oplus b = rel(a, b)(b \oplus a), \tag{A2}$$
$$a \oplus (b \oplus c) = (a \oplus b) \oplus rel(a, b)c, \tag{A3}$$

which express by means of $rel(a, b)$ a weak form of commutativity (A2) and associativity (A3).

The algebraic structure $(G, rel)$ consisting of the additive groupoid $G = (S, \oplus)$ and the relator *rel* is called an *organ*.

**Definition 2.1** *A linear relating computation refers to any algebraic computation taking place in an organ defined by the data* $\{\oplus, rel\}$ *satisfying the three axioms (A1), (A2), (A3).*

**Remark 2.3.1** In (Definition 2.7, Ungar 2008), the relator is called gyrator with $(A1) \iff (G5)$. Next $(A3) \iff (G3)$ is gyroassociativity and $(A2) \iff (G6)$ is gyrocommutativity which is optional in a gyrogroup. An organ is a gyrocommutative gyrogroup (Definition 2.8). And $\oplus$ is denoted either $+$ or $\oplus$ therein.

## 2.4 Some properties of the relator

The neutral 0 and the opposite $\ominus a$ are unique: left=right, and $a \ominus a = \ominus a \ominus a = 0$. The relator satisfies:

- $\ominus(a \oplus b) = rel(a,b)(\ominus b \ominus a)$, (Theorem 2.11)
  $= \ominus a \ominus b$ (Theorem 3.2)
- $rel^{-1}(a,b) = rel(\ominus b, \ominus a)$ (Theorem 2.32)
- $rel(b,a) = rel^{-1}(a,b)$ (Theorem 2.34)
  $= rel(a, \ominus rel(a,b)b)$ (Lemma 2.33)

- $$rel(\ominus a, a) = rel(\ominus a, a) = rel(0,a) = rel(a,0) = rel(0,0) = I \tag{2.1}$$

More in Table 2.2 (Ungar 2008).

The identities in (2.1) follow from the reduction axiom (A1). Because $\ominus a \oplus a = 0 \oplus 0 = 0$, $rel(\ominus a, a)$ and $rel(0,0)$ could be arbitrarily chosen in Aut $(S, \oplus)$. In full generality, 0 is a singularity with an indeterminate character. The indeterminacy disappears under the reduction axiom (A1).
We cite other consequences for (A1) which shed some light on organic computation in $G$. We assume below that the property $(\pi)$: $g \oplus g = 0 \iff g = 0$ holds for any $g \in G$. In a *multiplicative* algebra context, one would say that there exists no 2-torsion (Chatelin 2011 a). Observe that $(\pi)$ is satisfied in the Examples 2.1 to 2.3 given in Section 2.6.

**Proposition 2.1** *Given any* $g \neq 0$ *in* $G$*, there exists a unique half* $h$ *such that* $h \oplus h = h \hat{+} h = g$.

*Proof.* $\hat{+}$ is defined in (2.6) below. Use (A1) and $(\pi)$, see Theorem 3.34. □

There are two important corollaries (Ungar 2008):
(i) $rel(a,b) \neq \ominus I$ (Theorem 3.36),
(ii) $rel(a,b)b = \ominus b \Longrightarrow b = 0$ (Theorem 3.37).

## 2.5 The two basic equations associated with $\oplus$ and *rel*

Because $\oplus$ is not commutative we are led to consider $\mathcal{L} = \{L_a = a \oplus \cdot; a \in G\}$ and $\mathcal{R} = \{R_a = \cdot \oplus a; a \in G\}$ Left– (resp. right–) addition $\oplus$ is abbreviated $L\oplus$ (resp. $R\oplus$).
We consider the left and right linear equations associated with $a, b$ in $G$.

$$L_a x = a \oplus x = b, \tag{2.2}$$
$$R_a y = y \oplus a = b, \tag{2.3}$$

Each of them has the unique solution

$$x = \ominus a \oplus b, \tag{2.4}$$
$$y = b \ominus rel(a, b)a. \tag{2.5}$$

The equality (2.5) suggests to consider the composite map $\oplus rel(\cdot, \cdot)$ as an *induced* addition $\hat{+}$ defined by

$$(a, b) \in G \times G \mapsto a \,\hat{+}\, b = a \oplus rel(a, \ominus b)b. \tag{2.6}$$

The corresponding subtraction is denoted $\hat{-}$. Then (2.5) can be rewritten as $y = b \,\hat{-}\, a$.

Three properties are noteworthy:
- Aut $(S, \oplus) =$ Aut $(S, \hat{+})$,
- $\hat{+}$ is classically *commutative* (Theorems 2.38 and 3.4).
- $\ominus a = \hat{-} a$ by (2.1).

The concept of an *organ* is determined by the two data: $\oplus$ and the associated relator (as a map into the automorphisms for $\oplus$). In the relating perspective, the source notion is the *relator* which rules its associated addition $\oplus$. This addition precedes the secondary addition $\hat{+}$, which is induced by $R\oplus$ and the relator combined together.

The concept of an organ reduces to the classical concept of an abelian additive group when the primitive operation is associative and commutative (hence $\oplus = \hat{+}$), that is when the range $\boldsymbol{R}$ reduces to $\{I\}$.
By expanding its range to the larger subset $\boldsymbol{R} \subsetneqq$ Aut $(S, \oplus)$ ($\ominus I \notin \boldsymbol{R}$) the relator controls the weak (or relative) commutativity and associativity of $\oplus$, thus introducing inhomogeneity and anisotropy in the organic structure. This has the

additional benefit to induce the existence of $\hat{+}$, another addition which is classically commutative.
In other words, the expansion $\{I\} \to \boldsymbol{R}$ loosens the rigid structure of an abelian group and provides the more flexible, relating, structure of an organ.
We observe that since $\boldsymbol{R}$ is a proper subset of Aut $(S, \oplus)$, the role of the relator is to *reduce* the variety of possible automorphisms. The standard group appears as a limit case corresponding to the ultimate reduction $\boldsymbol{R} = \{I\}$ which is tautological.
This explains why the concept of an organ is better suited than a group to describe an organic logic which is evolutive by essence.

Organic Information Processing (IP) is a *dynamical* process which reflects the dynamics of the *relator*. Its operations in $G$ consist of $\oplus$, $\hat{+}$ and their automorphisms. One can view an organ as a new algebraic species, some kind of a "fieldoid", based on the groupoid, in which $\hat{+}$ plays the role attributed to $\times$ in an ordinary field (group-based) structure. The main difference with a field is that the neutral 0 (identical for $\oplus$ and $\hat{+}$) replaces the unit $1 \neq 0$. The analogy is described in the

**Remark 2.5.1** The induction $\{R\oplus, rel\} \to \hat{+}$ is analogous to the creation of the product $n \times a$ by $n$ repeated additions of the real number $a$. In this most familiar case, the relator stems from an iterated addition. The induction can also be compared with the way by which the *complex* $\times$ is recursively induced by $\{+$, conjugation, iteration $k-1 \to k\}$ in Dickson algebras $A_k \cong \mathbb{R}^{2^k}$, $k \geq 1$ (Chatelin 2011 a).

**Remark 2.5.2** In Remark 2.3.1, we noted that the axiom $(A2) \iff (G6)$ is *optional* in a gyrogroup. To better appreciate the concept of a relator for an organ deprived of $(A2)$ it is interesting to contrast it with the way alternativity, a weak form of associativity is expressed in the multiplicative algebra of octonions $\mathbb{G}$ (Chapter 9 in Chatelin 2011 a, Section 9.4).
In additive parlance, weak associativity is ruled by (A3).And in multiplicative notation for the *alternative* algebra $\mathbb{G}$ (Lemma 9.4.1):

$$a \times (b \times \gamma) = ((a \times (b \times \gamma)) \times \gamma^{-1}) \times \gamma, \quad \gamma \neq 0.$$

Recalling that $\ominus\gamma$ becomes $\gamma^{-1}$, one sees that the second formula related to $\times$, and determined by multiplication by $\gamma$ and $\gamma^{-1}$ only, is much more intricate than the first which writes simply $(a \oplus b) \oplus rel(a,b)\gamma = a \oplus (b \oplus \gamma)$.

## 2.6 Three basic examples

The following Examples are found in Sections 3.4, 3.8 and 3.10 respectively of (Ungar 2008) The explicit formula for $x \oplus y$ entails the determination of $rel(x, y)$.

**Example 2.1** The subgroup of all Möbius transformations of the complex open unit disk $D = \{z \in \mathbb{C}; |z| < 1\}$ into itself is defined by : $(a, z) \mapsto e^{i\theta} \frac{a+z}{1+\bar{a}z}$ for $a, z \in D$ and $\theta \in \mathbb{R}$. If we set $a \oplus z = \frac{a+z}{1+\bar{a}z}$, the relator is defined by $rel(a, z) = \frac{1+a\bar{z}}{1+\bar{a}z} \in$ Aut $(D, \oplus)$. Hence clearly $a \oplus z = rel(a, z)(z \oplus a)$. Endowed with $\oplus$ the unit disk becomes an organ. △

**Example 2.2** Let $c$ be the vacuum speed of light. We set $B_c = \{x \in \mathbb{R}^3; \|x\| < c\}$ to represent the ball of relativistically admissible velocities.
Einstein's law of addition of velocities $x, y \in B_c$ is

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}} \left[ x + y + \frac{1}{c^2} \frac{\gamma_x}{1+\gamma_x} x \wedge (x \wedge y) \right]$$

where $\gamma_x = \left(1 - \frac{1}{c^2}\|x\|^2\right)^{-1/2}$ is the inverse of Lorentz contraction.
Using Grassmann identity in $\mathbb{R}^3$:

$$x \wedge (y \wedge z) = < x, z > y - < x, y > z,$$

(Lamotke 1998, Chapter 7, p. 207), one can also write

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}} \left[ x + \frac{1}{\gamma_x} y + \frac{1}{c^2} \frac{\gamma_x}{1+\gamma_x} < x, y > x \right]$$

The two velocity components, parallel and orthogonal to the relative velocity between inertial systems, were given by Einstein in his 1905-epoch-making paper. The latter formula is valid for $n \geq 2$.
Einstein's addition is ruled by a relator which is the rotation: $y \oplus x \mapsto x \oplus y$ with axis parallel to $x \wedge y$ through the angle $\varepsilon$, $0 \leq |\varepsilon| < \pi$ (Borel 1913, Silberstein 1914). $\varepsilon$ is related to $\theta = \measuredangle(x, y)$ in the following way (Ungar 1988,1991): $\varepsilon = 0$ for $|\theta| \in \{0, \pi\}$ and for $|\theta| \in ]0, \pi[$ $x$ and $y$ are independent, yielding:

$$\cos\varepsilon = \frac{(\rho + \cos\theta)^2 - \sin^2\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

$$\sin\varepsilon = -2\frac{(\rho + \cos\theta)\sin\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

with $\rho^2 = \frac{\gamma_x+1}{\gamma_x-1}\frac{\gamma_y+1}{\gamma_y-1}$, $\rho > 1$, and $|\varepsilon| < |\theta|$.
When $\|x\|$ and $\|y\|$ tend to $c^-$, $\gamma_x$ and $\gamma_y$ tend to $\infty$ and $\rho \to 1^+$. Then $\cos\varepsilon \to \cos\theta$ and $\sin\varepsilon \to -\sin\theta$. △

**Example 2.3** $V = \mathbb{R}^n$, $n \geq 2$ is the euclidean linear vector space with scalar product $< \cdot, \cdot >$. Let be given $\lambda$, $0 < \lambda < \infty$, and define $v_\lambda = \frac{1}{\lambda}v$ for $v \in V$, $\beta_v = \left(1 + \|v_\lambda\|^2\right)^{-1/2}$. We consider

$$u \oplus v = \left( \frac{1}{\beta_v} + \frac{\beta_u}{1+\beta_u} < u_\lambda, v_\lambda > \right) u + v$$

defined for $u, v \in V$. For $n = 3$ and $\lambda = c$, this additive law governs the relativistic addition of *proper* velocities expressed in traveller's time. The relator is again a rotation. △

The reader can check that in each example above $x \oplus y$ is symmetric in $x$ and $y$ iff $x$ and $y$ are *dependent*.

## 2.7 Liaison $\Lambda$ between $rel$, $\oplus$ and $\hat{+}$

To the linear equations (2.2), (2,3) for $\oplus$, we add the third equation for $\hat{+}$

$$a \hat{+} \hat{x} = \hat{x} \hat{+} a = b \tag{2.7}$$

which admits the unique solution

$$\hat{x} = \ominus (\ominus b \oplus a) = b \ominus a. \tag{2.8}$$

Observe that $x = rel(\ominus a, b)\hat{x}$ by (A2).

Each of the solutions $x, y$ and $\hat{x}$ is obtained by a respective call to the three following cancellation laws for $\oplus$ and $\hat{+}$:

- left cancellation for $\oplus$ : $a \oplus (\ominus a \oplus b) = b$ (2.9)
- right cancellation for $\oplus$ : $(b \hat{-} a) \oplus a = b$ (2.10)
- cancellation for $\hat{+}$: $(b \ominus a) \hat{+} a = a \hat{+} (b \ominus a) = b$ (2.11)

Identities (2.10) and (2.11) express a link by means of the relator between $R\oplus$ and $\hat{+}$ which is not present in (2.9) concerning $L\oplus$.
If one uses $x, y$ and $\hat{x}$, the three identities become respectively

$$a \oplus x = b \tag{2.12}$$
$$y \oplus a = b \tag{2.13}$$
$$\hat{x} \hat{+} a = a \hat{+} \hat{x} = b \tag{2.14}$$

This notational artifact separates $R\oplus$ and $\hat{+}$ in (2.10), (2.11) which appear now as (2.13) = right cancellation for $\oplus$, (2.14)=cancellation for $\hat{+}$.
None of the two writings is a faithful description of the computational reality which is, by essence, *connected*. Whichever writing is chosen, the reader should keep in mind that a liaison based on $rel(a, \cdot)$ exists between $\cdot \oplus a$ and $\cdot \hat{+} a = a \hat{+} \cdot$ for $rel(a, \cdot) \neq I$ when the linear cancellation laws are at work. This liaison reflects the existence of the relator which regulates any relating computation in its organ. The liaison concerns $L\oplus$ as well. Indeed, the equality (2.8) $\hat{x} = b \ominus a$ suggests to consider the equation involving $L_b$:

$$L_b \tilde{x} = b \oplus \tilde{x} = a$$

whose solution is $\tilde{x} = \ominus b \oplus a = \ominus (b \oplus a) = \ominus \hat{x}$.

**Definition 2.2** *We call liaison* $\Lambda(rel, \oplus, \hat{+})$ *the computational consequences of the three fundamental cancellation laws (2.9), (2.10) and (2.11).*

The computational dynamics of organic IP results from the shifts $L\oplus$, $R\oplus$ and the automorphisms of $G$. Given $a$ and $b$, we shall be concerned in Sections 4 and 7 with the evolution of $\hat{x} = b \ominus a$ (resp. $y = b \hat{-} a$) when a left (resp. right) shift by an arbitrary $g \in G$ is realised simultaneously on $a$ and $b$. For future reference we mention the two results valid for $g \in G$:

- $(g \oplus b) \ominus (g \oplus a) = rel(g, b)(b \ominus a)$ (Theorem 6.12) (2.15)

- $(a \hat{-} b) = (a \oplus h) \hat{-} (b \oplus g)$ with $h = rel(a, b)g$ (Theorem 6.14). (2.16)

# 3 Metric cloths

## 3.1 The normed vector space frame

Let $V$ be a linear vector space over $\mathbb{R}$ with finite dimension $n \geq 2$, scalar product $< a, b >$ for $a, b \in V$ and derived norm $\|a\| = \sqrt{< a, a >}$.
The addition $+$ and scalar multiplication are standard operations in $V \cong \mathbb{R}^n$. Let $\lambda$ be given, $0 < \lambda < \infty$ and set $B_\lambda = \{x \in V; \|x\| < \lambda\}$. We suppose that the ball $B_\lambda$, or $V$ itself, are endowed with the *organic* structure $(G, \oplus)$ with relator $rel$, where $G$ represents $B_\lambda$ or $V$ as the case may be. The neutral 0 for $G$ is identified with $0 \in V$.
The linear vector space $V$ is the frame of the organ $G$ iff the relator preserves the scalar product: $< rel(u, v)x, rel(u, v)y >=< x, y >$ for any quadruple $(u, v, x, y) \in G^4$. It follows that $\|rel(x, y)\| = 1$ for $x, y \in G$. Hence the range $\boldsymbol{R}$ of the relator is a subset of the group of *isometries* of $G$. Note that $\boldsymbol{R} \subset O(n)$ does not contain $-I$ because of (A1).
We assume moreover that if $x$ and $y$ are linearly dependent in $G$ then for $x = ry$, $r \in \mathbb{R}$ (say), $(ry) \oplus y = y \oplus (ry)$. Hence $rel(ry, y) = rel(x, y) = I$ ($\Longrightarrow x \oplus y = x \hat{+} y$). The formula for $\oplus$ becomes symmetric in $x$ and $y$. The property is satisfied for the 3 Examples given in Section 2.6. We shall see how this assumption enables simplexity to take place in an axial fashion.

## 3.2 The scalar multiplication $\otimes$

We suppose that $G$ admits a scalar multiplication $\otimes : \mathbb{R} \times G \to G$ such that

- $1 \otimes a = a$,
- $(r_1 + r_2) \otimes a = (r_1 \otimes a) \oplus (r_2 \otimes a)$,
- $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$, $a \in G$, $r_1, r_2 \in \mathbb{R}$,
- for $r$ and $a \neq 0$ $\frac{|r| \otimes a}{\|r \otimes a\|} = \frac{a}{\|a\|}$,
- $rel(u, v)(r \otimes a) = r \otimes (rel(u, v)a)$ for $u, v, a \in G$, $r \in \mathbb{R}$,
- $rel(r_1 \otimes u, r_2 \otimes u) = I$, $u \in G$, $r_1, r_2 \in \mathbb{R}$.
- $\|r \otimes a\| = |r| \otimes \|a\|$, $r \in \mathbb{R}$, $a \in G$.

## 3.3 $n = 1$: the measuring rod $M = \{\pm\|a\|, a \in G\}$

All elements in $M$ are colinear, hence the relator image reduces to $\{I_1 = 1\}$, and $\oplus = \hat{+}$ on $M$. $M$ is a 1D-linear vector line equipped with $\oplus$, $\otimes$ and $\|\cdot\|$ deriving from $G$ and $V$. The 3 operations usually *differ* from the standard operations $+, \cdot, |\cdot|$ defined on $\mathbb{R}$.

## 3.4 $n \geq 2$: the $V$-framed metric cloth $W$

We suppose that $\|a \oplus b\| \leq \|a\| \oplus \|b\|$, $a, b \in G$.
The structure $W = (G, \oplus, \otimes)$ obeying the assumptions above is a *metric cloth* in the normed vector frame $V$. The cloth $W$ is organically and metrically woven by $\{\oplus$, relator, $\otimes, \|\cdot\|\}$. This is Ungar's gyrovector space in the carrier $V$ (Definition 6.2, Ungar 2008).

**Example 3.1** The scalar multiplication for the organ $B_c$ in Example 2.2 is such that $r \otimes 0 = 0$, $r \otimes x = \mu(r)x$ for $0 \neq x \in B_c$. We set $x_c = \frac{1}{c}x$, then

$$\mu(r) = \frac{1}{\|x_c\|} \tanh(r \tanh^{-1} \|x_c\|), \quad r \in \mathbb{R}.$$

Then $B_c$ becomes the $\mathbb{R}^3$-framed cloth $W_E$ (based on Einstein's addition) which is an alternative framework for Special Relativity in Physics, classically presented by means of Lorentz transformations in the field of quaternions $\mathbb{H}$.
Let $q = (c\alpha, X)$ be given in $\mathbb{H}$, with real part $c\alpha$, $\alpha \in \mathbb{R}$ and imaginary part $X$ in $\mathbb{R}^3$. Then $q^2 = c^2\alpha^2 - \|X\|^2 + 2c\alpha X$. A Lorentz transformation in $\mathbb{H}$ leaves invariant the quantity

$$\Re q^2 = c^2\alpha^2 - \|X\|^2 = f \text{ constant for all } q \in \mathbb{H}$$

(Poincaré 1905). Observe that $\|X\|^2 = c^2\alpha^2 - f$ and $\|\Im q^2\|^2 = 4c^2\alpha^2(c^2\alpha^2 - f)$ are nonnegative iff $c^2\alpha^2 \geq f$ which is always satisfied when $f \leq 0$.
By (11.2) in (Ungar 2008), the Lorentz transformation *without* rotation is a boost $L(u)$ for $u \in B_c$ such that, for $u_c = \frac{1}{c}u$, $q_c = \frac{1}{c}q = (\alpha, X_c)$

$$L(u)q_c = (\gamma_u[\alpha + < u_c, X_c >], \gamma_u u[\alpha + \frac{\gamma_u}{1 + \gamma_u} < u_c, X_c >]).$$

Then by (11.10) for $u, v \in B_c$ we get the composition law:

$$L(u)L(v) = L(u \oplus v)rel(u, v) = rel(u, v)L(v \oplus u).$$

The general case (transformations with rotations in $SO(3)$ is given in (11.15), (11.20).
These formulae shed an interesting light about the connection between hypercomputation in $\mathbb{H}$ based on $\times$ and computation in the cloth $W_E$ based on Einstein addition $\oplus_E$. The connection is developed in (Chatelin 2011c). △

Because $a - a = 0$ in $V$, $-a = (-1) \times a = \ominus a$ in $W$. In general $r \boxtimes (a \oplus b) \neq (r \boxtimes a) \oplus (r \boxtimes b)$, unless $a$ and $b$ are dependent. Scalar multiplication distributes axially: $r \boxtimes [(r_1 \boxtimes a) \oplus (r_2 \boxtimes a)] = (rr_1) \boxtimes a \oplus (rr_2) \boxtimes a = (rr_1 + rr_2) \boxtimes a$.
The automorphisms of $W$ form the group Aut $(W)$: they consist of automorphisms of $G$ which preserve also the scalar multiplication $\boxtimes$ and the scalar product $< \cdot, \cdot >$.
The identification $-a = \ominus a = \hat{-} a$ which holds in $W$ provides more insight on the induced addition $\hat{+}$ by considering the mirror equation for (2.2) where $a$ and $b$ are exchanged:

$$b \oplus \tilde{x} = a. \tag{3.1}$$

**Lemma 3.1**

$$\tilde{x} = -\hat{x} \tag{3.2}$$

*Proof.*(3.1) yields $\tilde{x} = -b \oplus a$ by (2.4) and $\hat{x} = b \ominus a$ by (2.8). Now $\hat{x} = -(-b \oplus a) = -\tilde{x}$. □

In the larger context of a cloth, the liaison $\Lambda$ includes $+$, as illustrated by the identification $\hat{x} = -\tilde{x}$.

**Definition 3.1** *A linear weaving computation refers to any algebraic computation taking place in a metric cloth $W$ defined by the data $\{rel, \hat{+}, \mathbb{R}, V\}$.*

The set of operations that we shall consider in weaving Information Processing (WIP) is restricted to $\mathrm{Op}(W) = \mathcal{L} \cup \mathcal{R} \cup \mathrm{Aut}\ (W)$.

**Definition 3.2** *The weaving information processing WIP in a metric cloth $W$ is realised in $W$ by means of* $\mathrm{Op}(W)$.

We shall study by geometric means the results of WIP. The metric cloth $W$ inherits from its euclidean frame not only a scalar product/norm, but also its *affine* essence with respect to a *real* parameter. Therefore the geometry derived from a cloth is based on *lines* (as affine functions of a real parameter) and in particular on *geodesics* (for which the triangle inequality becomes an equality). In what follows, we build on Ungar's vision based on physical insight. We develop some aspects of the role of geometry in WIP. The existence of the three operations $\oplus$, $\hat{+}$, $+$ entails a complex structure for cloth geometry which sheds some light on the polymorphic nature of *non euclidean* geometry.

# 4 The metrics associated with $\oplus$ and $\hat{+}$

## 4.1 Definition

We revisit the four linear equations (2.2), (2.3) (2.7) and (3.1) and their four solutions $x$ (2,4), $y$ (2.5), $\hat{x}$ (2.7) and $\tilde{x}$ (3.1). A simplification occurs because $\|rel(a,b)\| = 1$ for $x = rel(-a,b)\hat{x}$, hence $\|x\| = \|\hat{x}\| = \|\tilde{x}\| \neq \|y\|$. Thus one can associate *two* metrics in $W$ with the three cancellation laws. They are given by

$$\mathring{d}(a,b) = \|-a \oplus b\| = \|b \ominus a\|, \tag{4.1}$$

$$\hat{d}(a,b) = \|b \hat{-} a\|. \tag{4.2}$$

The reason for the upperscripts stems from the respective triangle inequalities

$$\mathring{d}(a,c) \le \mathring{d}(a,b) \oplus \mathring{d}(b,c), \tag{4.3}$$

$$\hat{d}(a,c) \le \hat{d}(a,b) \hat{+} \hat{d}(b,c). \tag{4.4}$$

The two metrics are invariant under Aut $(W)$. Invariance under left shift in $\mathcal{L}$ holds for $\mathring{d}$ by (2.15): $\|b \ominus a\| = \|(g \oplus b) \ominus (g \oplus a)\|$ for any $g \in G$.
$\hat{d}$ is not $\mathcal{L}-$invariant in general: with $a" = g \oplus a$ and $b" = g \oplus b$, $\|b \hat{-} a\| \neq \|b" \hat{-} a"\|$ for an arbitrary $g \in G$.
Regarding $\mathcal{R}$-invariance for $\hat{+}$ (based on $\oplus g$), if $rel(a,b) = I$, then: $a \hat{-} b = (a \oplus g) \hat{-} (b \oplus g)$ implies $\mathcal{R}$-invariance for $\hat{+}$. This is always true when $a$ and $b$ are dependent. In general (2.16) holds with $\|h\| = \|g\|$. The topic will be developed further in Section 7.1.

## 4.2 Liaison vs. metric entanglement

What is the influence of metrisation on the liaison $\Lambda(rel, \oplus, \hat{+}, +)$? It is twofold:

(i) The metric $\mathring{d}$ induces an illusory unification based on the norm equalities $\|x\| = \|\hat{x}\| = \|\tilde{x}\|$ between the three *distinct* solutions $x \neq \hat{x} = -\tilde{x}$ for (2.2), (2.7) and (3.1). Thus empirical deductions based on $\mathring{d}$ are *ambiguous*: the equal norms can be attributed to *three* different causes. We call this effect the *metric entanglement* $\&(\Lambda, \mathring{d})$ which transforms the computational liaison $\Lambda$ into an equality about norms.
(ii) Both metrics $\mathring{d}$ and $\hat{d}$ are *woven* by the relator: they do not provide the absolute certainty of classical analysis resulting from $\| \cdot \|$ in the presence of the *unique* standard addition $+$.

Not too surprisingly, we see that weaving computation in the context of a metric cloth offers a freedom of *choice* rather than the absolute certainty of classical mathematics, a situation already encountered with hypercomputation (Chatelin 2011 a). In the sequel, we investigate the impact of weaving on cloth geometry based on $\mathring{d}$ and $\hat{d}$.

**Remark 4.2.1** *On the notational dilemma*
It is important to keep in mind that in the ambiguous context of weaving computation the notation itself is, by force, ambiguous. For example the notation $\mathring{d}$ and $\hat{d}$ was suggested by the triangle inequalities (4.3), (4.4). But, of course, the notation $\mathring{d}$ reflects $\phi$ through its two aspects (i) $L\phi$, and (ii) $\hat{+}$ which combines $R\phi$ and the relator. And $\hat{d}$ reflects the *unique* aspect $R\phi$ converted into $\hat{+}$. In the difficult task to capture as best as possible the subtle relational interplay between $\phi$ and $\hat{+}$ by symbolic means, cloth geometry will prove to be a meaningful tool.

# 5 About the lines passing through 2 distinct points

## 5.1 Introduction

Let be given $a \neq b$. In euclidean geometry there exists a unique straight line passing through $a$ and $b$, which can be represented by the affine function: $t \in \mathbb{R} \mapsto a + (b - a)t \in \mathbb{R}^n$: the point $a$ (resp. $b$) corresponds to $t = 0$ (resp. 1).
The straight line is the geodesic of the euclidean metric.
The segment $[a, b]$ is defined by $0 \leq t \leq 1$. It has a unique midpoint $m_{ab} = a + \frac{1}{2}(b - a) = \frac{1}{2}(b + a) = m_{ba}$. This simple euclidean picture will be modified in cloth geometry since there exist *more than one* line passing through two points due to the existence of more than one cancellation law.
In what follows we restrict our attention to the three fundamental (cancellation) laws (2.9), (2.10), (2.11) that we put at the foundations of our geometric study. The three laws are ordered respectively as first, second and third. They define *three*

types of lines numbered 1,2,3. It is important to distinguish whether $a$ and $b$ are dependent or not.

## 5.2 Three fundamental lines through $a \neq b$, $a$ and $b$ independent

To each fundamental law we associate a unique fundamental line passing through $a$ for $t = 0$ and $b$ for $t = 1$. These lines are given by the table below

| symbol | definition | representation, $t \in \mathbb{R}$ | |
|---|---|---|---|
| $L_1 = L\text{-}L_{ab}$ | left-line for $L\oplus$ | $a \oplus (-a \oplus b) \boxtimes t$ | (5.1) |
| $L_2 = R\text{-}L_{ab}$ | right-line for $R\oplus$ | $(b \,\hat{-}\, a) \boxtimes t \oplus a$ | (5.2) |
| $L_3 = \hat{L}_{ab}$ | line for $\hat{+}$ | $(b \ominus a) \boxtimes t \,\hat{+}\, a = a \,\hat{+}\, (b \ominus a) \boxtimes t$ | (5.3) |

We call $a$ the *origin* of the 3 lines ($t = 0$). The three distinct solutions $x, y, \hat{x}$ are the respective coefficients of $t$ for the lines. The 3 representations can be rewritten respectively under the form: $a \oplus x \boxtimes t$, $y \boxtimes t \oplus a$, $\hat{x} \boxtimes t \,\hat{+}\, a = a \,\hat{+}\, \hat{x} \boxtimes t$.

**Lemma 5.1** *(i)* $a \,\hat{+}\, \hat{x} \boxtimes t = a \oplus x_1 \boxtimes t$ *with* $x_1 = rel(a, -b)\hat{x} = x$.
*(ii)* $\hat{x} \boxtimes t \,\hat{+}\, a = \hat{x} \boxtimes t \oplus a_2$ *with* $a_2 = rel(b, -a)\hat{x}$.
*(iii) Moreover* $\|\hat{x}\| = \|x\| = \|a_2\|$*:* $x$ *and* $a_2$ *are rotated about* $O$ *from* $\hat{x}$ *through opposite angles.*

*Proof.*(i) $a \,\hat{+}\, \hat{x} \boxtimes t = a \oplus (rel(a, -\hat{x})\hat{x}) \boxtimes t$ by (2.6) with $rel(a, -b \oplus a) = rel(a, -b)$ by (A1). And $rel(a, -b)\hat{x} = rel(-a, b)\hat{x} = x$.
(ii) $\hat{x} \boxtimes t \,\hat{+}\, a = (b \ominus a) \boxtimes t \oplus rel(\hat{x}, -a)\hat{x}$ by (2.6) and $rel(b \ominus a, -a) = rel(b, -a)$.
(iii) Clear when we observe that $rel^{-1}(a, -b) = rel(b, -a)$. □

The line $\hat{L}_{ab}$ can be interpreted as a *modified* (i) left- or (ii) right- line for $\oplus$.
We use the generic notation $L_{ab} = L(a, x)$ where $x$ is the coefficient of the parameter $t$ in the equation for the line passing through the origin $a(t = 0)$ and $b(t = 1)$. Hence $\hat{L}_{ab} = L_3 = L(a, \hat{x})$, $\hat{x} = b \ominus a$.

**Lemma 5.2** *(i)* $\hat{L}_{ab} = L\text{-}L(a, x) = L\text{-}L_{ab}$ *with* $x = rel(a, -b)\hat{x}$, $b = a \oplus x$.
*(ii)* $\hat{L}_{ab} = R\text{-}L(a_2, \hat{x}) = R\text{-}L_{a_2 b_2}$ *with* $a_2 = rel(b, -a)\hat{x}$, $b_2 = \hat{x} \oplus a_2$.

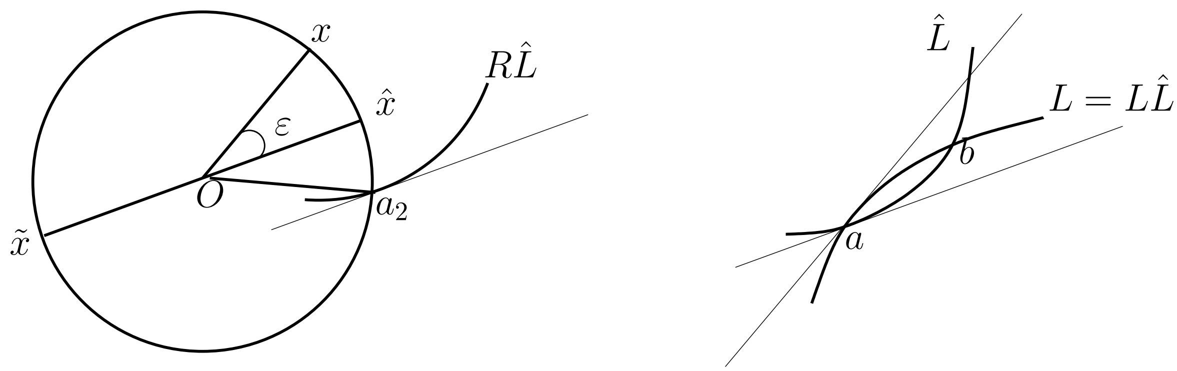


**Figure 5.1**: $\hat{L} = \hat{L}_{ab}$ *and its geodetic components:* $L = L\hat{L} = L\text{-}L_{ab}$ *and* $R\hat{L} = R\text{-}L(a_2, \hat{x})$.

*Proof.*Apply Lemma 5.1. For $t = 1$, (i) $a \oplus x = b$, (ii) $\hat{x} \oplus a_2 = b_2 \iff \hat{x} = b_2 \hat{-} a_2$.
□

The line $\hat{L}_{ab}$ is the source of two distinct images, left and right. In the left (resp. right) image, the origin $a$ is preserved (resp. moved to $a_2$) and the coefficient $\hat{x}$ is moved back to $x$ (resp. preserved).
The line $\hat{L} = \hat{L}_{ab}$ is a *composite* construction resulting from $\oplus$ and $rel(a, -b)$ which can be decomposed into its left and right geodetic components. Quite remarkably, the left component is $L\text{-}L_{ab}$ itself. The right component $R\hat{L}$ can be characterised by the rotation $\hat{x} \mapsto x = rel(a, -b)\hat{x}$ about $O$ through the angle $\varepsilon$. Then $a_2$ is rotated through $-\varepsilon$. See Figure 5.1.
There are altogether *four* lines of interest associated with a pair $(a, b)$: the three lines through $a, b$ plus the right image $R\hat{L}$ through $a_2, b_2$.

## 5.3 $a$ and $b$ are dependent

When $a$ and $b$ are dependent, the 3 points $O, a, b$ are collinear. An *essential simplification* takes place: the *four* lines above coalesce geometrically into *one* euclidean straight line through $O$.

**Lemma 5.3** *If $a$ and $b$ are dependent, then $rel(a, b) = I$, $a_2 = x = y = \hat{x}$, and $b_2 = 2 \otimes x$.*

*Proof.* By assumption $rel(a, b) = I$ then $a \oplus b = a \hat{+} b$, hence $x = y = \hat{x} = a_2 = b \ominus a$. The 3 lines $L_1, L_2, L_3$ coalesce into one euclidean straight line through $O$ if $x \neq 0 \iff b \neq a$, since $a$ and $x$ are dependent.

(i) Because $x = \hat{x}$, the left image for $\hat{L}$ coalesces with itself.
(ii) For the right image $a_2 = x$ and $b_2 = 2 \otimes x$ yielding the equation $x \otimes (t+1)$. The right image is geometrically identical to the axis passing through $O, a$ and $b$. But its equation $(t+1) \otimes x$ differs from $a \oplus t \otimes x$, unless $a = x \Longleftrightarrow b = 2 \otimes a$. □

**Corollary 5.4** *(i) If $a \neq 0$ and $b$ are dependent such that $b = l \otimes a$; $l \in \mathbb{R} \backslash \{1\}$, the three lines through $a$ and $b$ and the right image for $\hat{L}$ coalesce geometrically into a unique euclidean straight line. The analytic coincidence does not hold for the right image unless $l = 2$.*
*(ii) If $l = 1$, the unique line is reduced to the point $\{a\} \neq 0$, and the right image to $\{0\}$.*

*Proof.* (i) Clear by Lemma 5.3. If $b = l \otimes a$, $a = \frac{1}{l-1} \otimes x$ and the common equation is $\left(\frac{1}{l-1} + t\right) \otimes x$.
For $l \neq 2$, the analytic difference between the two equations is $x \ominus a = (l-2) \otimes a = \left(\frac{l-2}{l-1}\right) \otimes x$.
(ii) $l = 1 \Longleftrightarrow x = 0$. □

When $a$ and $b$ are *dependent*, the nature of the geometric lines differs markedly; it is reminiscent of that encountered in euclidean geometry. It is the linear *independence* of $a$ and $b$ which forces the lines to *bend*, indicating a nonlinearity in disguise.
In what follows, $a$ and $b$ are assumed to be independent, unless otherwise stated.

# 6 About midpoints on a curvilinear segment

There are 3 types of fundamental curvilinear segments $(a, b)$ to consider. We first assume that $a$ and $b \neq 0$ are independent.

## 6.1 Midpoints on $L_1$ and $L_2$ for $\oplus$

In Chapter 6, Ungar shows that a unique midpoint exists for (5.1) by Theorems 6.53, 6.34 and for (5.2) by Theorem 6.74:

- $m_{ab}^L = a \oplus x \otimes \frac{1}{2} = \frac{1}{2} \otimes (a \hat{+} b) = b \ominus x \otimes \frac{1}{2} = m_{ba}^L$, (6.1)
  $\|a \ominus m_{ba}^L\| = \|b \ominus m_{ba}^L\| = \|x\| \otimes \frac{1}{2}$

• $m^R_{ab} = y \otimes \frac{1}{2} \oplus a = b \ominus y \otimes \frac{1}{2} = m^R_{ba}$, with $\|y\| \neq \|x\|$ (6.2)

The equality $m^L_{ab} = m^L_{ba} = \frac{1}{2} \otimes (a \hat{+} b)$, suggests that $a$ and $b$ could play a more symmetric role in the definition of the left line $L_1$ for $\oplus$ under an appropriate change of parameter.

**Lemma 6.1** *The line L-$L_{ab}$ can be represented in the four equivalent forms:* $a \oplus x \otimes t = a \otimes (1-t) \oplus b \otimes t$, $x = -a \oplus b$, *and* $b \oplus \tilde{x} \otimes t' = b \otimes (1-t') \oplus \otimes t'$, $\tilde{x} = -b \oplus a$, *with* $t + t' = 1$.

*Proof.*$a \oplus (-a \otimes t \oplus b \otimes t) = a \otimes (1-t) \oplus b \otimes t$ since $rel(a, a) = I$.
When $t'$ replaces $t$, $a$ and $b$ are exchanged. □

Letting $t = t' = \frac{1}{2}$ yields $m^L_{ab}$ which admits the fully symmetric representation $\frac{1}{2} \otimes (a \hat{+} b)$. This reflects an essential property of the scalar multiplication $\otimes$ by 2 (Theorem 6.7, Ungar 2008).

$$2 \otimes (a \oplus b) = a \oplus (2 \otimes b \oplus a) = a \hat{+} (a \oplus (2 \otimes b)) \tag{6.3}$$

for any $a, b \in W$. In (6.3), $2 \otimes a$ is split so that $a$ occurs in two places in the rhs of $2 \otimes (a \oplus b)$, yielding three terms.

We have proved the remarkable

**Theorem 6.2** *For any two independent points* $0 \neq a \neq b$ *the three additions* $L\oplus$, $R\oplus$ *and* $\hat{+}$ *provide the same arithmetic mean on L-$L_{ab}$:*

$$m^L_{ab} = \frac{1}{2} \otimes (a \oplus b) = \frac{1}{2} \otimes (b \oplus a) = \frac{1}{2} \otimes (a \hat{+} b).$$

*Proof.*Clear. Observe that, in addition to the above coincidences, and (6.1), we also have $m^L_{ab} = b \oplus \tilde{x} \otimes \frac{1}{2} = a \ominus \tilde{x} \otimes \frac{1}{2}$. □

No such remarkable property holds for R-$L_{ab}$. The identities about $m^R_{ab}$ given in (6.2) cannot be complemented in general.

## 6.2 On the line $\hat{L}$ for $\hat{+}$

The third type of curvilinear segment on $\hat{L}$ defined by (5.3) has *two* pseudo-means: $\hat{m}_{ab} = \hat{x} \boxtimes \frac{1}{2} \hat{+} a$ differs from $\hat{m}_{ba} = b \hat{-} \hat{x} \boxtimes \frac{1}{2}$ (Section 6.13). However, $\|x\| = \|\hat{x}\|$ guarantees the equality of the respective distances $\|a \hat{-} \hat{m}_{ab}\| = \|b \hat{-} \hat{m}_{ba}\| = \|\hat{x}\| \boxtimes \frac{1}{2}$ and of their counterparts on $L$-$L_{ab}$.

**Lemma 6.3** *The two pseudo-means $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$ are such that*

$$\|x\| \boxtimes \frac{1}{2} = \mathring{d}(a, m^L_{ab}) = \mathring{d}(a, \hat{m}_{ab}) = \mathring{d}(b, \hat{m}_{ba}).$$

*Proof.*Clear by (6.1). □

When $a$ and $b$ are independent, there are four means: $m^L$ on $L$-$L_{ab}$, $m^R$ on $R$-$L_{ab}$, $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$.

## 6.3 $a$ and $b$ are dependent

**Lemma 6.4** *If $a \neq 0$, $b = l \boxtimes a$, the four means coalesce into a single point $m = \frac{1}{2} \boxtimes (a \hat{+} b)$ on the unique line $L_{ab}$.*

*Proof.*Use Lemma 5.3 and Corollary 5.4. □

# 7 About geodesics, lines and triangles

## 7.1 Two types of geodesics coexist

Among the three fundamental lines passing through $0 \neq a$ and $b$ independent, only the *first two* are geodesics (Theorems 6.48 and 6.78): for any $c$ on $L$-$L_{ab}$ (resp. $R$-$L_{ab}$) the triangle inequality (4.3) (resp. (4.4)) reduces to an equality. The third fundamental line $\hat{L}_{ab}$ is *not* a geodesic. This justifies the existence of *two* pseudo-means (Lemma 6.3).

**Proposition 7.1** *The two additions $\oplus$ and $\hat{+}$ coalesce on the geodesic $L$-$L_{ab}$.*

*Proof.* By a dichotomy argument based on Theorem 6.2: $x \oplus y = x \hat{+} y$ for any $x, y$ between $a$ and $b$ on $L$-$L_{ab}$. □

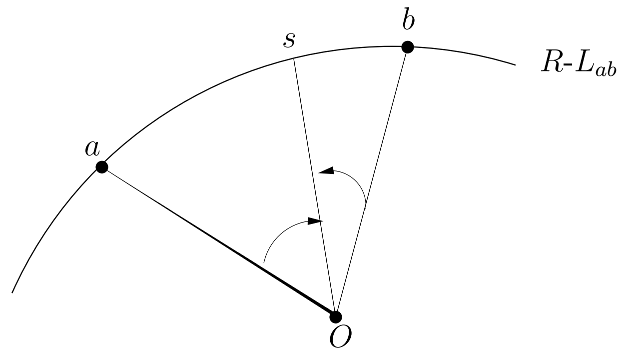


**Figure 7.1**: $a, b,\ s \in R\text{-}L_{ab}$

Proposition 7.1 indicates that a sort of "differential" commutativity holds for $x \oplus y$ when $x$ and $y$ vary on $L_1$. Given $a$ and $b$ linearly independent the geodesic for $\mathring{d}$ through $a, b$ describes the unique locus of points for which $\oplus$ is *commutative*, hence $\oplus\ =\ \hat{+}$ locally (on $L_1$). This mechanism underlies the emergence of the role of commutativity for addition in classical mathematics.

We discover below that the geodesics for $\hat{d}$ play a very different connecting role in IP by revisiting (2.16) above:

**Proposition 7.2** *The geodesic $R\text{-}L_{ab}$ is such that for any $w \in W$ and $s \in R\text{-}L_{ab}$, then*

$$b \,\hat{-}\, a = (b \oplus rel(b, s)w) \,\hat{-}\, (a \oplus rel(a, s)w) \tag{7.1}$$

*Proof.* See Theorem 6.76 in (Ungar 2008) and Figure 7.1. □

The relation (7.1) is one possible form of the kind of right shift-invariance enjoyed by $\hat{+}$ when $a$ and $b$ are independent (2.16). The coefficient $y = b \,\hat{-}\, a$ is *invariant* when the same right shift chosen in $\{\cdot \oplus rel(\cdot, s)w,\ w \in W,\ s \in R\text{-}L_{ab}\}$ is equally applied to $a$ and $b$, see Figure 7.1. This *exact*, albeit limited, kind of $\mathcal{R}$-invariance for $\hat{G}$ under right shift should be contrasted with the metric $\mathcal{L}$-invariance for $\mathring{G}$ (which hides the rotation $rel(g, b)$ in (2.15)).

**Definition 7.1** *Given $a \neq b$, the property (7.1) for $w \in W$, $s \in R\text{-}L_{ab}$ defines the homotopic link between $a$ and $b$ assumed to be independent.*

Any $s$ on $R\text{-}L_{ab}$ is uniquely defined by $t \in \mathbb{R}$ through (5.2) which defines the map:

$$t \in \mathbb{R} \mapsto y(t) = y \otimes t \oplus a,\ t \in \mathbb{R},\ y = b \,\hat{-}\, a.$$

At any $(t, w) \in \mathbb{R} \times W$ we consider in $W$

$$z_a(t) = rel(a, y(t))w, \quad z_b(t) = rel(b, y(t))w,$$

with $z_a(0) = z_b(1) = w$. By (7.1), $b \,\hat{-}\, a = (b \oplus z_b(t)) \,\hat{-}\, (a \oplus z_a(t))$ for all $t \in \mathbb{R}$, with $\|z_a(t)\| = \|z_b(t)\| = \|w\|$ for any $w \in W$.

The homotopic link between $a$ and $b$ is ruled by the two values $rel(a, y)$ and $rel(b, y)$ for the relator. Indeed, $rel(a, (b \,\hat{-}\, a) \otimes t \oplus a) rel(b \,\hat{-}\, a, a) = I$ by (2.16) in Ungar (2008), and $rel^{-1}(b \,\hat{-}\, a, a) = rel(-a, a \,\hat{-}\, b)$ (Section 2.4).

**Proposition 7.3** *When $w$ varies on the sphere $S_r = \{x, \|x\| = r\}$ for $0 < r < \lambda$, the homotopic link between $a$ and $b$ maintains $z_a(t)$ and $z_b(t)$ on $S_r$ for all $t \in \mathbb{R}$.*

*Proof.*Clear from the above discussion. □

When $w$ is arbitrary in $W$, the double equality $\|w\| = \|z_a(t)\| = \|z_b(t)\|$ holds for any $t$, and hides the actual source of the homorphic link (7.1) between $a$ and $b$ which resides in the relator at the pairs $(a, b \,\hat{-}\, a)$ and $(b, b \,\hat{-}\, a)$.

In conformity with the notation adopted in Section 4, we set $\overset{\circ}{G} = \overset{\circ}{G}(a, b)$ for $L$-$L_{ab}$ $\hat{G} = \hat{G}(a, b)$ for $R$-$L_{ab}$. This unsatisfactory notation exemplifies the notational dilemma expressed in Remark 4.2.1 about the difficulty to represent intrinsically complex facts in the relating domain by a simple enough symbol.

Proposition 5.2 can be rephrased as follows. The line $\hat{L}_{ab}$ has *two geodetic images* $\overset{\circ}{G}(a, b)$ and $\hat{G}(a_2, b_2)$ which are two different versions of itself.

## 7.2 Weaving computation and broadcasting information

The broadcasting of information from $a$ to $b$ uses the *real* parameter $t$ in $\mathbb{R}$ to channel through the two kinds of geodesics $\overset{\circ}{G}$ and $\hat{G}$ with distinct capabilities.

1) For a geodesic $\overset{\circ}{G}$, $\|b \ominus a\|$ is invariant under left shift. We say that $\overset{\circ}{G}$ *radiates metric* information. In other words, $\overset{\circ}{G}$ is a channel which is blind to rotations performed on the results produced by WIP: it is a *normative* channel. Because the two additions $\oplus$ and $\hat{+}$ yield identical results for any pair of points on itself, $\overset{\circ}{G}$ draws the *commutative* path from $a$ to $b$.

2) By comparison, the geodesic $\hat{G}$ through $a$ and $b$ (when independent) is a channel which *selects*, from the whole of WIP results, only the ones which enjoy a homotopic link (according to Definition 7.1). We say that $\hat{G}$ *emanates* limited *exact* information.

The line $\hat{L}$ given by (5.3) cannot broadcast information since it is not a geodesic. But its two geodetic images are suitable for broadcasting. Channel $\mathring{G} = L\text{-}L_{ab}$ broadcasts the left image which generally differs from $\hat{L}_{ab}$ for $t \neq 0$ and 1. The right image is broadcast through $R\hat{L} = R\text{-}L_{a_2b_2}$ which passes through $a_2 \neq a$ $(t = 0)$ and $b_2 \neq b$ $(t = 1)$ in general. This computational property lends weight to the notion of "action at a distance" for information, a notion which is too often ruled out a priori in science.

By contrast, if $a$ and $b$ are *dependent*, $a \neq b$, there exists a *unique* channel because $\mathring{G}$, $\hat{G}$ and $\hat{L}$ coalesce with the right geodetic image for $\hat{L}$ (Corollary 5.4). Action is local, but analysis $\neq$ geometry unless $a \neq 0$ and $b = 2 \boxtimes a$, or vice-versa.

It appears that there are *three* ways by which information can be broadcast from $a$ to $b$:
(i) If $a$ and $b$ are independent, the information potential is *twofold*, deriving from the two metrics $\mathring{d}$ and $\hat{d}$.
(ii) If $a$ and $b$ are dependent, the geometric potential derives from a *unique* norm. However the remaining analytic difference $a \ominus x$ vanishes iff $b = 2 \boxtimes a \neq 0$ only.

## 7.3 Geometries of the triangle

There are three geometric aspects stemming from *lines* through $a, b$ independent deriving from $L\oplus$, $R\oplus$, $\hat{+}$ which are connected and at times entangled. A very different aspect emerges when $a$ and $b$ are dependent, a situation which we leave aside in this section.
Section 6.1 has shown that any *geodetic* segment defined by *two* distinct points $a$ and $b$ has a midpoint. In order to discriminate further, we consider *three* non collinear points $a, b, c$ and the two associated fundamental triangles with vertices at $a, b, c$ and whose sides are geodetic segments corresponding to $\mathring{d}$ and $\hat{d}$ respectively. Such triangles are *geodetic* triangles of type 1 and 2 respectively.
Non-geodetic triangles of the third type are constructed by means of non-geodetic lines $\hat{L}$.

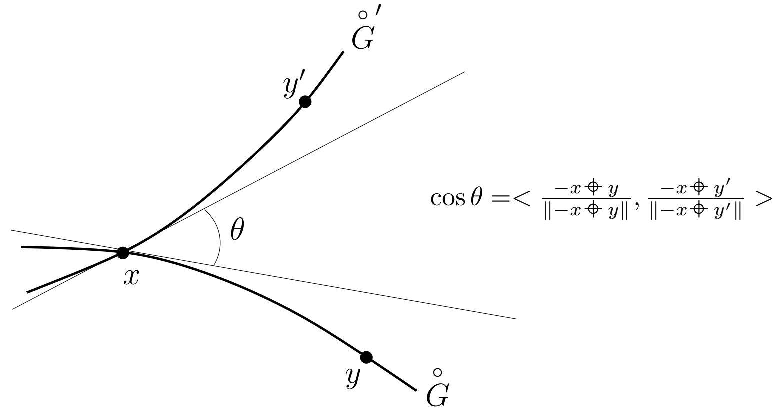


**Figure 7.2**: $\overset{\circ}{G}$ and $\overset{\circ}{G}{}'$ intersect at $x \neq 0$ under the angle $\theta$, $0 \leq \theta \leq \pi/2$

In euclidean geometry, any triangle has concurrent medians meeting at the centroid. In non-euclidean geometry for a cloth, we restrict our attention to triangles of each type 1,2 and 3.
According to Section 6.13 in (Ungar 2008), only $L$-triangles have a centroid. $R$-triangles have medians but no centroid. And non-geodetic triangles of type 3 have no medians.

**Example 7.1** *On geodetic triangles and squares of type 1.*
Let be given $\lambda$, $0 < \lambda < \infty$ and $B_\lambda = \{x \in \mathbb{R}^n, \|x\| < \lambda\}$, $n \geq 2$. Then for $x \in B_\lambda$ and $x_\lambda = \frac{1}{\lambda}x$, $\|x_\lambda\| < 1$. Using Ungar's Definition 3.4, we set

$$x \oplus y = \frac{1}{C}\left(Ax + By\right),$$

$A = 1 + 2 < x_\lambda, y_\lambda > + \|y_\lambda\|^2$, $B = \left(1 - \|x_\lambda\|^2\right)$, $\quad C = 1 + 2 < x_\lambda, y_\lambda > + \|x_\lambda\|^2 \|y_\lambda\|^2$.
Then we use the metric $\overset{\circ}{d}$ defined by $\oplus$ to measure angles and distances for the geodesics $\overset{\circ}{G}$, see Chapter 8 in (Ungar 2008), and Figure 7.2.
We consider, in the hyperbolic plane, two geodesics of type 1 for $\oplus$, which intersect at $x \neq 0$ at an angle $\theta$, $0 \leq \theta \leq \pi/2$. The Riemann metric associated with $\oplus$ is *conformal* to the metric $dx^2 = \sum_{i=1}^{N} dx_i^2$ of the Euclidean space $\mathbb{R}^n$, $x = (x_i)$ (Section 7.3, p. 247). Hence $\theta$ measures the euclidean angle between the intersecting tangents at $x$.
We distinguish two cases below:
(i) $0 \leq \theta \leq \pi/3$. One can construct *equilateral triangles* (*three* angles$= \theta$) with defect $\delta_3 = \pi - 3\theta$, $0 \leq \delta_3 \leq \pi$ and side length $\|a\|$ such that $\rho_3(\theta) = \frac{\|a\|}{\lambda} = \sqrt{2\cos\theta - 1}$ (Theorem 8.56), with $\rho_3 = 1$ (resp. 0) for $\theta = 0$ (resp. $\pi/3$), meaning that $\|a\| = \lambda$ (resp. $\lambda \to \infty$).
One can equally construct *squares* (*four* angles$= \theta$) with defect $\delta_4 = 2\pi - 4\theta$, $\frac{2}{3}\pi \leq \delta_4 \leq 2\pi$, side length $\|b\|$, $\frac{1}{\lambda}\|b\| = \rho_4(\theta) = \sqrt{\cos\theta}$ and diagonal length $\|c\|$, $\frac{1}{\lambda}\|c\| = \sqrt{\frac{2\cos\theta}{1+\cos\theta}}$ ((8.260) in Section

8.18). Hence $\frac{\lambda}{\sqrt{2}} \leq \|b\| \leq \lambda$, $\sqrt{\frac{2}{3}}\lambda \leq \|c\| \leq \lambda$.
(ii) $\frac{\pi}{3} < \theta \leq \pi/2$. No equilateral triangle can be constructed because $-1 \leq \rho_3^2(\theta) < 0$. Only remain squares with defect $0 \leq \delta_4 < 2\frac{\pi}{3}$, and $0 \leq \rho_4(\theta) < \frac{1}{\sqrt{2}}$. △

The distinction $\theta < \frac{\pi}{3}$ or not in Example 7.1 reveals an interesting epistemological property. When $\theta \leq \frac{\pi}{3}$, the intersection point $x$ can be related in *two* ways to points on each geodesic at the same relative distance ratio $\rho$:
1) either by using an implicit equilateral triangle with $0 \leq \rho_3 \leq 1$,
2)or by using an implicit square with $1/\sqrt{2} \leq \rho_4 \leq 1$.

In other words, the implicit connection of $x$ to *three* other points (square) rather than *two* (triangle) forbids the ratio $\rho$ to tend to 0: necessarily $\rho \geq \frac{1}{\sqrt{2}}$.
This restriction holds as long as $\theta \leq \frac{\pi}{3}$. When $\frac{\pi}{3} < \theta \leq \frac{\pi}{2}$, the connection to three points is the only possible connection, yielding $0 \leq \rho < \frac{1}{\sqrt{2}}$.
This sheds some light on what it may mean to consider the intersection $x$ as immersed in a *causal space* with 3 or 4 dimensions. When $0 \leq \theta \leq \frac{\pi}{3}$ the immersion in 3D (resp 4D) enables continuity to take place in the limit $\lambda \to \infty$ (resp. imposes discreteness because $\rho \geq \frac{1}{\sqrt{2}}$). When $\frac{\pi}{3} < \theta \leq \frac{\pi}{2}$, the immersion in 4D is *by necessity*; it yields continuity as $\theta \to \pi/2$ ($\lambda \to \infty$ is the euclidean limit).
This shows that the pair $(x, \theta)$ should be considered: one cannot decide between 3 or 4 dimensions for the analysis of $x$ without the information about $\theta$ that either $\theta \leq \frac{\pi}{3}$ or $\theta > \pi/3$. It is quite remarkable that the *quantisation* which can take place when $\theta \leq \pi/3$ is the result in hyperbolic geometry of a 4D-perspective for an inherently 3D-phenomenon. It is clear that for $\theta > \pi/2$, the same reasoning applies by considering regular polygons with more than 4 vertices.

At this point, we may pause and ponder on the seemingly counter-intuitive fact that the algebraically poorest $L\Phi$ provides the geometrically richest of the three geometric representations of a triangle, the one closest to the familiar euclidean triangle.
Similar phenomena are actually *ubiquitous* in hypercomputation (Chatelin 2011 a). This expresses the creative tension between algebra and geometry telling us that:

$$\text{less algebraic rules} = \text{more geometric options.}$$

# 8 Four aspects in cloth geometry

The classical (commutative and associative) addition + underlies the familiar euclidean space, the geometric frame of classical vector calculus.
A nonclassical addition $a \oplus b$ such that $rel(a, b)$ is an isometry in $\boldsymbol{R} \subset O(n)$ transforms the euclidean geometry into a computational construction with *four* different geometric aspects which are related and metrically entangled by $\boldsymbol{R}$.
In the most primordial aspect, the geodesics $\overset{\circ}{G}$ are directly derived from $L\oplus$ and provide the picture which has the largest amount of geometric information, corresponding to classical hyperbolic geometry. Weaving computation in a metric cloth reveals in fact a more complex, multifaceted, picture for *cloth geometry.* The nature of the lines through the points $a$ and $b$ depends on their being linearly independent (I) or not (II).

| line type | metric | line | triangle |
|---|---|---|---|
| 1 | $\overset{\circ}{d} = \\| -a \oplus b \\|$<br>geodesic $\overset{\circ}{G}$<br>= Channel 1<br>metric radiation | dichotomy[1]<br><br>$\oplus = \hat{+}$ [1] | centroid [2] |
| 2 | $\hat{d} = \\| b \mathbin{\hat{-}} a \\|$<br>geodesic $\overset{\circ}{G}$<br>= Channel 2<br>limited exact emanation | dichotomy [1] | no centroid [2] |
| 3 | $\overset{\circ}{d} = \\| b \ominus a \\|$<br>line $\hat{L} \neq$ geodesic | two pseudo-midpoints<br>left geodetic image $\overset{\circ}{G}$<br>right geodetic image<br>through $a_2$ and $b_2$ | no medians |

Table 8.1.: (I) *a and b independent*

A partial summary (limited to lines and triangles) is given in Table 8.1 for $a$ and $b$ independent. Then Table 8.2 lists for $a$ and $b$ dependent some features of the fourth

[1]on a geodetic segment
[2]for a geodetic triangle

- The four lines (including $\overset{\circ}{G}$ and $\hat{G}$) coalesce into a unique straight channel through $O$ for $l \neq 1$.
- For $l = 2$, there is *analytic* coincidence as well.
- The four points resulting from the three dichotomies coalesce into one.
- For $l = 1$, the three lines are reduced to the point $\{a\}$ and the right image to the origin $O$ $(a \neq 0)$.

Table 8.2.: (II) $a \neq 0$, $b = l \boxtimes a$, $l \in \mathbb{R}$.

aspect reduced by *linear dependence* to a pencil of lines through $O$ $(a \neq b)$ or to a set of points $(a = b)$.
Because the non-geodetic lines do not enjoy the unique midpoint property, Ungar(2008) leaves out the third aspect, to focus on the first two (p. 205).
He gives three concrete examples in Chapters 4 and 6. The first one corresponds to Example 7.1; it extends for $n \geq 2$ the Möbius transformation given in Example 2.1 for $n = 2$. The 2nd example generalises to $n \geq 2$ the Einstein addition given in Example 2.2 for $n = 3$. The third example, also of physical origin, was presented in Example 2.3. We shall discover later that these three basic examples shed an essential light on the first two aspects of WIP dealing with geodesics.

## 8.1 The first aspect based on $\overset{\circ}{d}$

Because of the geodetic differential identification $\oplus = \hat{+}$ taking place in the first aspect (Proposition 7.1), each choice of $\oplus$ yields a cloth geometry whose *first* aspect exhibits some of the properties found in axiomatic hyperbolic geometry (where models are all equivalent).
It is noteworthy that the three concrete examples do correspond to the three best known models, respectively the Poincaré ball-, Beltram-Klein ball-, hyperboloid, models. One marvels at the fact the Beltrami model (1868) anticipates by almost 40 years its most fundamental *raison d'être* to be provided in 1905 by Einstein.
The Poincaré and Klein ball-models $(n \geq 2)$ are related to the hyperboloid one *projectively* (by means of $(t, x_1, \cdots, x_n)$). For $n = 3$, the asymptotic cone for the hyperboloid is understood as the light-cone $c^2t^2 = x_1^2 + x_2^2 + x_3^2$ in the Minkowski time-space interpretation of Special Relativity.
The equivalence between the two ball-models is remarkably simple: with respective notation $P$ (for Poincaré) and $E$ (for generalised Einstein, i.e. Beltrami-Klein) the

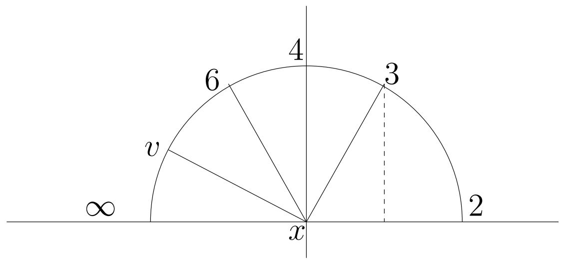


**Figure 8.1**: Organic causality: $1 \to v-1, v \geq 2$ at $x$.

correspondence is a homothety:

$$x_E \mapsto x_P = \frac{1}{2} \boxtimes x_E \Longleftrightarrow y_P \mapsto y_E = 2 \boxtimes y_P.$$

(Table 6.1, Section 6.21 in Ungar 2008).

Because the geodesics $\overset{\circ}{G}$ radiate *metric* information only, it is possible that the geometric causality at the intersection $x$ of two geodesics of type 1 is based on the implicit construction of a regular polygon $P_v$ with $v$ vertices and angle $0 \leq \theta \leq \varphi(v) = \pi \frac{v-2}{v}$, $v \geq 2$, between two adjacent sides. The formula for $\varphi(v)$ is based on the assumption that for a regular hyperbolic $v-$polygon ($v > 2$), the defect is $\delta_v(\theta) = (v-2)\pi - v\theta$ with $0 \leq \delta_v(\theta) \leq (v-2)\pi$ where the maximum value is the sum of the $v$ angles in a euclidean $v-$polygon. This geometric construction based on $\theta$ defines an *organic causality* 1 to $v-1$, $2 \leq v < \infty$, see Figure 8.1.

**Example 8.1** *On quantisation and $P_v$-causality*
We consider a regular polygon $P_v$ with $v$ vertices, $v \geq 2$ and angle $\theta$. The limit case $v = 2$ corresponds to $\theta = 0$: the geodesics are tangent at $x$, and $\rho_2(0) \in [0,1]$ ($0 \leq \|a\| \leq \lambda$). When the case $\theta = 0$ is analysed in 3D ($v = 3$) then the value 1 is the only possibility if we use the formula $\rho_3(\theta) = \sqrt{2\cos\theta - 1}$. This means $\|a\| = \lambda$: the two points are on the boundary of $B_\lambda$. Example 7.1 has shown that when $0 \leq \theta \leq \pi/3$ is analysed in 4D ($v = 4$) then $\frac{1}{\sqrt{2}} \leq \rho \leq 1$.
We assume that for $v \geq 3$, the map: $\theta \in [0, \varphi(v)] \mapsto \rho_v(\theta) \in [0,1]$ is *continuous*. When $v \to \infty$, $\varphi(v) \to \pi$ and $P_v$ becomes a circle. In a $P_v$-analysis, we suppose that $0 \leq \theta \leq \varphi(v-1)$, then $0 < l(v) \leq \rho_v \leq 1$, where $l(v)$ is the value of the relative side length $\rho_v(\theta)$ at $\theta = \varphi(v-1)$. Accordingly $0 \leq \rho_v(\theta) \leq l(v)$ for $\varphi(v-1) < \theta \leq \varphi(v)$. We get $l(3) = 1 > l(4) = \frac{1}{\sqrt{2}}$, and $\lim_{v\to\infty} l(v) = 0$ by continuity. Hence the quantisation effect disappears in the limit. The effect is the result of an organic causal analysis based on $P_v$ with $3 \leq v < \infty$, when $\theta \leq \varphi(v-1)$. It can be eliminated by a shift to a continuous straight line ($v = \infty$); then continuity takes over and replaces discreteness.
The reader is referred to (Chatelin 2011c) for a detailed study of the case where $\theta$ is obtuse ($\theta > \pi/2$ hence $v > 4$) but the analysis uses 4 causal dimensions (i.e. squares $P_4$ hence 3 causes).

The number 4 is a computational suggestion stemming from Fourier analysis of complex signals (Chapter 10 in Chatelin 2011a), uncovering the role of $\cos\theta \left(= \rho_4^2(\theta)\right)$.

△

## 8.2 About the second aspect based on $\hat{d}$

Because of the existence of the liaison $\Lambda$, it is not clear that the equivalence of models which may hold in the 1st aspect remains valid for the other aspects (no geodetic identification between $\oplus$ and $\hat{+}$).

Ungar(2008) gives the geodesics $\hat{G}$ and $\mathring{G}$ in the three concrete examples cited above.

**Example 8.2** We list the euclidean nature of geodesics $\mathring{G}$ and $\hat{G}$ for $n = 2$ corresponding to the three additions given in Section 2.6.

- In the Poincaré disc-model, with $\lambda = 1$, $\mathring{G}$ (resp. $\hat{G}$) is a euclidean circular arc which intersects the unit circle orthogonally (resp. at two diametrically opposite points).
- In the Belrami disc-model, $\mathring{G}$ (resp. $\hat{G}$) is a euclidean straight line segment (resp. an elliptic arc which intersects the boundary circle at two diametrically opposite points).
- In the hyperboloid model in $\mathbb{R}^2$, $\mathring{G}$ (resp. $\hat{G}$) is a hyperbola whose asymptotes meet at $O$ (resp. a straight line). The model was used by Helmholtz as early as 1870, and by W. Killing, and K. Weierstrass in 1872. △

In all three examples ($n \geq 2$) the second aspect reveals a curious *resurgence* of flatness into hyperbolicity. Geodetic triangles of type 2 may exhibit certain properties which are considered classically as characteristics of *euclidean* geometry (Section 8.28 in Ungar 2008). One can cite the two most emblematic ones:

- parallelism is supported,
- any triangle angle sum equals $\pi \Longleftrightarrow \delta_3 = 0$.

We add that the gaussian curvature computed in (Ungar 2008) for $\hat{d}$, $n = 2$, in the three examples, is *positive*, a fact classically associated with *elliptic* geometry.
More about the epistemological significance for weaving computation in Section 9.

## 8.3 The third aspect deriving from $\hat{+}$

The third aspect is by far the most challenging from a computational point of view. An in-depth treatment is beyond the scope of this report. We can only mention some properties exhibited by $\hat{L}$ which deserve serious attention:


a) base 3 b) base 2

**Figure 8.2**: $a$ and $b$ are dependent

1) Each line $\hat{L}$ with origin $a \neq 0$ has *two images* in the first and second geometries (Proposition 5.2). Therefore to any pair $(a, b)$ one can associate *four* lines, of which three are geodesics, the only non-geodetic being $\hat{L}_{ab}$.
2) Lemma 6.3 shows that dichotomy on a segment $(a, b)$, $a$ and $b$ independent, can be metrically ambiguous with $\mathring{d}$.

## 8.4 The fourth aspect deriving from linear dependence

If $W$ is perceived as a pencil of straight lines through the origin $O$, then the geometry is strictly reduced to that of the measuring rod $M$. The situation arises in two ways:
1) $a \neq 0$, $b = l \boxtimes a$, for $l \neq 1$, $rel(a, b) = I$. The origin $O$ appears as the unique source of analytic and geometric information for $l = 2$.
When $l = 1$, $W$ is perceived as a totally *discontinuous* set of points, a situation which evokes non interacting dimensionless particles.
2) $a = 0$ and $b \neq 0$, $rel(0, b) = I$, and $x = \hat{x} = b = a_2$, $b_2 = 2 \boxtimes b$.
Then geometry and analysis can never agree. The three lines have the common equation $b \boxtimes t$ and the right image corresponds to $b \boxtimes (t + 1)$: the parameter $t$ is shifted by $+1$.

It is tempting to interpret the two cases 1) $a \neq 0$ and $b \neq 0$ for any $l \notin \{0, 1\}$ and 2) $a = 0$ and $b \neq 0$ in terms of measurement on the corresponding line.
1) $O$, $a$ and $b$ are the three collinear points such that if $l = 2$, there is analytic *and* geometric coincidence. This evokes the base 3, see Figure 8.2 a). When $l$ is an integer $> 2$, one can think of base $l - 1$, yielding no analytic coincidence.
2) There are only $O$ and $b \neq 0$, *without* analytic coincidence. This corresponds to the base 2. See Figure 8.2 b).
This sheds an interesting new light on the difference in computing capabilities between base 2 and 3 respectively. See Chapters 6 and 8 in (Chatelin 2011 a).
The creative tension between the bases 2 and 3 in *one* dimension is reminiscent of the *choice* in causal dimension between 3 and 4 offered to analyse $x$ at the intersection of two geodesics of type $\mathring{G}$ in a hyperbolic plane with $\mathring{d}$ metric, when the angle $\theta$ between the two tangents satisfies $0 \leq \theta \leq \pi/3$. (Example 7.1 and Figure 7.2).

## 8.5 Shifting towards unity

Let us consider again the additions of Examples 2.2 ($n = 3$) and 2.3 ($n = 2$). The Einstein addition corresponds to an *outer observation* of motion in $\mathbb{R}^3$ (the observer is at rest, i.e the observation is *objective*). The latter addition corresponds to an *inner* observation of motion in $\mathbb{R}^n$, $n \geq 2$ (the observer is moving, i.e. the observation is *subjective*).

In the first (resp. second) type of observation, the geodesic $\overset{\circ}{G}$ (resp. $\hat{G}$) is a euclidean straight segment (resp. line). This remarkable fact has an important consequence. It enables the observer to operate a change of origin in the frame $V$. Putting the origin on the line or segment defined by $a$ and $b$ unifies the two previously distinct channels by making $a$ and $b$ appear linearly *dependent*.
In the case of inner observation one can choose the new origin such that $b = 2 \otimes a$ since $\hat{G}_{ab}$ is unlimited. This may not be always possible for outer observation because $\overset{\circ}{G}_{ab}$ is a limited segment.
The shift of origin is a move towards *unity* in two ways. First, it unifies the twofold channel into a unique one. Second, when $\theta = 0$ the causality for $v = 2$ is the classical 1 to 1 causality of science. The linear causality used in science appears in WIP as a particular case of organic causality which can only be valid when $\theta = 0$, reducing the pair $(x, \theta)$ to $(x, 0)$. The single datum $x$ is an *incomplete* information in WIP, which ignores $\theta$.
We believe that the shift of origin is at work in *simplexity* (Chatelin 2011a,b), a non-reductionist change of perspective in order to *ease* computation while preserving the complex structure of the observed phenomenon. An example in astronomical observation is provided by the shift of focus from Earth to Sun to analyse the planetary movements in the solar system. The counter-intuitive heliocentric view was first voiced in Greece (Aristarchus of Samos, ca. -310 to -230), then confirmed some 1800 years later by Kepler in Prague using Brahe's data.

# 9 An epistemological appraisal

## 9.1 Hyperbolic geometry in Nature

A number of natural shapes exhibit, at least locally, a hyperbolic character in their geometry. The most famous example is a horse saddle or a mountain pass. Among other natural hyperbolic surfaces, one can cite lettuce leaves, coral reef or some species of marine flatworms with hyperbolic ruffles. According to W. Thurston, if one moves away from a point in hyperbolic plane, the space around the point expands exponentially. The idea was implemented in crochet in 1997 by D. Taimina by ceaselessly increasing the number of stitches in each row of her crochet model (Henderson and Taimina 2001). Experiments have shown that the visual information seen through the eyes and processed by our brain is better explained by hyperbolic geometry (Luneburg 1950). This explains the popularity of hyperbolic browsers among information professionals (Lamping et al. 1995, Allen 2002).

## 9.2 Axiomatic vs. cloth geometries

The classical concept of a *group* underlies the three geometries which can be axiomatically derived from three versions of the parallel postulate: by a point not on a given line, one can draw a number $p$ of parallels to the line with $p \in \{0, 1, \infty\}$. The best-known case $p = 1$ corresponds to a linear vector space endowed with a scalar product and derived norm. The cases $p = 0$ (elliptic) and $p = \infty$ (hyperbolic) are modifications of the euclidean case, each with many equivalent models.

By comparison, cloth geometry is derived from a metric cloth framed in a linear normed space with dimension $n \geq 2$, and based on an organ $G(\oplus$,relator). It is not axiomatically defined, but is a *computational construct* based on $\oplus$ and on the corresponding choice of isometries for the relator. The computation results in a polymorphic geometry in which the relator blurs the clear-cut distinctions created by axiomatisation based on an abelian group. For example, $p = \infty$ and $p = 1$ can be co-existing properties. Depending on the choice $\boldsymbol{R}$ of isometries, the computed geometry will exhibit *new non-euclidean* features, among which some are already well-known in hyperbolic or elliptic geometries defined axiomatically.

## 9.3 Cloth geometry in the mind

In (Chatelin 2011 a,b) we have argued that hypercomputation in multiplicative Dickson algebras is part of the algorithmic toolkit for the human mind. Experimental evidence provided by Special Relativity indicates that the mental construction of the outside 3D-reality is controlled by cloth geometry based on Einstein addition.

**We posit** that, more generally, there exists a commonly shared set of relators for mind computation. This would explain why most people agree on the general appearance of the 3D- landscape, if not on all the details. Two eye-witnesses never agree on the minute details about the scene they both observed at the same place and time.

The existence of a common cloth geometry in 3D is the reason why we, human beings, have the feeling that we share more or less the same external reality, our habitat called Nature. As for the inner world inside each of us, it differs widely from one individual to the next. Why? Because the number $n$ of dimensions for the frame is not bound to be 3 anymore, but may vary arbitrarily at will, $n \geq 2$.
Cloth geometry provides a plausible mechanism for outer action and inner understanding after observation (Section 8.5). In WIP perspective, both processes result from a drive in the mind toward alignment. Both processes involve the participation of the observer's mind (at a conscious or unconscious level). We note once more that the observer is *free* to choose to relate $a$ and $b$ by outer or inner observation. However the reader should remember that the physical reference $\lambda = c$ for the speed of light is imposed by physical reality and defines the limit of observable velocities. No such constraint exists for inner observation; in other words the reference $\lambda$ is self-imposed (or chosen).

## 9.4 On the Poincaré vs. Einstein debate about Relativity and Geometry

During the first two decades of the 20th century the intellectual debate about the "true" nature of physical space was structured around Poincaré (and his legacy after 1912) and Einstein, see (Paty 1992). These giants stood at the two endpoints of a continuum of ideas running from Mathematics to Physics. The issues at stakes have been heatedly debated. They can be easily understood by means of hypercomputation and cloth geometry.
On the one hand Poincaré had an axiomatic vision of Geometry based on *groups* (Poincaré 1902). In special relativity he proved the dynamical invariance of physical laws for Mechanics and Electromagnetism (slightly ahead of Einstein). The relativistic dynamics presented in (Poincaré 1905) bears on group theory and (implicitly) on the field $\mathbb{H}$ of quaternions, two advanced mathematical notions which are now common in theoretical physics. His work wraps up more than 250 years of discoveries about the baffling behaviour of light (Auffray 2005). Poincaré is often criticised because – as Lorentz, Maxwell and Fresnel did before him – he occasionally mentions *ether*, a notion which is considered obsolete in current physics. We remark that in the cognitive perspective of cloth geometry in the mind, a reference is required for weaving computation, whatever name is given to it, ether, consciousness ....
On the other hand Einstein did not at first feel the need for a non-euclidean geometry, because he only slowly became aware of the physical consequences of his

non symmetric composition law. Together with Ehrenfest, Max Born and others, he realised that an accelerated motion would not permit exact rigidity for the moving body, but would imply elastic deformations and possible explosion. In order to save the relativity principle (by showing that it can apply to all kinds of motions including accelerations) Einstein had to *modify the geometry*, thus uncovering the full breadth of the 1905 paper.

Following (Paty 1992 ), we may say that: "Poincaré thought Physics with his geometric mind, as much as Einstein viewed Geometry through his physicist's eyes".

The principle of relativity has been observed in light phenomena since the 17th Century. In this intellectual odyssey, history has chosen to emphasise the year 1905 and the sole contribution of Einstein, This is an ironical twist of fate since the version of Special Relativity which survives today in textbooks bears out the group structure of Lorentz transformations due to Poincaré, while it overlooks the information role of Einstein's non commutative addition of 3-vectors in the construction of the human image of the world.

In retrospect, one realises that relativity has two intricate aspects based on *two* algebraic structures: the metric cloth $W_E$ (based on $\oplus_E$) envisioned by Einstein *and* the noncommutative field $\mathbb{H}$ (based on $\times$) implicit in Poincaré.

A thorough comparison between the *distinct* computational roles played by these two structures is given in (Chatelin 2011c).

## 9.5 Einstein's vision of Relativity

In 10 years (1905-1915) Einstein's vision evolved from the commonly shared euclidean view to a highly personal one. By transmuting ideas borrowed from Riemann and Poincaré he was led to General Relativity in 1916. This larger vision he would maintain and refine for the rest of his life (Einstein 1921). Hence his work presents a remarkable continuity of thought since the day he planted the seed of Relativity by positing that admissible velocities do *not* add in a symmetric fashion. The simplicity of this idea – so daring at the time – should strike a chord in any mathematically inclined mind! Simplicity is not triviality ...; it means depth and beauty, conferring a flavour of eternity to Einstein's revolutionary idea. The new idea ran against a couple of centuries of scientific development for physics, which had climaxed in the 19th century with a commutative addition for 2- or 3-vectors in classical Mechanics, symbolised by the parallelogram law. It is fair to say that there exists a world of difference between the two physics papers authored by Einstein and Ungar which are 83 years apart: the difference illustrates the progress of algebraic knowledge in the 20th century. More than a century had to elapse to allow the slow coming of age of the idea relativity: from its birthplace in experimental physics into its original habitat in the human mind. This evolution would not surprise the perceptive Mach who wrote in *Die Mechanik* (1883): "We should not consider as *foundations* for the real universe the auxilliary intellectual means that we use for the *representation* of the world on the *stage of thought." (italics in original).*

The relativistic formula is routinely put to good use by engineers in telecommunications and space industries. But is it really understood? A look at textbooks for physics undergraduates casts some doubts. The pristine clarity of Einstein's addition is obscured behind the cloud of Lorentz transformation and its inherent technicalities. The essence is lost in the mist of Minkowski's 4D-spacetime. It is not uncommon to find only the symmetric formula (valid for parallel velocities) as any quick websurf will confirm. It is no coincidence that history has chosen to tout the (physically more difficult) equation $E = mc^2$, which is but one of the many consequences of Einstein's source law of addition.
We find another clue to the incomplete understanding of relativity if we look at Quantum Mechanics. Einstein was never convinced by the theoretical status of this small scale domain of Physics. He acknowledged that the equations worked extremely well but he maintained that the theory was *incomplete*. And indeed quantisation can follow from relativity (Sections 7.3 and 8.1). Einstein's views on QM were not taken seriously by the physicist community; however, his deeply intuitive understanding of relativity did not lead him astray.

The result of this unsatisfactory state of affairs is that relativity is not yet fully embraced: it is, at best, interpreted as an exotic law of Nature, with no other consequences on everyday life than the use of cellular phones and GPS devices. Relativity is not perceived as giving us a clue about the ways by which the human mind builds its *"imago mundi"*, its image of the world (Chatelin 2011a,b). The role of relativity in western science is confined to physics research (nanoscale or high energy) in order to develop ever more sophisticated technologies. More than one century after Einstein's groundbreaking discovery, relativity has not yet been taken seriously by social scientists. They do not venture beyond the overly simplified version that is called *relativism*, a mental construct which does not do justice to the philosophical depth of relativity.
Information Processing is of paramount importance for human affairs. Information-based activities such as education, medicine, economy and ecology, could benefit greatly from a new relativity-based scientific approach to cognition.

# 10 Cloth geometry and human Information Processing

We conclude by a short recollection of some of the arguments which show that relativity is an ubiquitous property of the reality that we build and experience on earth. In the course of the report some major principles for WIP have emerged:
1) Given two points $0 \neq a$ and $b$ which are independent, there are 4 ways to relate $a$ to $b$ by lines, 3 of which are geodesics, including 2 geodetic images of the non geodetic line. These four ways reduce to 1 when $a \neq 0$ and $b$ are independent (Section 5.3). Section 8.1 has indicated that there exists an implicit organic causality which is 1 to $v-1$, $v \geq 2$, in general. For $v \geq 3$, the rules are that (1) there are more than one cause to any information, (2) action at a distance is ubiquitous.
2) The geodesics are channels which either radiate or emanate information. The third (non geodetic) line, has 2 geodetic versions which channel two different informations about itself. This expresses the twofold dynamical potential of weaving computation in cloth geometry when $a$ and $b$ are independent.
3) When $a$ and $b$ are dependent, we get clues about the respective roles of base 2 and base 3 in Nature's computation.
4) Empirical evidence based on measurements can be problematic due to metric entanglement (Section 4). Empirical sciences will always need revision for methodological reasons. They are temporary mental constructs which reflect the current know-how of the day, always in a state of becoming.
5) If the external reality can appear, up to a point, to approximately obey general physical laws, such a reduction is radically impossible for the inner reality of each individual mind, who is freely choosing its values for $n \geq 2$.

Any serious theory of human knowledge acquisition should not ignore the lessons above taught by cloth geometry. Weaving computation is performed by the mind to extract *meaning* from observation. Lessons 1), 2) and 3) are about the construction of this meaning in the human mind. And lessons 4) and 5) expose some of the limitations of any naively scientist and purely empirical approach to life on earth. Assigning a *unique* cause $a$ to an observation $b$ is possible only when $a$ and $b$ are *linearly dependent*, that is in the limited context of a purely linear theory. Linear independence between $a$ and $b$ implies that the information is processed through a twofold potential. Under observation, this information processing (which includes action at a distance) can sometimes be interpreted as *creativity*. In molecular biology, the result of action at a distance of an information is often observed as a genetic modification usually called mutation.

Moreover to analyse a phenomenon $x \neq 0$ at the intersection of two geodesics associated with $\oplus$, one needs information about the angle $\theta$ between the geodesics (the legendary "hidden" variable). If $\frac{\pi}{3} < \theta \leq \frac{\pi}{2}$,$v = 4$ and 3 causes of geometric origin are required for the analysis. If $0 < \theta \leq \frac{\pi}{3}$, the choice is possible between 2 or 3 causes. Quantisation occurs when $x$ is analysed in $\mathbb{R}^4$ but $\theta \leq \frac{\pi}{3}$. Continuity in 4D is obtained only in the euclidean limit $\theta \rightarrow \frac{\pi}{2}$. This organic causality view challenges the conventional theory of randomness in science where $\theta$ is ignored.

It is discomforting to note that our post-modern society – increasingly based on "information" – ignores the message of relativity. Through massive computerisation, its scientific agenda is to reduce quality to quantity, with the predictable outcome that all human creativity will be eradicated and mankind will be transformed into an army of robots acting in an artificial "linearly rational" reality. The growing awareness against this nightmarish future, akin to Brave New World (A. Huxley), may not be as unscientific as it is often described. If the 2D-chart is not the 3D-territory, even less so a one-dimensional ranking scale. The value of human life cannot be judged on a cost/benefit scale! The healthy reaction to say: "No" is deeply rooted in the feeling that man is neither a machine or a market product. Cloth geometry in the mind provides a welcome demonstration (based on computation and geometry only) that this feeling is indeed mathematically sound. The future need not be gloomy because computers can do much more for us than to participate in the domination of a financial techno-structure which heightens the imbalance of the world to unbearable extremes.
They are versatile tools which can prove irreplaceable to implement cloth geometry and the organic logic of life to the greater benefit of all life on earth. A deeper understanding of Nature's reality is needed to cope with the physical limitations of our planet. We have shown why the principle of relativity in the mind is one of the keys which are necessary to live up to the scientific challenges of the 21st century.

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