An Invitation to Qualitative Computing

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It presents an historical survey of the evolution of mathematical computation since Sumer four millennia ago. It shows how the laws of logic have evolved in the mind of mathematicians. This evolution has been driven by three major ideas. First idea: *multiplication*, that is nonlinearity, leading to $\sqrt{2}$ (irrationality in Ancient Greece) and $\sqrt{-1}$ (complexity in Renaissance Italy). Second idea: *zero*, a difficult concept (there exists the non existence is paradoxical) turned into a number in India (7th century). Third idea: *actual infinity* (the “inverse” of zero) embraced by Middle Ages Europe. This was the milestone that unleashed the power of modern analysis and made possible the rapid development of western technology.

This evolution, which appears necessary only in retrospect, was not at all obvious and demanded an immense courage and a lot of soul-searching on the part of mathematicians in the past. The evolution is fragile and may stop at any time. For example, it was dramatically slowed down at the beginning of the 20th century when logic took precedence over computation, leading to a purely axiomatic view of mathematics with many adverse consequences for a sound understanding of life on earth. Thanks to countless unfashionable numerical analysts working at the margins of main-stream mathematics, the spirit of mathematical computation was kept alive during the reign of the all-axiomatic ideology. New fashion and the pressing need to overcome the limitations of our planet may soon call it back in the open.

Keywords: Qualitative Computing, numeracy, irrational, complex, zero, actual infinity, hypercomplex numbers, numerical linear algebra, backward analysis test.

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“My illustrious sister, holy Nisaba,  
Is to receive the 1-rod reed.  
The lapis-lazuli rope is to hang from her arm.  
She is to proclaim all the great divine powers.  
She is to fix boundaries and mark borders.  
The planning of the gods’ meals is to be in her hands.”  

Verses 406-411 in *Enki and the World Order*,  
Sumer, ca. 2000 BC.

This introductory Chapter sketches a few landmarks in the historical evolution of Computation. Computing is a human activity much older than any historical records can tell, as testified by stone and bone tallies found in prehistoric sites. The poem quoted above illustrates the fact, surprising to a modern eye, that the art of computing was regarded as a divine feminine attribute four millennia ago! In sumerian cities, a number of women ran schools where scribes-to-be would learn to read, write and compute [Robson (2008)]. Computing is still today one of the first skills taught to small children.

*Why* do we compute? What does it mean to compute? To what end? Everyone senses that there is more to the matter than the multiplication tables. Computing is one possible form of the “libido scienti”, the quest for knowledge, which dwells in every human being. But because our technological society wants to compute more and more efficiently, quickly and accurately, the philosophical question “why compute” tends to be relegated to the background. The engineering question “how to compute” takes over in practice.

The meaning of the very act of computing recedes from sight as we are increasingly surrounded by a digital world which separates us from our basic nature.

It is more than ever vital that we examine Computation through the double lenses of *why* and *how*. This will lead to a better understanding of intensive computer simulations in particular, and of the dynamics of evolution in general. The why and how in computation have many different facets, which are interrelated. We refer to them under the global umbrella title **Qualitative Computing**. The technical aspects (about the how) are developed in the following chapters. The opening and concluding chapters put the emphasis on the why. Qualitative Computing is the branch of mathematics which extends analysis and algebra by looking specifically at how the laws of classical computation (Euler-Cauchy-Riemann-Jordan-Puiseux) are
modified when mathematical computation does not take place over a commutative field.

[Qualitative Computing is a mathematical field much wider than the domain created by social scientists which goes under the same name. In social sciences, qualitative computing refers only to the use of computers to create new epistemological data and ontological arguments to study language and meanings.]

1 The art of computing before the 20th century

1.1 Numeracy is not ubiquitous

It is well known that the art of computing did not develop everywhere on planet earth. Unlike language, the potential to compute may remain dormant, favouring the development of other skills. Remarkably, when it does appear, computing may precede reading and writing. Below is a quotation from Duhalde, Description de la Chine et de la Tartarie chinoise, p. 293, 1735, cited in [Lucas (1891), p.XVII]:

“Les premières colonies qui vinrent habiter le Se Tchuen n’avaient, pour toute littérature, que quelques abacé arithmétiques faits avec de petites cordes nouées, à l’imitation des chapelets, à globules enfilés, avec quoi ils calculaient et faisaient leur comptes dans le commerce. Ils les portaient sur eux et s’en servaient quelquefois pour agrafer leur robe; du reste, n’ayant pas de caractères, ils ne savaient ni lire, ni écrire."

While the Chinese calculators used the bases 2 and 10, the Babylonians were skilled in base 60 – hardly suggested by the 10 fingers! They used a mixed positional representation which combined base 60 and base 10, and had a special mark for missing digits. This notation survived through greek astronomy until today (minutes and seconds).

1.2 $\sqrt{2}$: an irrational consequence of nonlinearity

According to the Greeks, the world could be ordered by means of the sequence $2, 3, 4, \ldots$ of natural integers (arithmos) which derived from the unit 1, not itself considered an ordinary number. It is said that the proof of the irrationality of

\[\text{The first societies which went to inhabit the Se Tchuen had as literature only a few arithmetical abaci consisting of pearls on knotted threads similar to rosaries, with which they calculated and kept their trade accounts. They wore them and sometimes used them to fix their robes; anyway, not having symbols, they did not know how to read or write.}\]
\( \sqrt{2} \) created a scandal: the inconspicuous diagonal length of the unit square was “alogos” (without name) because it did not obey the law of rationality: it lived outside the “logos”, the world of ratios of integers. This story (reported centuries after the event) is almost surely apocryphal. It is more likely that the Greek thinkers interpreted the irrationality of \( \sqrt{2} \) as an incompleteness result (see Plato’s dialogue between Socrates and Meno’s slave described in Remark 9.6.1). In modern terms, the quadratic equation \( x^2 = 2 \) has no rational solution. Alternately, plane geometry cannot be fully captured by integers evenly spaced on the real line. Natural integers are not the whole story: geometry exceeds rationality.

That the greek numberworld exceeded the set of rational numbers is clear from Book V in Euclid’s *Elements* (3rd century BC) which presents a general theory of proportions designed to compare magnitudes (largely developed by Eudoxus of Cnidus). And two major scientists from the hellenistic world, Heron in Alexandria and Archimedes in Syracuse devised efficient algorithms to compute approximate values for the irrational numbers \( \sqrt{2} \) and \( \pi \) respectively, which improved on the early babylonian values. The greek insight into complex numbers is illustrated by the Antikythera astronomical calculator (constructed between 150 and 100 BC) which was miraculously rescued from deep sea in 1900. It took more than a century to be restored and deciphered, demonstrating the hellenistic knowledge of elliptical orbits for the Moon and the Sun [Marchant (2008), pp. 253-254].

**Remark 1.2.1** The dichotomy rational/irrational in numbers stems from the comparison between the lengths \( a, b, c \) for the vectors \( x, y, x + y \) in the plane. When \( x \) and \( y \) are colinear, then for \( a \) and \( b \) integral, both ratios \( \frac{c}{a} \) and \( \frac{c}{b} \) are rational because \( c \) equals \( a + b \) or \( |a - b| \). This is not the case when \( x \) and \( y \) are orthogonal. The rule becomes quadratic: \( a^2 + b^2 = c^2 \).

Tradition (Cicero, Plutarch) has attributed the rule to Pythagoras, but its origin can be traced back to the Stone Age. Both megalithic standing stones [Thom (1964)] and sumerian clay tablets [Conway and Guy (1986), pp. 173-176] have been tentatively reinterpreted in the light of pythagorean triangles. No record has been found of a general proof older that of Euclid (Book I, no. 47). Most probably, the visual geometric proof with the square \( c \times c \) inscribed inside the square \( (a + b) \times (a + b) \) was compelling evidence of the correctness of the rule \( (a + b)^2 = a^2 + b^2 + 2ab \) commonly used in Mesopotamian scribal houses.

In China, the oldest mathematical text known is the *Chou pei suan ching* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven). It collects the mathematical knowledge accumulated in China until the time of Confucius (6th century BC), the time of Pythagoras in southern Italy. The visual proof was so popular in China that the diagram for the triple 3-4-5 bears the special name *hsuan-thu* in the *Chou pei* [Swetz and Kao (1977), pp. 14-16]. The first general proof known is due to Zha Shang (3rd century AD). The *Chou pei* displays a remarkable trigonometric knowledge based on circles and right triangles which rings an amazingly modern tone. It anticipates by some 2400 years the invention/discovery of complex integers expressed in the trigonometric form due to de Moivre (early 18th century). In the Indian subcontinent
also, mathematics and astronomy developed in parallel. They consisted of a set of empirical rules (with no formal justification) orally transmitted to students in the form of Sanskrit verses [Plofker (2008)].

1.3 Zero: thinking the unthinkable

Beyond its role as a number, zero is above all a formidable philosophical concept. It takes great intellectual courage to state, as an evident truth, the obvious contradiction: “Nonexistence exists and is called Zero”.

Aristotle took the risk, got scared and shied away from the logical impossibility. A computational path into the impossible was gradually opened by various Indian thinkers, until Brahmagupta turned “sunya” (void) into the number 0 by showing how to do arithmetic over \( \mathbb{Z} \) (on goods, debts and “nothing”) in a book dated 628 A.D.

When unleashed and tamed by computation, zero allowed the art of computing, initiated in India (under the name chakra-valla, or cyclic process) to blossom in the Middle East (Baghdad) and Central Asia, thanks to the highly successful arabic adaptation by Al Khwarizmi (hence algorithm) in the 9th century. It was handed down to Europe between the 12th and 15th centuries where it was called Algebra, but was received with great suspicion. The italian abacists did not need 0 to compute with their abaci. The new written arithmetic, done with pen and paper, eventually won acceptance because it was without rival to balance the account books of European merchants while keeping track of all transactions. But the “real” existence of negative numbers was still a matter for debate in the middle of the 17th century. Far from being naive or silly, the debate indicates clearly the limits of mathematics when it comes to philosophy. Turning the concept of “sunya” into the number 0 does not exhaust the philosophical conundrum: how to conceive of nonexistence without calling it into existence?

1.4 \( \sqrt{-1} \): a complex consequence of nonlinearity

In the 16th century, a new difficulty arose in Italy while solving \( x^2 + 1 = 0 \), or \( x^2 = -1 \): how could a square number be negative? Out of necessity, Cardano (1545) created the symbol \( \sqrt{-1} \) to represent one of the two roots of the above (quadratic) equation. Even with the computing rules enunciated by Bombelli (1572), the strange \( \sqrt{-1} \) was treated with extreme reservations, as an impossible or imaginary oddity.

This resistance was justified: \( \sqrt{-1} \) cannot be a real number lying on the real line \( \mathbb{R} \) since its square is negative. The difficulty, as with \( \sqrt{2} \), stems from the quadratic
character of the equation to be solved: respectively \( x^2 + 1 = 0 \) and \( x^2 - 2 = 0 \). It took another 300 years for scientists to fully realise that a complex number \( a + ib = a + \sqrt{-1}b \) with \( a, b \) real represents the point with coordinates \((a, b)\) in the plane \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \). In other words, real numbers have 1 dimension but complex ones have 2 dimensions. In the skilled hands of Euler and Cauchy complex numbers became indispensable to describe many wave phenomena in physics (electricity, magnetism, sound, light ...). These successes made them quickly popular among engineers.

1.5 Infinity: decoding divergent series

Infinity is not a number, it is a pregnant mathematical concept related to the impossibility of any division by 0. The ancient Indians clearly distinguished between potential and actual infinity, which they also called “whole”. They pondered its many paradoxical aspects (Isha Upanishad, ca. 4th to 3rd century BC).

The lack of the number 0 in Greece severely limited the ability to think about infinity in any other way than as an unreachable potentiality. The difficulty is vividly illustrated by Zeno’s paradox about Achilles racing the tortoise. The need to evaluate areas and volumes forced Archimedes and his arabic successors to work their way around the difficulty by exhaustion. The fact that Archimedes computed the sum of the geometric series with ratio 1/4 in this way is no small feat.

But the concept of infinity as an actual limit only emerged in Europe in the 14th century leading to important activity on infinite processes such as series. Nicole Oresme (1323-1382) proved the divergence to \( \infty \) of the harmonic series \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots \).

It is after Euler and Abel that it became pressing to clarify the conditions under which infinite series would converge. And when they diverged, was it possible to assign a finite value by expanding the rules for summation? A most famous example is provided by the series \( s \in \mathbb{C} \mapsto \sum_{n \geq 1} \frac{1}{n^s} \) defined for \( \Re s > 1 \) and expanded as \( \zeta(s) \) into \( \mathbb{C}\setminus\{1\} \) by Riemann (1859) using analytic continuation. In a weak sense, the series has the two values \( \{\infty, \zeta(s)\} \) for \( \Re s \leq 1, s \neq 1 \). In the usual sense, it has the unique value \( \infty \) for \( s = 1 \), and \( \zeta(s) \) for \( \Re s > 1 \), with \( 0 < |\zeta(s)| < \infty \). The complex expansion of the domain for \( \zeta \) establishes the validity of the conjecture in [Euler(1749)] that \( \zeta(-2n) = 0, n \geq 1 \).
2 The unending evolution of logic due to complexification

The evolution of the concept of number which has occurred over some 3 to 4 millennia is quite impressive. It is the breathtaking result of many multicultural cross-fertilizations between thinkers from Sumer, Greece, Egypt, Baghdad, India, China and central Asia. Later the torch moved westward, towards Europe and North America.

2.1 Classical analysis

The principles of real and complex analysis were firmly established during the 18th and 19th centuries. These successes led most mathematicians to believe that the construction of numbers had been satisfactorily completed with the complex numbers (which represent the topological closure of the algebraic numbers). The only attempt to look at the Fundamental Theorem of Algebra (FTA) beyond $\mathbb{C}$ [Eilenberg and Niven (1944)] remained an idle curiosity. But, in spite of human prejudice, the construction of new numbers by complexification never stops, as we shall see below.

2.2 The creative role of zero

We saw that zero is born out of an irreducible contradiction. The number zero is a creation of the human mind, calling the nonexistent into existence. This reaches far deeper than the sign zero which signals missing digits in the positional representation. This explains why a supreme philosopher such as Aristotle – who wanted to describe what “is” – could not accept 0 in the event. But once 0 is accepted in computation, equations can be written down. And algorithmic computing can be set into motion as soon as infinite processes can be thought of.

2.3 The evolutive pressure of paradoxes on logic

The two symbols $\sqrt{2}$ and $\sqrt{-1}$ appeared paradoxical at first: they broke certain computing rules which were agreed upon at the time, so much so that they were rejected for a while as unsuitable for the task of understanding computation.

1) $\sqrt{2} \neq \frac{a}{n}$: $\sqrt{2}$ falsifies the implicit assumption that any geometric length is the ratio of two integers $n$ and $m$. 

2) $\sqrt{-1}$ breaks the explicit rule that a square is positive.

In each case, the apparent paradox was resolved by expanding the conceptual frame for computation. Irrational numbers were no problem in Baghdad in the 10th century. Both paradoxes above are an invitation from computation to enlarge the frame of thought, to know better and modify the laws which call for a careful adaptation which cannot be arbitrary.

In each case, the source of the paradox lies in Multiplication: the product $x \times x = x^2$ does not behave as expected; something new happens. And wonder of wonders: the surprises do not stop with dimension 2 and Cardano’s creation of $i = \sqrt{-1}$.

2.4 Hypercomplex numbers of dimension $2^k$, $k \geq 2$ (1843-1912)

Multiplication over complex numbers of dimension 2 is possible and remains commutative. Is multiplication possible over numbers represented as vectors of arbitrary dimension $n$ while preserving the potential to compute? Not unless $n = 2^k$, $k \geq 0$, as discovered by three spirited adventurers.

For $k = 2$ by R. W. Hamilton (October 1843): $\times$ is not commutative over the quaternions in $\mathbb{H}$,

for $k = 3$ by J. T. Graves (December 1843): $\times$ is not associative over the octonions in $\mathbb{G}$,

for $k \geq 4$ by L. E. Dickson (1912): $\|x \times y\| \neq \|x\|\|y\|$ and zerodivisors exist in any quadratic Dickson algebra $A_k$ of dimension $2^k \geq 16$.

[A complete description of hypercomplex numbers is given in Chapter 2.]

As is often the case, history repeated itself. At the end of the 19th century, a fierce battle was raging between pro- and anti-quaternionists [Crowe (1994)] which ended in the temporary victory of the easy-to-grasp vectors in $\mathbb{R}^3$ over the quaternions in $\mathbb{H} \cong \mathbb{R}^4$.

None of the three valiant pioneers – Hamilton, Graves and Dickson – did succeed in convincing their peers that their innermost conviction of the necessity to go beyond complex numbers was of any value. Complex numbers were good enough for everyday science. Why bother with difficulties such as noncommutativity, nonassociativity and still worse? This book will show why. It will vindicate the computational intuition of the inventive trio who were too far ahead of their times. Around 1900, times were not yet ripe for a theory of computation centered on multiplica-
tion because linear vector spaces were the exciting novelty then: quaternions were ridiculed by most and octonions ignored by all!

3 The 20th century

3.1 A paradigm shift

At the dawn of the 20th century, an event of momentous proportions took place which was to shape the post-modernist Zeitgeist [Gray (2008)]. Scientific logic, which so far had been second to the art of computing, took precedence over computation and imposed its iron law. The notion of mathematical computability, crafted over millennia, was abandoned in favour of mechanical computability, also called “effective” computability because it could be delegated to a machine.

The coup de force was accomplished in the name of rigour, neglecting the fact that rich polysemic notions are necessarily ambiguous. Only trivialized notions can be crystal clear. The inventive mathematical spirit was to be cut to size to survive the scrutiny of a machine!

The British logician A. N. Whitehead knew all too well the risks of perfect rigour, having coauthored with B. Russell the 3 volumes of the Principia Mathematica (1910-1912). He later wrote in Adventures of Ideas: “Insistence on clarity at all costs is sheer superstition as to the mode in which human intelligence functions.”

To put it bluntly, man is not a machine! The anti-mechanistic stance had such prestigious advocates as Poincaré, Borel and the fierce fighter Brouwer. But none of them could win over the naive scientism of the time, fully under the spell of the machine. By a twist of fate, at the time mathematics was being purified of all ambiguity, most experimental sciences were infiltrated by the highly ambiguous notion of randomness.

3.2 Fixing the laws of logic a priori

Section 1.2 has illustrated how the laws of computational logic residing in the minds of mathematicians have evolved in the course of history under the pressure of computing paradoxes. Of course, such an evolution is an open-ended process which comes from the inner necessity to adapt logic to the increasing complexity of computing in any dicksonian algebra $A_k$ when it lacks commutativity for $k \geq 2$ and associativity for $k \geq 3$ (with $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$).
During the first third of the 20th century, the obsession with rigour was so strong that any logical theory would ban paradoxes. But paradoxes are the true source for the autonomous evolution of mathematics. Fixing the rules of logical inference a priori is the surest way to disallow mathematical creativity. This seems a steep price to pay for rigour! And this may account for the impenetrability of most new mathematical theories which, during the last century, were not born out of inner necessity.

3.3 The eclipse of the art of computing

Because of the structuralist and axiomatic diktats that came into fashion to deal with Mathematics in the 1930s, the art of computing underwent a long eclipse. It became dormant. Its development essentially stalled after the remarkable work of Cauchy and Riemann on analytic functions of a complex variable which paved the road opened by Euler in the 18th century. By tacit consent, multiplication of numbers is to this day very often considered commutative. There is one brilliant exception to be described next in Section 3.4.

“Objection” will say the skeptic, explaining that “computers are everywhere. The world around us is being digitized at lightning speed, and we live under the rule of the quantity”.

True, but blind mechanical calculation is a far cry from the creative art of computing for many reasons. Let us mention three of them.

1) The intellectual and mechanical odyssey of the western mind rests on the rejection of new paradoxes, forgetting that life itself is paradoxical. This blindness does not come for free; there is a price to pay for ignoring the paradoxes of life.

In nonlinear sciences, this became visible in the 1970s when a number of “chaotic” phenomena were uncovered by experiments which challenged the classical theory of mechanics. As the past century drew to a close, more and more clues were found in lab experiments which indicated that a strict ban on paradoxes is untenable in the experimental sciences. Many natural phenomena seem to be ruled by an emergent meta-principle of organization. The new scientific frontier at the beginning of the 21st century is best described by “self-organization” and “complexity”. In order to cope with increasing complexity, living organisms display an ability to simplify the rules without reducing the complexity. This emergent feature of life, known as “simplicity” [Berthoz (2009)], realises a synthesis between self-organization and complexity, as well as between induction and deduction. A typical example of simplicity is the Copernican model in astronomy, whose deeper simplicity enabled Ke-
pler to propose his three laws, later superseded by Newton’s one law of gravitation. At the time of Copernicus, the geocentric model provided an equally good agreement with astronomical data. But its complicated structure limited severely its predictive power.

2) Digital reality is but a simulation of reality, a make-believe. *Simulation* expresses the know-how; it describes only the “how” from outside. But *knowledge* is of a different kind which cannot be reduced to simulation. Knowledge is from within; it tells more than simulation about what is really going on inside. Simulation is an objective activity which comes from without, from outward and cannot reach deeper than the descriptive level. The chart is not the territory.

3) The notion of effective computability (based on Turing machines) has no epistemological value (Chapters 7, 8 and 12). It took only 4 decades (1936-1977) to prove that it can only produce an *aggravated* form of randomness within formal axiomatic systems [Chaitin (1977,1987)]. For a more philosophical discussion of artificial intelligence, see [Dreyfus (1979), Penrose (1989)].

### 3.4 The rise of numerical linear algebra

During the 20th century, numerical analysis has been one of the very few domains of mathematics to stay immune from the pervading all-axiomatic fashion. This, of course, was interpreted as a weakness [Dieudonné (1987)]. Much to the dismay of the majority, some mathematicians remained deaf to the siren song of axiomatisation: these unfashionable craftsmen went on quietly practising their art much in the spirit of the 19th century.

With the advent of electronic computers, they were on the right track to propose methods and algorithms which could tackle problems that defeated exact methods of analysis. Thanks to such people working at the margins of main-stream mathematics, the art of computing was kept alive, burning on a low flame. With the growing availability of computers in the past 60 years, the new domain, now known as *numerical linear algebra*, emerged to become the theoretical backbone for the proper use of modern scientific computers. These machines are not useful tools for computation unless they are run with *reliable numerical software* (see Section 4.2 below). Scientific computers are now indispensable for our technology-based development. The reason for this sweeping success rests on their use of scientific notation for numbers – a fact which is either ignored or misinterpreted! Despite the prevalent prejudice against the *floating point representation* of machine numbers, this *scientific notation* is the magic by which a mere mechanical piece of hardware can
metamorphose into a computer endowed with an *epistemological value* (Chapter 8).

In meeting the demands of scientific computing, little-known mathematical breakthroughs are at least as important as the well-advertised hardware innovations. Very few mathematicians are familiar with the two mathematical results which make possible the reliable resolution of the most basic problems in matrix theory, i.e. solving linear systems and computing eigenvalues.

On the one hand, solving a set of $m$ linear equations in $n$ unknowns ($m \geq n$) uses an algorithmic idea due to Laplace (1812) which *factors* the $m \times n$ array $A$ of real or complex data as the product $A = QR$, where $Q$ is an $m \times n$ orthonormal matrix ($Q^H Q = I_n$, where $Q^H = (\bar{Q})^T$ denotes the conjugate transpose of $Q$, $I_n$ being the identity matrix of order $n$) and $R$ is upper triangular of order $n$. When $m > n$, the least-squares solution of $Ax = b$ satisfies the linear system of $n$ equations given by $Rx = Q^H b$ [Laplace (1820), Langou (2009)].

On the other hand, Schur (1909) established that a general square matrix is always *unitarily* similar to a triangular matrix, revealing its eigenvalues on the diagonal. This property justifies the QR algorithm, a revolutionary method devised by Wilkinson and his collaborators at NPL, Teddington, in the early 1960s to compute the complete spectra of matrices. They exploited an idea independently published in 1961 by Vera Kublanovskya in Russia and John Francis in England. The two were standing on the shoulders of H. Rutishauser in Switzerland. Why revolutionary? Because the QR method is in fact a *global* method which amounts to computing all roots (without omission) of the characteristic polynomial. The novel method is again a remarkable example of simplicity: Newton’s iteration (which is only a *local* method) multiplies data of dimension 1 (scalars over $\mathbb{R}$ and $\mathbb{C}$), whereas the QR method multiplies data of dimension 2 (square matrices, i.e. arrays of scalars). This algorithmic feat realizes a *paradigm shift* in the art of computing. In the 1960s, for the first time in computational history, it became possible to compute the totality of the zeros of an arbitrary polynomial as the spectrum of its companion matrix. The convergence properties were elucidated in [Parlett (1968)]. The QR method, which is the *constructive* version of the fundamental theorem of algebra over $\mathbb{C}$, is based on the theorems of Laplace and Schur.

**Remark 3.4.1** 1) The QR method for computing the spectrum of $A$ is based on the Laplace factorization $A = QR$ (with $m = n$). The basic method computes the sequence of matrices unitarily similar to $A = A_0 = Q_0 R_0$ and defined by $A_k = R_{k-1} Q_{k-1} = Q_{k-1}^H A_k A_{k-1} = Q_k R_k$ for $k \geq 1$. The sequence can converge (up to a diagonal unitary similarity) to a triangular Schur form for $A$ under specific assumptions [Chatelin (1988, 1993)].
2) The Laplace algorithm for the factorization $A = QR$ is most often attributed to Gram and Schmidt in the engineering literature. In mathematics, the factorization $A = QR$ goes under the name of “Schmidt orthogonalization”.

3) The approach of Laplace to the least-squares problem is a better alternative than that of Gauss which yields the classical $n$ normal equations $A^H Ax = A^H b$. Each method realises an orthogonal projection of the problem on the range subspace $\text{Im} A$. But only Laplace uses an orthonormal basis to represent the projected problem in $\text{Im} A$. Therefore Laplace’s version is vastly more robust than Gauss’ to uncertainty in the data. The condition number of $R$ is that of $A$, whereas the one of $A^H A$ is that of $A$ squared. This squaring may have a devastating effect in practice when data are strongly correlated. This fact is rarely mentioned in mathematical presentations of the least-squares problem.

It may seem strange to an outsider that neither the mathematical stature of Laplace and Schur, nor the deep computational significance of their results were sufficient reasons to win them the recognition of pure mathematicians. These two theorems – with simple proofs and momentous algorithmic consequences – are almost never included in the college-level curricula for schools of mathematics worldwide. This simple fact reveals how deeply ingrained in the western mind is the dichotomy between theory and practice!

Despite the complete lack of understanding on the part of the mathematical establishment, the numerical linear algebra community goes on unabated in its endeavour, rightly proud of its remarkable achievements such as the highly acclaimed packages LAPACK and Matlab. What made these achievements possible? The field offers a unique blend of matrix algebra and mathematical analysis over $\mathbb{R}$ or $\mathbb{C}$ together with the experimental spirit of engineering, harbouring no dogmatic views about what is allowed. It became clear in the 1980s that good time-efficiency could be obtained on parallel machines when the basic “numbers” for the algorithms were themselves matrices. Today, the numerical linear algebra software field exploits to its fullest the capabilities of algebraic computation over the noncommutative algebra of matrices. This explains the algorithmic success of another of Schur’s discoveries, known as the Schur complement formula [Schur (1917-1918)], cf. Chapter 7.

Many users and mathematicians alike have an instinctive distrust of scientific computers because of their inexact arithmetic. This is viewed as a serious limitation casting doubts on the validity of all computed results. But such a reaction may not be as “rational” as it seems at first sight! Although counter-intuitive, it is proved (and verified everyday in practice) that finite precision arithmetic can be an asset for computation. The Krylov-based methods, which are today without rival to deal numerically with extremely large matrix problems, are a famous example of the
phenomenon [Parlett (1980)]. Another aspect of the phenomenon is described in Chapter 10 (Examples 10.5.2 and 10.6.1).

3.5 Contemporary experimental sciences

Classical physics (including relativity) proposes an interpretation of the physical world set in a linear vector space, with 3 dimensions for space and 1 dimension for time. This interpretation has been extremely successful in mechanics and lies at the bottom of our modern technology. Its very success overshadows the fact that a linear framework may not be the most appropriate one to interpret nonlinear phenomena. In a truly nonlinear process, 3 or 4 dimensions may best be interpreted as 3 or 4 generators for real Dickson algebras. Considering up to 3 generators defines the three complex division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{G}$. But if one wants to consider $k \geq 4$ generators, the algebras $A_k$ are without division. In the 20th century, geometers (notably E. Cartan) and physicists chose to expand the associative framework of quaternions (finally accepted) to that of Clifford algebras which remain associative (Chapter 2, Section 2.2.5). Because of the imposed associativity, octonions are de facto outlawed and computation is severely limited. Clifford algebras remain at the descriptive level of geometry; they cannot display the full gamut of nonlinear computational phenomena found in Dickson algebras. A strong computational necessity to go beyond associativity is expressed in theoretical physics [Baez (2001)]. The necessity is even stronger in life sciences [Woese (2004,2007), Alon (2007)].

4 Back to the art of computing

Experimental sciences indicate that it is time to free the complexity of computation in Dickson algebras $A_k$ from the arbitrary limit imposed by the bound $k \leq 2$ (most often $k \leq 1$). The driving force behind the art of computing is the key notion of the noncommutative multiplication of vectors or of square matrices which constitutes the axis about which this book turns.

4.1 Hypercomputation in Dickson algebras

The Dickson algebras form an unbounded sequence of quadratic algebras $A_k$ consisting of vectors of dimension $2^k$, $k \geq 0$, on which multiplication is defined recursively. Each $A_k$ is a complexified version of $A_{k-1}$, $k \geq 1$, with $A_0 = \mathbb{R}$ and $A_1 = \mathbb{C}$. In sharp contrast with conventional wisdom, the computing potential of $A_k$ – far from
being hindered – increases spectacularly with \( k \).

The algebra \( \text{Der}(A_k) \) consisting of linear derivations defined on \( A_k \) is the foundation for the epistemological theory of hypercomputation in \( A_k, k \geq 0 \) (Chapter 3).

i) The two fields \( A_0 = \mathbb{R} \) and \( A_1 = \mathbb{C} \) are irreducible by algebra alone. Together with \( \infty \), they form the rational basis \( \{\mathbb{R}, \mathbb{C}, \infty\} \) for computation that we call Reason. By using analysis, Reason based on algebra can be reduced further to \( \{\mathbb{Q}, i\} \), the deeper notion of Rational Core (within Reason) so clearly perceived by Pascal.

ii) The field \( A_2 = \mathbb{H} \) is exceptional: it is reducible by \( \mathbb{C} \) (\( \mathbb{H} = \mathbb{C} \times \mathbb{C} \)) or by \( \mathbb{R} \) (\( \mathbb{H} = \mathbb{R} \oplus 3\mathbb{H} \)).

iii) Nonassociative computation in \( A_k, k \geq 3 \), is paradoxical (Chapter 5). Each algebra \( A_k \) is twice reducible: \( A_k \supset A_{k-1} \supset A_{k-2} \), and computation in \( A_k \) can be described by means of coefficients always in \( \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \), when \( k \geq 3 \).

Relaxing the constraints on \( \times \) which hold for \( k \leq 2 \) opens Pandora’s box of measurement paradoxes and provides a freedom of choice between several computational routes. Good or bad? This is for the user to decide. The variety of possibilities seems overwhelming at first. It is organized by an organic logic based on the arithmetic potential of the first four division algebras: from \( \mathbb{R} = A_0 \) to \( G = A_3 \). This organic logic is founded on measurements related to the multiplication map defined by \( a \neq 0 \), that is \( L_a : x \mapsto a \times x \), which is a linear map in \( A_k \). For \( k \leq 3 \), \( L_a \) has for unique singular value the euclidean norm \( \|a\| > 0 \); but for \( k \geq 4 \), there can exist \( 2^{k-3} \) distinct singular values \( \geq 0 \) which differ from \( \|a\| \). Moreover the results may depend on the computational route, and may even be hypercomplex and uncountable! This is one of the surprises that the Fundamental Theorem of Algebra keeps in store when set in noncommutative algebras. The internal clockwork of hypercomputation is guided in part by such measures which modify the local 3D-geometry defined at \( a \). This results in an expanded logic which provides an arithmetic basis for the emergence of simplexity in life’s complex processes, and in highly unstable phenomena.

The computational journey into nonlinearity in the framework of Dickson algebras is \textit{endless}. At every level \( k \geq 4 \), one gets new vistas, each richer than before. The book offers glimpses of the ever changing territory. New computational principles emerge at each level \( k \geq 2 \) which may supersede some others valid at a lower level \( k' < k \). For example, if we drop commutativity in \( \mathbb{H} (k = 2) \) then the discrete can emerge from the continuous by exponentiation (a generalization of \( e^{i \pi/2} = i^n \)). Discrete exponentiation sheds light on the measuring role of the real zeros for the \( \zeta \) function.

Without associativity \( (k \geq 3) \), there are several different ways to compute the
multiplicative measures of vectors which may agree only partially with each other. This creates paradoxes and new options as well. As a rule, the emergence of paradoxes goes hand in hand with an increase in the freedom of choice. This freedom of choice provides a rational basis for the many fuzzy phenomena encountered in experimental sciences at a small scale: they are currently attributed to randomness, as in statistical physics, quantum mechanics, or genetic mutation. However, the proverbial God (i.e. the computing spirit) does not play dice in mathematical computation, but rather offers an ever richer variety of computational options to choose from. Hypercomputation supports the old adage: “Variety is the spice of life.”

Caveat. The words “hypercomputation”, “computability” and “complexity, complexification” are used throughout the book in their classical mathematical sense. They should not be confused with the same words used in Computer Science. In this specific context, the words applied to programs for Turing machines acquire a meaning which differs greatly from the mathematical one. Algorithmic complexity is more akin to “complication” (Leibniz, Von Neumann) than it is to mathematical complexity.

4.2 Homotopic Deviation in associative linear algebra over \( \mathbb{C} \)

When vectors lie in a linear space with dimension \( n \neq 2^k \), they do not multiply. Does this mean that we have to give up multiplication? Not at all: endomorphisms are represented by square matrices of order \( n \) which do multiply non trivially for all \( n \geq 2 \). Associativity is preserved, but not commutativity. The theory of Homotopic Deviation for square matrices over \( \mathbb{C} \) was developed in the Qualitative Computing group at Cerfacs during the decade 1998-2007 (Chapter 7). It analyzes the spectral properties of the linear coupling \( A(t) = A + tE \), where \( A \) is the original matrix, \( E \) is the singular deviation matrix with rank \( r \), \( 1 \leq r < n \) and \( t \) is the coupling parameter which varies in the completed field \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \). In particular, the theory describes the 2-way flow of spectral information which takes place at the limit eigenvalues of \( A(t) \) which stay at finite distance when the intensity of the coupling \( |t| \to \infty \), realizing a synthesis between \( A \) and \( E \). The flow is top-down from the higher level \( n + r > n \) to level \( n \), and bottom-up from the lower level \( r < n \) to level \( n \). The top-down flow corresponds to homotopic deduction, a particular case of the well-known mathematical deduction. The bottom-up flow corresponds to the homotopic induction, a nonlinear aspect of algebraic creativity which reaches beyond what is known as mathematical induction. Homotopic creativity is at work.
in the phenomenon of self-organization described by many experimentalists, which emerges by coupling two nonlinear processes.

4.3 Understanding why and explaining how

The dominant role of machines and techniques is the fuel which fed the rapid development of the scientific know-how for the past three centuries. This know-how is incorporated into scientific theories which are destined to come and go as the explanations evolve. A theory stays alive as long as it explains or predicts satisfactorily the body of experimental data in the domain it is intended for. The general criterion for acceptance is based on the following simple idea:

Given a problem to be solved, a tentative solution (produced by computer programs or by experiments) is accepted if it can be interpreted as the exact solution for a nearby problem of the same kind, whose distance from the original one is, at most, of the order of the uncertainty on the data of the problem.

This backward analysis test is the key concept which enables software developers to assess the validity of results obtained with an arithmetic of limited precision. The validation theory for numerical software was initiated in the 1950s by Givens and Wilkinson [Chaitin-Chatelin and Frayssé (1996)]. In the specific domain of the assessment of numerical software, the uncertainty on the input data is of the order of the accuracy of the arithmetic of the computer. The intricate software implementation of this principle plays an essential role in enabling scientific computers to output results which are meaningful in high tech industries in spite of the limited accuracy of their arithmetic. The book will show it to be a vital part of the backward analysis necessary to give meaning to mathematical computation beyond commutativity (Chapters 5 and 7).

All theories – expressing the current know-how in whatever aspect of experimental sciences – are subjected to the above retrospective test, where the problem to be solved is to perform an experiment. The data predicted by the theory should match the experimentally produced data within experimental precision. A theory is discarded after many repeated failures to pass the test. The breathtaking technological achievements of the last two centuries had a predictable epistemological outcome. Today, most scientists downplay the philosophical role of understanding why a phenomenon exists; they limit their ambition to the engineering aspect of
knowing how an existing mechanism works. Inevitably, with no more access to the
why, the how withers and meets its own limits. It is not surprising that so many
of the experiments in life sciences defy the current theoretical doxa (Sections 3.3
and 3.4). After 200 years of “rational” management, the ecological situation of the
whole planet shows no lasting sign of improvement.

A possible way out of the epistemological bottleneck is to keep a better bal-
ance between the why and the how. By turning their gaze again towards the art of
computing, by freeing themselves from the arbitrary limits of linear vector spaces,
scientists will unleash the full power of computation. This will bring to light many
unheard-of phenomena. Some of these may be the source of long awaited technolog-
ical breakthroughs. These grand expectations are rooted in the preceding Sections
4.1 and 4.2. Creativity implies that discoveries may show up in places and ways
which are the least expected. For example, a fresh arithmetic look at the most fa-
miliar complex plane reveals the deeper notion of organic integrality in \( \mathbb{C} \) (Chapters
3, 10, 11 and 12). This opens a window on the dynamics of organic intelligence
in the dicksonian numbers.

### 4.4 Qualitative Computing

The subject of Qualitative Computing covers theoretical and practical aspects of
nonlinear computation. Multiplication is the lead actor: multiplication of num-
bers, vectors and matrices. The theoretical aspects which have been chosen for
presentation in the book describe hypercomputation over vectors in Dickson alge-
bras (Chapters 2 to 6, 9 and 11), the theory of Homotopic Deviation for matrices
(Chapter 7), and Fourier analysis for complex signals (Chapter 10). As for Chapter
8, it addresses more practical aspects. Inter alia, it clarifies why the scientific com-
puter not only is a most efficient tool to speed-up intractable computations in every
corner of our technological society, but also has an epistemological potential which
begs to be put to good use in our attempt to decipher the organic evolution of life.
Then, the final Chapter 12 concludes with organic intelligence for dicksonian num-
bers and wraps up some of the lessons in computation that were taught in the book.
One of the fundamental lessons is that small integers have a definite personality of
their own which reveals itself through computation. This qualitative approach to
numbers, which can be traced back to Pythagoras, is overshadowed by the definitely
quantitative vision of numbers in modern science, where randomness prevails. This
is clearly illustrated by Borel’s concept of a normal number. The leading opinion
among computer scientists is that the world can be reconstructed from only the two
bits 0 and 1 of their computer. The reader will discover why this belief logically ex-
cludes $\infty$, a notion irreplaceable in mathematical analysis (Chapter 3). The reader will understand that not only 0 and 1, but also 2 and 3, do play a crucial role in the recursive architecture of nonlinear computations. The number 3 in particular is related to the endless creativity displayed by Nature (Chapter 6). No less than four distinct modalities are necessary to set the natural evolution of numbers into a motion which enables the discovery of an everchanging numerical landscape. The two modalities of binary logic rule stability and invariance in numbers, giving to the whole mathematical building the logical status of a giant tautology. By contrast, the Sharkovski order (derived from the fixed points of $x = f^{(n)}(x)$) reveals an algorithmic connection between the actual $\infty$ and the four dichotomies (rational/irrational, algebraic/transcendental discrete/continuous, real/complex) that numbers possess as building blocks for classical calculation (arithmetic, calculus, analysis). The logistic computation realises an approximate numerical synthesis between Uniqueness and Multiplicity by the magic of successive iterations over $\mathbb{R}$ in finite precision. And this magic is revealed by the evolution of a real parameter. During the process, all integers are called into an orderly existence, starting from 2 and ending at 3, with the help of a countable infinity of calls to $\infty$. The majestic beauty of the source of mathematical creativity is revealed by computation. Numbers need not be confined to commutative fields. They can be vectors or matrices in algebras equipped with a noncommutative multiplication that is the starting point for evolution.

Another essential lesson is that the evolution of multiplication through complexification is bountiful: new properties keep emerging as the dimension $2^k$ for the algebra $A_k$ increases without bound. The internal clockwork for the organic logic is driven by arithmetic in the alternative – but nonassociative – ring of octonionic integers in eight dimensions. In parallel, it is driven continuously by analysis in the octonionic subalgebras of $A_k$, $k \geq 4$.

Only a few snapshots of the infinite computational landscape lying beyond modern calculus and analysis can be shown in one single book. The door is open for further investigation. If scientists want it to happen, the multimillenary evolutive art of computing will come back to life, awoken from a long sleep of more than a century. As Fermat once wrote to Huyghens (August 1659): “Multi pertransibunt et augebitur scientia”

Concluding Remark

This introductory chapter was written in 2009. Since then, the scope of Qualitative Computing has been developed into Relating Computation by considering the possibility that not only multiplication but also addition be non standard. See Cerfaçs Tech. Rep/PA/11/27 and 37.

Many will pass away and knowledge will grow.