Centre Européen de Recherche et de Formation Avancée en Calcul Scientifique

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Technical Report TR/PA/14/55

Publications of the Parallel Algorithms Team

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ABOUT A FAMILY OF LOGISTIC EQUATIONS BASED ON A RELATIVE DISTANCE TO $\frac{1}{2}$ RAISED TO A POSITIVE POWER

M. MONSERRAT RINCON-CAMACHO [(1),(2)], FRANÇOISE CHATELIN [(1),(2)], AND PHILIPPE RICOUX [(3)]

Abstract. We study the fixed-point equation, given for a fixed $l > 0$ by:

$$x = h(1 - |2x - 1|^l), \;\; x, \; h \in \mathbb{R},$$

where $|2x - 1| = \frac{|x-\frac{1}{2}|}{\frac{1}{2}}$ represents the relative distance of x to the mean value of 0 and 1 which are the fixed points of multiplication. The particular cases $l = 1$ and 2 are classical. This work intends to look at the question: "How much of the specific behaviour for $l = 1$ and 2 remains valid when the exponent l varies freely in $]0, \infty[$?". Some preliminary answers are given, which are derived either from theory ($l \in \mathbb{N}^*$, $1 \leq l \leq 5$) or from numerical simulation ($0 < l < 1$ or $l > 5$).

Keywords: Fixed points 0 and 1 for real multiplication, relative distance to the mean, logistic family, path of homogenisation, degree of heterogeneity, symbiosis, wavetracks, backward and forward chaos, Aristotle's logic, natural logic.

1. Introduction

1.1. **An introductory example: the classical logistic equation.** Let us consider the family $h \mapsto hf(x) = 4hx(1 - x)$ where x, $h \in \mathbb{R}$. The fixed point equation

$$x = hf(x) = f_h(x) \tag{1.1}$$

is the well known *logistic equation* studied between 1970 and 1990, [Feigenbaum, 1979, May et al., 1974, Nagashima and Baba, 1999]. It is easy to prove that equation (1.1) is satisfied when $x_{(1)}(h) = 0$ or $x_{(2)}(h) = 1 - \frac{1}{4h}$. In Figure 1 (a), we see in red the fixed points corresponding to the intersection of the curves $y = x$ (in black) and $y = f_h(x)$ for different values of h. However, the solutions may not be always reached in the limit of the fixed-point iteration

$$x_0 = \frac{1}{2}, \;\; x_{n+1} = hf(x_n) = f_h(x), \;\; n \geq 0, \;\; h \geq 0. \tag{1.2}$$

Indeed, given an initial point x_0, it is possible to reach or not a fixed point after some iterations depending on h. This behaviour is exemplified in Figure 1 (b). For the same initial point $x_0 = \frac{1}{2}$, we see that for h equal to 0.25, 0.45 and 0.65, the values of x_n in (1.2) reach a steady state after some iterations n, whereas for h equal to 0.75 or 0.95 the values of x_n are continuously oscillating. These convergence results are assembled in what is called an orbit diagram, see Figure 1 (c). For $h \in [0, 1]$, the values of x_n in (1.2) are depicted when $200 \leq n \leq 400$ and the initial condition is given by $x_0 = \frac{1}{2}$. The solution $x_{(1)}$ (resp. $x_{(2)}$) to equation (1.1) is reached when $h \in [0, \frac{1}{4}]$ (resp. $]\frac{1}{4}, \frac{3}{4}[$). Thus, through consecutive compositions there is an evolution and the identity function is reached in some sense for particular values of the parameter h. The restriction of h

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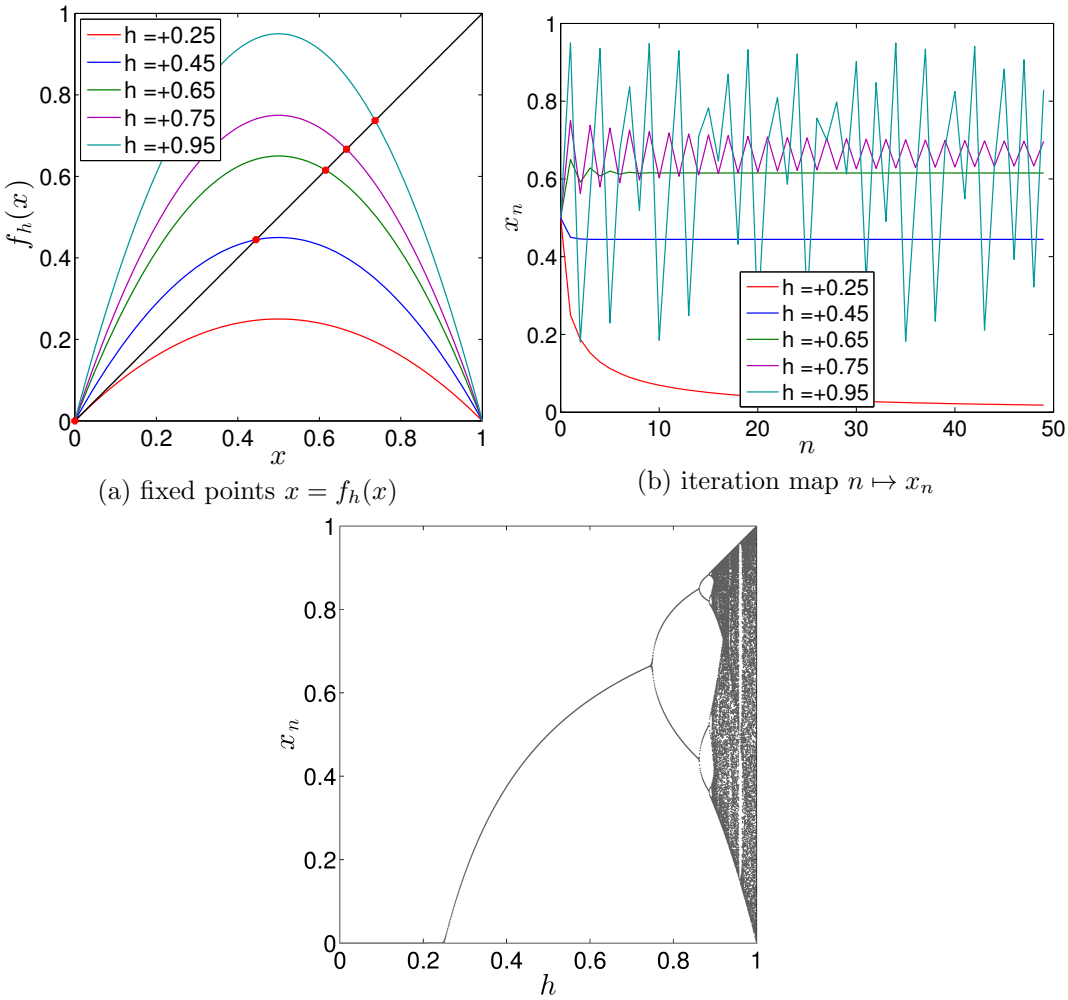
to $[0,1]$ allows one to display the *forward* orbit diagram only. See Section 4, Figure 16 (b) for the *backward* orbit diagram. For more details about the *numerical information* displayed by the logistic map see [Nagashima and Baba, 1999] and [Chatelin, 2012, Ch. 6 and 8].

In order to better understand this striking information processing phenomenon, we analyse in this report the fixed point equation (1.1) together with the Picard iteration (1.2) for a more general family of logistic in this report equations where $h \in \mathbb{R}$, of which the classical logistic map is a particular case.



(a) fixed points $x = f_h(x)$

(b) iteration map $n \mapsto x_n$

(c) orbit diagram ($200 \leq n \leq 400$)

FIGURE 1. The classical logistic map

Remark: We notice that the quadratic function $f(x) = 4x(1-x)$ appearing in the logistic equation has a maximum value 1 at the critical value $x = \frac{1}{2}$ taken as the starting point x_0 in (1.2).

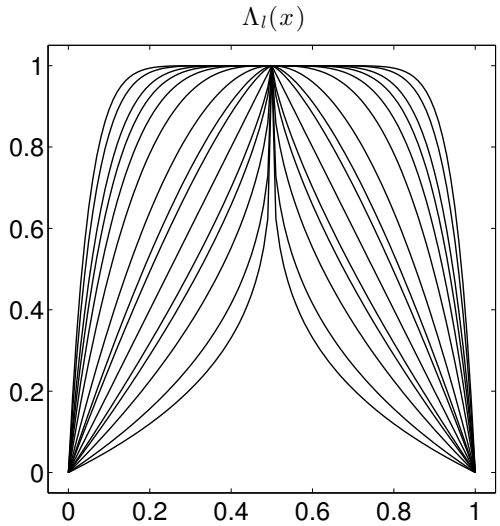


FIGURE 2. $\Lambda_l(x)$ for $l = \frac{1}{4},\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ 1,\ \frac{4}{3},\ \frac{3}{2},\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10$

1.2. **A family of logistic equations.** Let Λ_l be given by the following continuous function with the varying parameter $l > 0$:

$$\Lambda_l(x) = 1 - |2x-1|^l = 1 - e^{l \ln |2x-1|}. \tag{1.3}$$

When the parameter l is > 1 it quantifies the smoothness of the maximum 1 for $\Lambda_l(\frac{1}{2})$. See Figure 2. The parameter $h \in \mathbb{R}$ induces evolution in the following family of logistic equations:

$$x = h\Lambda_l(x). \tag{1.4}$$

This equation was introduced in [Nagashima and Baba, 1999, Ch. 3, p. 55, typo corrected] where as a preliminary result, it is indicated (Problem 8 in Chapter 3) that the Schwarz derivative $S(f) = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$ with $f(x) = h\Lambda_l(x)$, $l \in \mathbb{N}^*$, equals $-\frac{l^2-1}{(2x-1)^2}$, $x \neq \frac{1}{2}$. It is negative for $l > 1$, which implies observable periods. The null function $\mathbf{0}$ is a solution of (1.4) for all $l > 0$ since $\Lambda_l(0) = 1 - e^0 = 0$. The case $l = 2$ reduces to the classical logistic map presented in the previous section, and Λ_1 is the classical tent function.

1.3. **About functional fixed points.** The composition of continuous functions in $C^0(\mathbb{R})$

$$f \circ g = f(g(\cdot)), \quad f, g : \mathbb{R} \mapsto \mathbb{R}$$

is an analytic operation which models nonlinear evolution. There are two fixed points for this operation: the identity function $\mathbf{1} : x \mapsto x$ and the null function $\mathbf{0} : x \mapsto 0$. Both $\mathbf{1}$ and $\mathbf{0}$ satisfy $f \circ f = f$. Moreover, $\mathbf{1} \circ f = f \circ \mathbf{1} = f$ and $\mathbf{0} \circ f = \mathbf{0}$, $f \circ \mathbf{0} = f(0) \not\equiv \mathbf{0}$.

It was observed in [McCarthy, 1980] that $\circ$ reduces to $\times$ when f is the linear map $x \mapsto a \times x$ (denoted ax). Over linear functions, the identity function $\mathbf{1}$ (with constant derivative equal to 1) identifies itself with the constant unit function $x \mapsto 1$ (with derivative $= 0$). The corresponding identification of $\mathbf{0}$ with the zero function 0 is valid for any $f \in C^0(\mathbb{R})$. It is of computational significance that, over $\mathbb{R}$, 0 and 1 are the real fixed points of $\times$ which satisfy $x^2 = x \Leftrightarrow x \times (x-1) = 0$. The binary field $\mathbb{Z}_2 = \{0, 1\}$ is a model for the classical binary logic of Aristotle.

In general $\circ$ is neither commutative nor distributive with respect to $+$. This new multiplicative law endows the Banach space $C^0(\mathbb{R})$ with a *nonlinear* generalisation of multiplication. We shall denote $f \circ f$ as f^2 whenever the context prevents any ambiguity with numerical squaring

$a^2 = a \times a$. Thus $f^k = f \circ (f^{k-1})$, $k \geq 1$, and $f^0 = \mathbf{1}$. Clearly $x = f(x)$ implies $x = f^k(x)$, $k \geq 1$, hence in particular $f = f^2 \Leftrightarrow f \circ (f - \mathbf{1}) = (f - \mathbf{1}) \circ f = \mathbf{0}$.

It is possible to interpret the relation $f^2 = f$ for $\circ$ as a nonlinear generalisation of $x^2 = x$ for $\times$, where the identity map $\mathbf{1} : x \mapsto x$ becomes the function f in $C^0(\mathbb{R})$, or the family f_h depending on the parameter $h \in \mathbb{R}$. The role of 0 and 1 for $\mathbf{1}$ will be played by the fixed points x for f, or $x \mapsto x(h)$ for f_h. The logic of Aristotle is rooted in the identity map $\mathbf{1}$. This is the reason why mathematics is sometimes presented as a giant tautology. However, such a viewpoint on mathematics shared by many logicians is very restrictive. It does not account for the historical evolution. It does not account for the historical evolution of mathematical concepts always in a state of flux. Thinking of numbers for example, there is a long way from the whole numbers of Ancient Greece to the matrices of the $20^{\underline{\text{th}}}$ century [Chatelin, 2012]. And it was discovered experimentally at the end of last century that Nature itself provides evolutive physical phenomena which are better explained by a family of functions f_h rather than by $\mathbf{1}$, [Strogatz, 2001, Nagashima and Baba, 1999, chapter 4]. Because of their *evolutive* nature they obey alternative logics which *differ* greatly from that of Aristotle, which is the logic of *invariance*, the logic of *identity*.

In this work, we have chosen to study the evolution logic which obeys the law $h\Lambda_l$ because, for $l = 2$, $h\Lambda_2(x) = 4hx(1-x)$ is precisely the archetypal quadratic map ubiquitous in the physics of chaos whose mathematical study was pioneered by Feigenbaum. Because $\mathbf{1}$ does *not* depend on a parameter, the remarkable arithmetic and analytic processing of information which lies in the core of the sequence of iterates of f_h, when f_h has a negative Schwarz derivative [Nagashima and Baba, 1999, chapter 3, Section 3.2, pp. 51-57] has remained ignored until the 1980s in the West. After the seminal works of Fatou and Julia in the 1920s, the field was abandoned by most scientists, with the brilliant exception of Sharkovsky (1964) in Ukraine.

The introduction of the parameter h in (1.4) reveals the dynamics of the law of composition applied to $h\Lambda_l$ by means of the Picard iteration. Indeed its convergence/divergence can be analysed for $l > 1$ in terms of the fixed points of $(h\Lambda_l(x))^k$, $k \geq 1$, $l > 0$. Therefore, (1.4) can be interpreted as a model of *real* nonlinear evolution, under the law $h\Lambda_h$, away from the classical logic represented by $\mathbb{Z}_2$: the *numbers* 0 and 1 become the two real *functions* $h \mapsto x(h,l)$ which solve (1.4), $l > 0$. One solution is $\mathbf{0} = 0$ the other $h \mapsto x(h,l)$ satisfies $x = h$ for $h = \frac{1}{2}$ only. However the first wavetrack for the Picard iteration $x_0 = \frac{1}{2}$, $x_n = h\Lambda_l(x_{n-1})$, $n \geq 1$, is $w_1 = \mathbf{1}$, see Section 6. This indicates that the concept of unit 1 over $\mathbb{R}$ evolves into the identity function $\mathbf{1}$ over $C^0(\mathbb{R})$ when $\times$ is replaced by $\circ$, while 0 remains invariant ($\mathbf{0} = 0$).

1.4. **A relative vs. absolute evolution of numbers.** In [Rincon-Camacho et al., 2014b] we consider the family of logistic equations for $l > 0$

$$y = 1 - m|y|^l, \quad m \in \mathbb{R} \tag{1.5}$$

as another model for the numerical evolution of the fixed points 0 and 1 for multiplication under the variation of the real parameter m. Clearly $y(0) = 1$ for any $l > 0$.

The equation (1.5) appears in [Briggs, 1991]. It expresses that the distance $|y|$ from y to 0, raised to the power l, is equal to the relative distance $\frac{1-y}{m}$, that is

$$|y|^l = \frac{1-y}{m}, \quad m \neq 0. \tag{1.6}$$

The quotient $\frac{y-1}{m} = \frac{y(m)-y(0)}{m-0}$ is a divided difference whose value is $\tan\psi(m) = -|y|^l$, where $\psi(m)$ is the angle displayed on Figure 3.

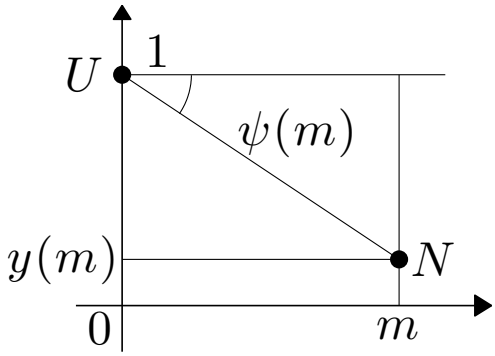


FIGURE 3. The angle $\psi(m)$

The point $N = (m, y(m))$ on the curve $m \mapsto y(m)$ is seen from the point $U = (0, 1)$ under the angle $\psi(m)$. See [Rincon-Camacho et al., 2014b] for a detailed analysis.

It is important to notice that firstly, $|y|$ is an *absolute* distance to 0, and that secondly, the information in (1.5) is provided by 0 and 1 only, which define the unit function $x \mapsto 1 \neq \mathbf{1}$ and the null function $x \mapsto 0$. Moreover, for $m = 1$, $G_l(1, y) = 1 - |y|^l$ is such that $G_l(1, 0) = 1$ and $G_l(1, 1) = 0$: $\{0, 1\}$ is a 2-cycle for $G_l(1, \cdot)$.

By comparison, it is useful to interpret the term $|2x - 1|$ in (1.3), as the *relative* distance of x to the mean value $\frac{1}{2}$ between 0 and 1:

$$E(x) = |2x - 1| = \tfrac{|x - \frac{1}{2}|}{\frac{1}{2}}.$$

The fixed point equation (1.4) is satisfied if and only if (iff)

$$E^l(x) = 1 - \frac{x}{h},\ h \neq 0 \tag{1.7}$$

which measures the *degree of heterogeneity* between x and h. The function $h \mapsto x(h)$ which satisfies the fixed point equation (1.4) and differs from $\mathbf{0}$ is called the *homogenisation path*. If $\frac{x}{h}$ gets close to 1 we say that a *symbiosis* between, or an *assimilation* of, the variables x and h takes place, in which case the homogenisation path gets close to the identity map $\mathbf{1} : h \mapsto x = h$. It achieves complete homogenisation at $(\frac{1}{2}, \frac{1}{2})$ for any $l > 0$. This function is the inverse of the map $x \mapsto h = \frac{x}{1-|2x-1|^l}$ which satisfies $h(0) = \frac{1}{2l}$, $l > 0$. We interpret the quotient $\frac{x}{h}$ as $\tan\theta(h)$, where the angle $\theta(h)$ is displayed on Figure 4. When $x(h) = h$ then $\theta(h) = \frac{\pi}{4}$. Eq. (1.7) can be rewritten

$$\tan\theta(h) = 1 - E^l(x). \tag{1.8}$$

The point $M = (h, x(h))$ on the homogenisation path $h \mapsto x(h)$ is seen from the origin $O = (0, 0)$ under the angle $\theta(h)$.

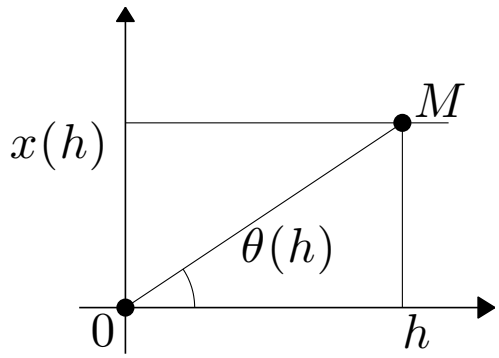


FIGURE 4. The angle $\theta(h)$

The deviation of $x(h)$ from $\mathbf{1}$, that is of $\theta(h)$ from $\frac{\pi}{4}$, is quantified by the opposite of the relative distance $E(x)$ to the *mean value* $\frac{1}{2}$ raised to the power $l > 0$. The interpretation of $\tan\psi(m) = -|y|^l$ is presented in [Rincon-Camacho et al., 2014b].

Observe that the analogous equations (1.6) and (1.7) differ actually in some important aspects. For example:

- the left-hand side distance in (1.6) (resp. (1.7)) is absolute (resp. relative) and taken to the value 0 (resp. mean value $\frac{0+1}{2} = \frac{1}{2}$),
- the right-hand side in (1.6) (resp. (1.8)) $= -\tan\psi(m)$ (resp. $\tan\theta(h)$),
- the angle $\psi(m)$ (resp. $\theta(h)$) is defined at the observation point $U = (0, 1)$ (resp. $O = (0, 0)$).

Therefore these equations define two distinct evolutions, respectively absolute (involving the two elements 0 and 1 of $\mathbb{Z}_2$) and relative (involving the three numbers 0, 1 and $\frac{1}{2}$).

We notice that the vector $\overrightarrow{OU}$ in $\mathbb{R}^2$, Fig. 3, can be given the algebraic interpretation of a *planar unit*, such as the complex unit i ($i^2 = -1$), the bireal unit u ($u^2 = 1,\ u \neq \pm 1$) or the dual unit n ($n^2 = 0,\ n \neq 0$), see [Chatelin and Rincon-Camacho, 2014,Rincon-Camacho et al., 2014a]. Such a threefold interpretation for $\overrightarrow{OU}$ enriches the linear vector structure of the numerical plane $\mathbb{R}^2$: it becomes an algebraic threefold 2D-structure endowed with a multiplication of a *triple* nature, see [Rincon-Camacho and Latre, 2013]

1.5. **Organisation of the Report.** We make a theoretical study of equation (1.4) for l integer, $1 \leq l \leq 5$ in Section 2. The degree of heterogeneity and symbiosis for this family of logistics are examined in Section 3. The Picard iteration on (1.4) is contrasted with the *explicit* real solutions of (1.4) for $l \in \{1 \ldots 5\}$ in Section 4, and with computed real solutions for $0 < l < 1$ and $l \to \infty$ in Section 5. The so-called wavetracks are introduced in Section 6. Finally some preliminary conclusions about future work are given in Section 7.

2. The positive exponent l is an integer ≤ 5

In this section we look for the real solutions $x(h)$ for the equation (1.4) when $l \in \{1, 2, 3, 4, 5\}$. In all cases we find that $x = \mathbf{0}$ is a solution to (1.4). The other possible *real* solution is the homogenisation path derived in closed form from the roots of polynomials of degree less or equal to 4 which depend on the parameter h. The venerable methods of Cardano and Ferrari (recalled in the Appendix) will allow us to present a complete discussion of the *nature*, either real (distinct or double) or complex, of the roots of (1.4). However in this report, we shall only consider the existence of *real* roots.

2.1. **Case $l = 1$.** Equation (1.4) becomes the known *tent map* for $l = 1$, see [Nagashima and Baba, 1999] and [Chatelin, 2012, Ch. 6]. Because of the absolute value we have two possibilities:

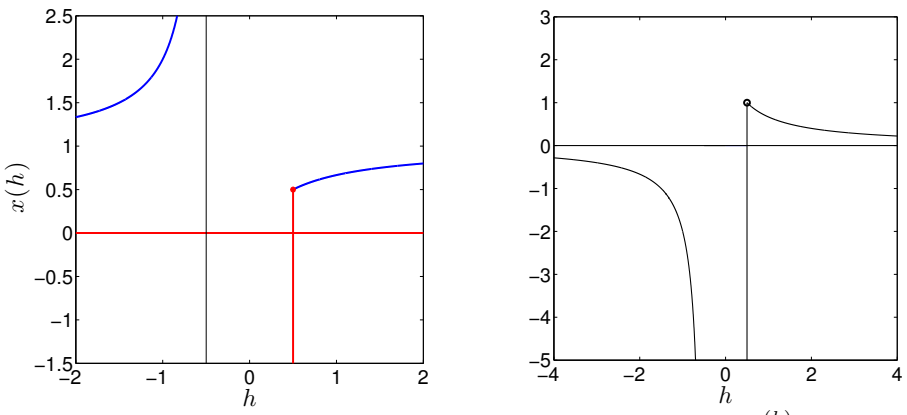
- If $x \leq \frac{1}{2}$, $\Lambda_1(x) = 1 - (1 - 2x)$ and

$$\begin{aligned} x &= h(1 - (1 - 2x)) \\ &= h(1 - 1 + 2x)) \\ &= 2hx. \end{aligned}$$

  Then, $x = h\Lambda_1(x)$ iff $x = 0$ or $h = \frac{1}{2}$, plotted in red in Figure 5.
- If $x > \frac{1}{2}$, $\Lambda_1(x) = 1 - (2x - 1)$ and

$$\begin{aligned} x &= h(1 - (2x - 1)) \\ &= h(2 - 2x). \end{aligned}$$

  Thus, $x = h\Lambda_1(x)$ iff $x = \frac{2h}{1+2h}$. In Figure 5 left, we see part of this hyperbola in blue for $x \geq \frac{1}{2}$. Figure 5 right displays $h \mapsto \tan\theta(h)$. Observe the cusps at $(\frac{1}{2}, \frac{1}{2})$



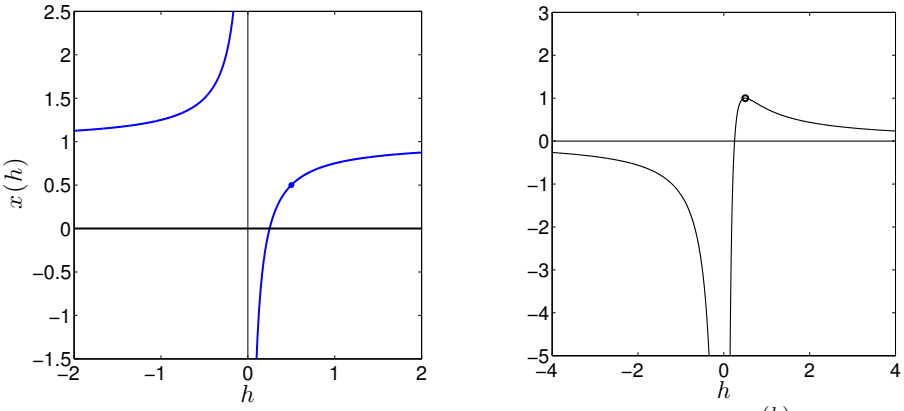
(a) 0 and $x(h)$ (b) $\tan\theta(h) = \frac{x(h)}{h}$ versus h

FIGURE 5. $l = 1$

2.2. **Case $l = 2$.** When $l = 2$, $\Lambda_2(x) = 1 - (1 - 2x)^2$, and equation (1.4) is as follows:

$$\begin{aligned} x &= h(1 - (1 - 2x)^2) \\ &= h(1 - (1 - 4x + 4x^2)) \\ &= h(4x - 4x^2) \\ &= 4hx(1 - x). \end{aligned}$$

As we mentioned before, this is the classical logistic equation described in the introduction. In this case, $x = h\Lambda_2(x)$ iff $x = 0$ or $x = 1 - \frac{1}{4h}$. This homogenisation path is the hyperbola plotted in Figure 6 left and the quotient $\frac{x}{h}$ on Figure 6 right. Both curves are smooth around $(\frac{1}{2}, \frac{1}{2})$.



(a) Homogenisation path (b) $\tan\theta(h) = \frac{x(h)}{h}$ versus h

FIGURE 6. $l = 2$

2.3. **Case $l = 3$.** Since the exponent in (1.4) is odd, the set of solutions is divided in two parts, for $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$.

- If $x \leq \frac{1}{2}$, then $\Lambda_3(x) = 1 - (1-2x)^2$ and the solution to equation (1.4) is

$$\begin{aligned} x &= h(1-(1-2x)^3) \\ &= h(1-(1-6x+12x^2-8x^3)) \\ &= h(6x-12x^2+8x^3). \end{aligned}$$

We have that $x = h\Lambda_3(x)$ iff $x = 0$ and $8x^2 - 12x + 6 - \frac{1}{h} = 0$ whose solutions are $x_1 = \frac{1}{4}\left(3 - \sqrt{\frac{2}{h} - 3}\right)$ and $x_2 = \frac{1}{4}\left(3 + \sqrt{\frac{2}{h} - 3}\right)$, which are real for $0 < h \leq \frac{2}{3}$. But only x_1 satisfies the condition $x_1 \leq \frac{1}{2}$ for $h \leq \frac{1}{2}$. This corresponds to the red curve plotted in Figure 8. Observe that $x_1(\frac{1}{6}) = 0$.

- If $x > \frac{1}{2}$, then $\Lambda_3(x) = 1 + (1-2x)^3$ and we have to solve the following fixed point equation

$$\begin{aligned} x &= h(1+(1-2x)^3) \\ &= h(1+(1-6x+12x^2-8x^3)) \\ &= h(2+6x-12x^2+8x^3). \end{aligned}$$

In order to find the homogenisation path we need to solve the following cubic equation by Cardano's method (see Appendix A):

$$4x^3 - 6x^2 + \left(\frac{1}{2h} + 3\right)x - 1 = 0. \tag{2.1}$$

By taking $a = 4$, $b = -6$, $c = \frac{1}{2h} + 3$ and $d = -1$, we obtain for $y = x + \frac{1}{2}$ the reduced equation:

$$y^2 + 3p(h)y + 2q(h) = 0, \quad q(h) = \frac{1}{2^5 h} - \frac{1}{2^4} = \frac{1}{2^4 h^2} h\left(\frac{1}{2} - h\right), \quad p(h) = \frac{1}{2^3 3h}$$

together with the discriminant (A.5):

$$D(h) = \frac{1}{2^9 3^3}\left(\frac{1}{h^3} + \frac{3^3}{2h^2} - \frac{2\cdot 3^3}{h} + 2\cdot 3^3\right)$$

(for the computation details, see Appendix C.1). The values of q, p and D depend on h, thus we need to study the sign of D depending on the values of h in order to determine the roots of equation (2.1).

Let us set $t = \frac{1}{h}$, then $D(h)$ becomes $f(t) = t^3 + \frac{3^3}{2}t^2 - 2 \cdot 3^3 t + 2 \cdot 3^3$. In order to determine the sign of $f(t)$ we find the roots of

$$t^3 + \frac{3^3}{2}t^2 - 2 \cdot 3^3 t + 2 \cdot 3^3 = 0 \tag{2.2}$$

by applying again Cardano's method. We set $a = 1$, $b = \frac{3^3}{2}$, $c = -2 \cdot 3^3$ and $d = 2 \cdot 3^3$, then

$$q = \frac{3^3 \cdot 71}{2^3}, \quad p = -\frac{3^2 \cdot 17}{2^2} \text{ and } D = 2 \cdot 3^6,$$

(see Appendix C.2 for details). Since $D > 0$ and $p < 0$, we are in case (ii) of Cardano's method and we have one real root. According to (A.6) and (A.11) we have

$$r = \frac{3}{2}\sqrt{17} \text{ and } \cosh\varphi = \frac{71}{17^{3/2}},$$

hence, $\varphi \approx 0.1607$. The real root of (2.2) is

$$t_0 = -2r\cosh\frac{\varphi}{3} - \frac{b}{3a} = -3\sqrt{17}\cosh\frac{\varphi}{3} - \frac{3^2}{2} \approx -16.88,$$

see (A.10) and (A.14). The sign of $f(t)$ is as follows

$$f(t) \geq 0 \text{ if } t \in [t_0, \infty[\quad \text{and} \quad f(t) < 0 \text{ if } t \in]-\infty, t_0[.$$

We derive the sign of $D(h)$ from the sign of $f(t)$. Set $h_0 = \frac{1}{t_0} \approx -0.0592$, then

$$D(h_0) = 0, \; D(h) < 0 \text{ if } h \in]h_0, 0[\quad \text{and} \quad D(h) > 0 \text{ if } h \in]-\infty, h_0[\cup]0, \infty[.$$

The signs of $q(h) = \frac{1}{2^5 h} - \frac{1}{2^4}$ and $p(h) = \frac{1}{2^3 \cdot 3h}$ are given by:

$$q(h) > 0 \text{ if } h \in]0, 0.5[, \quad q(\tfrac{1}{2}) = 0 \quad \text{and} \quad q(h) < 0 \text{ if } h \in]-\infty, 0[\cup]0.5, \infty[,$$

$$p(h) > 0 \text{ if } h \in]0, \infty[\quad \text{and} \quad p(h) < 0 \text{ if } h \in]-\infty, 0[.$$

The characterisation of the roots of (2.1) depending on the intervals of h ($]-\infty, h_0[$, $[h_0, 0[$, $]0, 0.5]$ and $h \in]0.5, \infty[$) is gathered in Table 1. According to the signs of D, q and p, we determine the 4 corresponding cases (i), (iii), (v) or (vi) of Cardano's method summarised below:

h		h_0		0		$\frac{1}{2}$	
$D(h)$	$+$	0	$-$	|	$+$	$+$	$+$
$p(h)$	$-$	$\vdots$	$-$	|	$+$	$+$	$+$
$q(h)$	$-$	$\vdots$	$-$	|	$+$	0	$-$
case	(v)	(iii)	(i)	|	(vi)	(vi)	(vi)

Since we have the restriction $x \geq \frac{1}{2}$, we compute the sign of $x_{1,2,3} - \frac{1}{2}$ to see whether the solution is acceptable or not.

h	$-\infty$		$h_0 \approx -0.06$		0		0.5		∞
D	$\vdots$	$+$	0	$-$	$-\infty \mid \infty$	$+$	$\vdots$	$+$	$\vdots$
q	$\vdots$	$-$	$\vdots$	$-$	$-\infty \mid \infty$	$+$	0	$-$	$\vdots$
p	0	$-$	$\vdots$	$-$	$-\infty \mid \infty$	$+$	$\vdots$	$+$	0
case		(ii)		(i)		(iii)		(iii)	
$\frac{q}{r^3}$	∞	$\cosh\varphi$	1	$\cos\varphi$	$0 \mid 0$	$\sinh\varphi$	0	$\sinh\varphi$	∞
φ	∞	$+$	0	$+$	$\frac{\pi}{2} \mid 0$	$+$	0	$+$	∞
$x_1 - \frac{1}{2}$	$\frac{1}{2}^+$	$+$	$\vdots$	$+$	$\mid$	$-$	0	$+$	$\frac{1}{2}^-$
$x_2 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$-$	$\mid$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$
$x_3 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$-$	$\mid$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$

TABLE 1. Cardano's method for equation (2.1)

We recall that in (2.1) $a = 4$ and $b = -6$ thus $-\frac{b}{3a} = \frac{1}{2}$ in (A.14). On the interval $]-\infty, h_0[$, the real solution x_1 is given by

$$x_1 = -2r\cosh\frac{\varphi}{3} + \frac{1}{2}.$$

This solution is plotted in blue in Figure 7 and it satisfies $x_1 \geq \frac{1}{2}$. On the interval $]h_0, 0[$ there are three real solutions

$$x_1 = -2r\cos\frac{\varphi}{3} + \frac{1}{2} \geq \frac{1}{2},\ x_2 = 2r\cos\frac{\pi-\varphi}{3} + \frac{1}{2} < \frac{1}{2},\ x_3 = 2r\cos\frac{\pi+\varphi}{3} + \frac{1}{2} < \frac{1}{2}.$$

They are plotted in blue, green and orange in Figure 7, respectively. We see that only $x_1 \geq \frac{1}{2}$. On the interval $]0, 0.5]$, the solution is

$$x_1 = -2r\sinh\frac{\varphi}{3} + \frac{1}{2}$$

and it is displayed in blue in Figure 7. We see that in this case x_1 is not acceptable since $x_1 \leq \frac{1}{2}$. Finally on the interval $]0.5, \infty[$, the solution x_1 in blue in Figure 7, satisfies $x_1 \geq \frac{1}{2}$. Thus, the final solution of $x = h\Lambda_3(x)$ when $x \geq \frac{1}{2}$ corresponds to the blue curve in Figure 8.

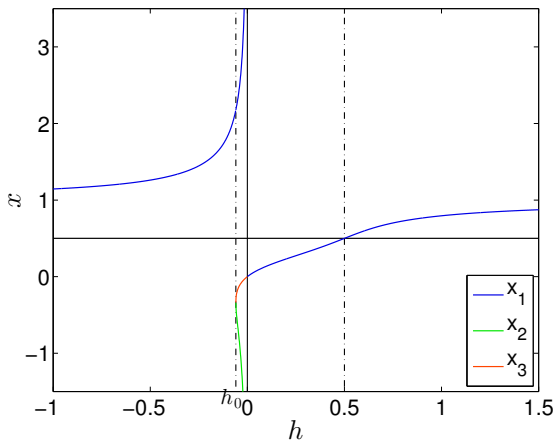


FIGURE 7. Solutions of equation (2.1)

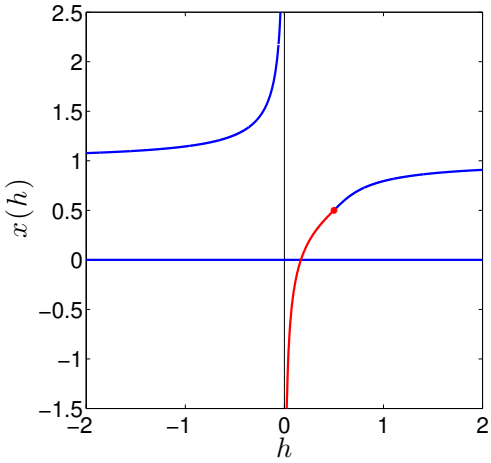


FIGURE 8. $x(h)$ for $l = 3$

2.4. **Case** $l = 4$**.** For $l = 4$, $\Lambda_4 = 1 - (1-2x)^4$ and the solution of (1.4) is given by:

$$\begin{aligned} x &= h(1-(1-2x)^4) \\ &= h(1-(1-8x+24x^2-32x^3+16x^4)) \\ &= h(8x-24x^2+32x^3-16x^4). \end{aligned}$$

Then $x = h\Lambda_4(x)$ iff $x = 0$ and

$$2x^3 - 4x^2 + 3x + \frac{1}{8h} - 1 = 0. \tag{2.3}$$

This equation is solved by setting $a = 2$, $b = -2^2$, $c = 3$, $d = \frac{1}{2^3 h} - 1$ and using Cardano's method (Appendix A). The details for computing q, p and D are given in Appendix D:

$$q(h) = \frac{1}{2^5 h} - \frac{5}{2^2 3^3} = \frac{1}{2^5 h}\left(1 - \frac{5\cdot 2^3}{3^3}h\right), \quad p = \frac{1}{2\cdot 3^2}, \quad |r| = \frac{1}{3\sqrt{2}}$$

and

$$D(h) = \frac{1}{2^{10}3^3}\left(\frac{3^3}{h^2} - \frac{5\cdot 2^4}{h} + 2^6\right) = \frac{1}{2^{10}3^3h^2}(3^3 - 5\cdot 2^4 h + 2^6 h^2).$$

The sign of D is that of the quadratic polynomial $h^2 - \frac{5}{4}h + \left(\frac{3}{4}\right)^3$, with discriminant $\Delta = \left(\frac{5}{4}\right)^2 - 4\left(\frac{3}{4}\right)^3 = \frac{1}{4^2}(25-27) < 0$. Hence $D(h)$ is positive and we have for all $h \neq 0$ only one real root. Since $p > 0$, we are in case (vi) of Cardano's method with $q(h) > 0$ for $h \in]0, \frac{27}{40}[$, $h' = \frac{27}{40} \approx 0.675$. Thus for $h \notin \{0, h'\}$

$$r(h) = \operatorname{sgn}(q(h))\sqrt{p} = \frac{\operatorname{sgn}(q(h))}{3\sqrt{2}}, \quad \sinh\varphi = \frac{q(h)}{r(h)^3} = 3^3 2^{3/2}\left|\frac{1}{2^5 h} - \frac{5}{2^2 3^3}\right| = \sqrt{2}\left|\frac{27}{2^4 h} - \frac{5}{2}\right| > 0.$$

Hence, according to (A.12) $y_1(h) = -2r(h)\sinh\frac{\varphi}{3} \neq 0$ and by (A.14)

$$x_1(h) = y_1(h) + \frac{2^2}{3\cdot 2} = -2r(h)\sinh\frac{\varphi}{3} + \frac{2}{3}. \tag{2.4}$$

The graph of the homogenisation path $x(h)$ given by (2.4) is displayed in Figure 9.

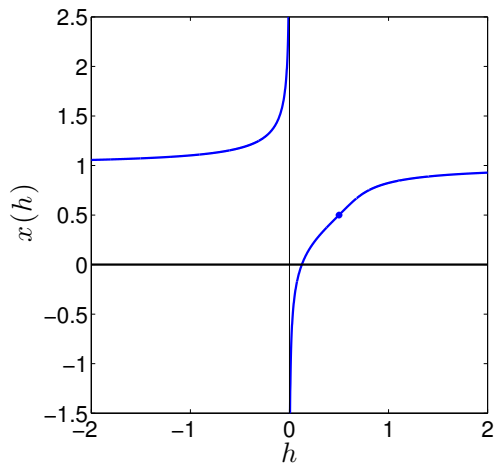


FIGURE 9. Homogenisation path for $l = 4$

Since $q(h) = \frac{1}{2^5 h}\left(1 - \frac{5\cdot 2^3}{3^3}h\right)$, $y_1(h) = 0$ for $h = h'$ and $x_1(\frac{27}{40}) = \frac{2}{3}$. Moreover $x_1(\frac{1}{8}) = 0$ is a double real root and $x_1(\frac{1}{2}) = \frac{1}{2}$ by direct check.

2.5. **Case $l = 5$.** When $l = 5$, we are able to solve the equation (1.4) in closed form only when $x \leq \frac{1}{2}$, since for $x > \frac{1}{2}$ we obtain an equation of degree 5 which is not solvable by radicals (Abel-Ruffini theorem, also proved independently by Evariste Galois).

If $x \leq \frac{1}{2}$, $\Lambda_5(x) = 1 - (1-2x)^5$ and equation (1.4) becomes

$$\begin{aligned} x &= h(1-(1-2x)^5) \\ &= h(1-(1-10x+40x^2-80x^3+80x^4-32x^5)) \\ &= h(10x-40x^2+80x^3-80x^4+32x^5). \end{aligned}$$

Then, $x = h\Lambda_5(x)$ iff $x = 0$ and we obtain a quartic equation $32x^4-80x^3+80x^2-40x+10-\frac{1}{h} = 0$ to be solved by Ferrari's method, see Appendix B. We solve the equation:

$$x^4 - \frac{5}{2}x^3 + \frac{5}{2}x^2 - \frac{5}{2^2}x + \frac{5}{2^4} - \frac{1}{2^5 h} = 0 \tag{2.5}$$

by setting $b = -\frac{5}{2}$ $c = \frac{5}{2}$ $d = -\frac{5}{2^2}$ $e = \frac{5}{2^4} - \frac{1}{2^5 h}$ in (B.1). We need to start by using Cardano's method to solve the auxiliary cubic equation given by (B.3):

$$2^3 z^3 - 2\cdot 5 z^2 + \left(\frac{3\cdot 5}{2^2} + \frac{1}{2^2 h}\right) z - \frac{5^2}{2^6} - \frac{3\cdot 5}{2^7 h} = 0, \tag{2.6}$$

we set $a = 2^3$, $b = -2\cdot 5$, $c = \frac{3\cdot 5}{2^2} + \frac{1}{2^2 h}$, $d = -\frac{5^2}{2^6} - \frac{3\cdot 5}{2^7 h}$ and we find that

$$q(h) = \frac{5^2}{2^{10}3^3} - \frac{5}{2^{11}3h}, \quad p(h) = -\frac{5}{3^2 2^5} + \frac{1}{2^5 3h}$$

and

$$D(h) = \frac{1}{2^{15}\cdot 3^3}\left(\frac{1}{h^3} - \frac{5\cdot 113}{2^7 h^2} + \frac{5^2\cdot 3^2}{2^5 h} - \frac{5^3}{2^5}\right),$$

see Appendix E.1 for details. The sign of $D(h)$ is given by setting $t = \frac{1}{h}$ and by studying the sign of

$$f(t) = t^3 - \frac{5\cdot 113}{2^7}t^2 + \frac{5^2 3^2}{2^5}t - \frac{5^3}{2^5}. \tag{2.7}$$

The roots of (2.7) are found again by Cardano's method, set $a = 1$, $b = -\frac{5\cdot 113}{27}$, $c = \frac{5^2 3^2}{2^5}$, $d = -\frac{5^3}{2^5}$ in (A.1). In this case we obtain

$$q = \frac{5^4 \cdot 13 \cdot 239}{3^3 \cdot 2^{21}}, \quad p = \frac{5^3 \cdot 211}{3^2 2^{14}}, \quad D = \frac{5^8}{3^3 \cdot 2^{21}},$$

see Appendix E.2. Since $D > 0$, $q > 0$ and $p > 0$ we are in case (iii) of Cardano's method, thus

$$r = \sqrt{p} = \frac{5\sqrt{5 \cdot 211}}{3 \cdot 2^7}$$

and

$$\sinh \varphi = \frac{|q|}{|p|^{3/2}} = \frac{5 \cdot 13 \cdot 239}{(5 \cdot 211)^{3/2}} \approx 0.4533, \quad \varphi \approx 0.43910$$

from where we obtain that

$$t_0 = -2r \sinh \frac{\varphi}{3} - \frac{b}{3a} = -\frac{2 \cdot \sqrt{5 \cdot 211}}{3 \cdot 2^7} \sinh \frac{\varphi}{3} + \frac{5 \cdot 113}{3 \cdot 2^7} \approx 1.341064.$$

The sign of (2.7) is:

$$f(t) \geq 0 \text{ if } t \in [t_0, \infty[\quad \text{and} \quad f(t) < 0 \text{ if } t \in]-\infty, t_0[.$$

Finally, the sign of $D(h)$ for $h_0 = \frac{1}{t_0} \approx 0.74233$ is given as follows:

$$D(h) \geq 0 \text{ if } h \in]0, h_0[\quad \text{and} \quad D(h) < 0 \text{ if } h \in]-\infty, 0[\cup]h_0, \infty[.$$

The signs of q and p are:

$$q(h) \leq 0 \text{ if } h \in]0, 0.9] \quad \text{and} \quad q(h) > 0 \text{ if } h \in]-\infty, 0[\cup]0.9, \infty[,$$

$$p(h) \geq 0 \text{ if } h \in]0, 0.6[\quad \text{and} \quad p(h) < 0 \text{ if } h \in]-\infty, 0[\cup]0.6, \infty[,$$

and

$$r(h) = \operatorname{sgn}\left(\frac{5^2}{2^{10} \cdot 3^3} - \frac{5}{2^{11} \cdot 3h}\right) \sqrt{\left| -\frac{5}{3^2 \cdot 2^5} + \frac{1}{2^5 \cdot 3h} \right|}$$

$$\frac{q(h)}{r(h)^3} = \frac{5\sqrt{|h|}|5 \cdot 2h - 3^2|}{2^3\sqrt{2}|3 - 5h|^{3/2}}$$

see Appendix E.3.

The real solutions to equation (2.5), depend on the quantity

$$\begin{aligned} A &= \pm\sqrt{8z + b^2 - 4c} \\ &= \pm\sqrt{8z + \frac{5^2}{2^2} - 2 \cdot 5} \\ &= \pm\sqrt{8z - \frac{3 \cdot 5}{2^2}} \end{aligned}$$

from (B.2) where z is a real solution of (2.6). In order to have A real $\neq 0$, we need that $8z - \frac{3\cdot 5}{2^2} > 0$, which yields the following constraint on z

$$z - c_A > 0,$$

where $c_A = \frac{3\cdot 5}{2^5} = \frac{15}{32} \approx 0.469$. These results are gathered in Table 2.

h	$-\infty$		0		$\frac{3}{5}$		$h_0 \approx 0.74$		$\frac{9}{10}$		∞
D	$\vdots$	$-$	$-\infty \mid \infty$	$+$	$\vdots$	$+$	0	$-$	$\vdots$	$-$	$\vdots$
q	$\vdots$	$+$	$\infty \mid -\infty$	$-$	$\vdots$	$-$	$\vdots$	$-$	0	$+$	$\vdots$
p	$\vdots$	$-$	$-\infty \mid \infty$	$+$	0	$-$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$
case		(i)		(iii)		(ii)		(i)		(i)	
$\frac{q}{r^3}$	$\frac{\sqrt{5}}{4\sqrt{2}}$	$\cos\varphi$	0	$\sinh\varphi$	∞	$\cosh\varphi$	1	$\cos\varphi$	0	$\cos\varphi$	$\frac{\sqrt{5}}{4\sqrt{2}}$
φ	1.16	$+$	1	$+$	∞	$+$	0	$+$	$\frac{\pi}{2}$	$+$	1.16
$z_1 - c_A$	$\vdots$	$-$	0	$+$	$\vdots$	$+$	$\vdots$	$+$	$\vdots$	$-$	$\vdots$
$z_2 - c_A$	$\vdots$	$+$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$-$	$\vdots$	$+$	$\vdots$
$z_3 - c_A$	$\vdots$	$-$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$

TABLE 2. Cardano's method for equation (2.6)

From equation (2.6), we recall that $a = 2^3$ and $b = -2 \cdot 5$, thus $\frac{b}{3a} = \frac{5}{3 \cdot 2^2}$ in (A.14). On the interval $]-\infty, 0[$, the only acceptable solution is given by

$$z_2 = 2r \cos \frac{\pi - \varphi}{3} + \frac{5}{3 \cdot 2^2},$$

where φ is given by $\cos\varphi = \frac{q}{r^3}$. This solution is plotted in green in Figure 10. For $h \in]0, 0.6[$, the only real solution is

$$z_1 = -2r \sinh \frac{\varphi}{3} + \frac{5}{3 \cdot 2^2}, \quad \sinh\varphi = \frac{q}{r^3}$$

and it satisfies $z_1 - c_A > 0$, as can be observed in Figure 10. On the interval $]0.6, h_0[$, we have also only one solution

$$z_1 = -2r \cosh \frac{\varphi}{3} + \frac{5}{3 \cdot 2^2}, \quad \cosh\varphi = \frac{q}{r^3}$$

and it also satisfies $z_1 - c_A > 0$, see Figure 10. For $h \in]h_0, 0.9[$ we have three real solutions and only z_1 satisfies the constraint $z_1 - c_A > 0$

$$z_1 = -2 \cos \frac{\varphi}{3} + \frac{5}{3 \cdot 2^2}, \quad \cos\varphi = \frac{q}{r^3},$$

displayed in blue in Figure 10. Finally on the interval $]0.9, \infty[$, we have also three real roots and z_2 plotted in green in Figure 10 satisfies $z_2 - c_A \geq 0$

$$z_2 = 2r \cos \frac{\pi - \varphi}{3} + \frac{5}{3 \cdot 2^2}.$$

In order to find the real roots of (2.5), using $B = \sqrt{8z - \frac{3 \cdot 5}{2^2}}$, we need to compute the discriminants (B.4) in Ferrari's method. They are plotted in Figure 11 and the results gathered in Table 3. Since only the discriminant Δ_- is positive on the interval $]0, h_0[$, only in this interval the roots x_1 and x_2 given by (B.5) are real. They are plotted in blue and green respectively in Figure 12. We observe then that only $x_2 \leq \frac{1}{2}$ for $h \in]0, 0.5[$ Thus the partial homogenisation path for $x = h\Lambda_5(x)$, $x \leq \frac{1}{2}$ is displayed in Figure 13, with $x(\frac{1}{10}) = 0$.

h	$-\infty$		0		$\frac{1}{2}$		$\frac{3}{5}$		$h_0 \approx 0.74$		$\frac{9}{10}$		∞
z	$\vdots$	z_2	$\vdots$	z_1	$\vdots$	z_1	$\vdots$	z_1	$\vdots$	z_1	$\vdots$	z_2	$\vdots$
Δ_-	$\vdots$	$-$	$-\infty \vdots \infty$	$+$	$\vdots$	$+$	$\vdots$	$+$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$
Δ_+	$\vdots$	$-$	$-\infty \vdots -\infty$	$-$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$	$-$	$\vdots$
$x_1 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$+$	$\vdots$	$+$	$\vdots$	$+$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$
$x_2 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$-$	0	$+$	$\vdots$	$+$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$
$x_3 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$
$x_4 - \frac{1}{2}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$	$\mathbb{C}$	$\vdots$

TABLE 3. Ferrari's method for equation (2.5)

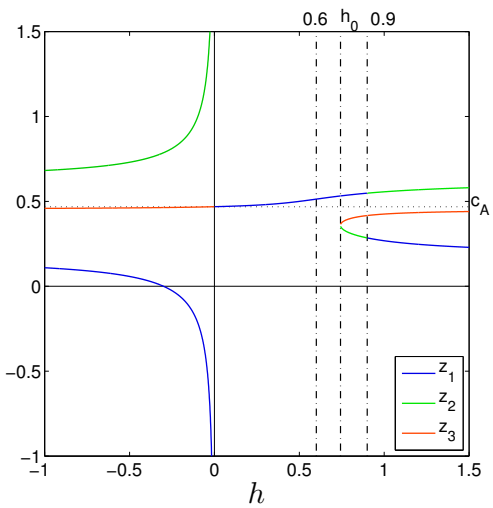


FIGURE 10. Solutions $z(h)$ of equation (2.6)

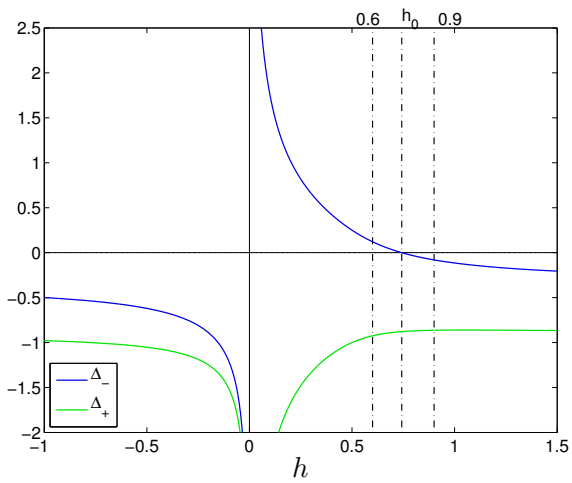


FIGURE 11. Δ_+ and Δ_-

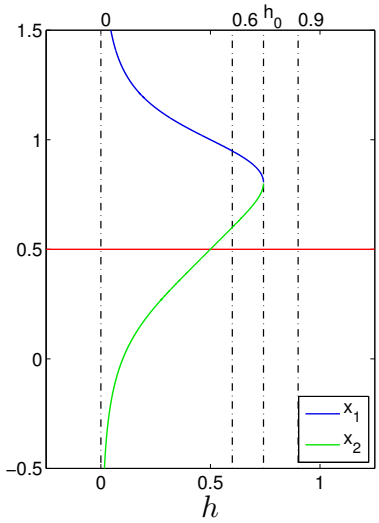


FIGURE 12. Solutions of equation (2.5)

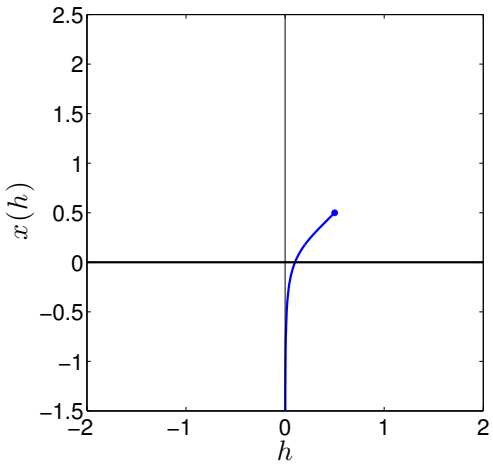


FIGURE 13. Solutions $x(h)$ for $l = 5$, such that $x \leq \frac{1}{2}$

3. Degree of heterogeneity and symbiosis/assimilation

Figure 14 displays the 5 homogenisation paths locally for $h \in [0.3, 0.7]$ and $l = 1$ to 5. The corresponding values for $1 - \frac{x}{h}$ are displayed in Table 4.

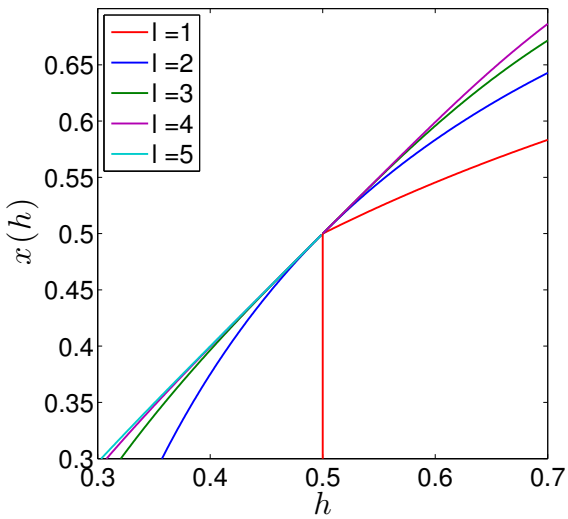


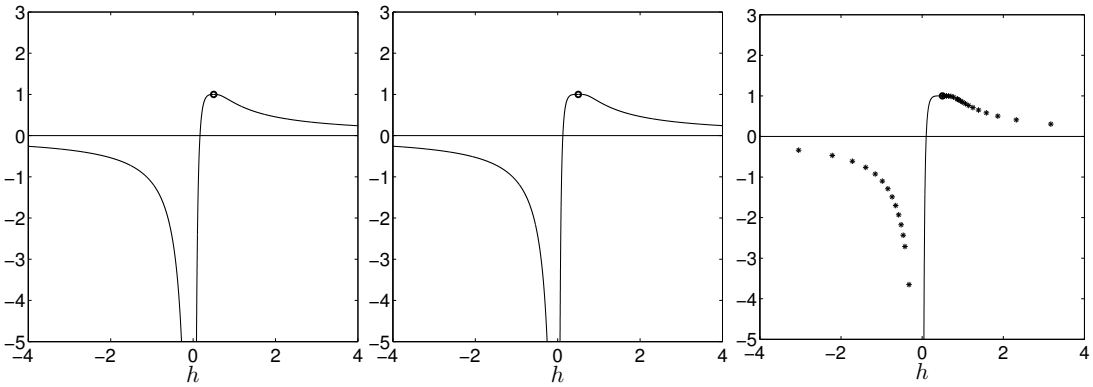
FIGURE 14. Exact homogenisation path near $(\frac{1}{2}, \frac{1}{2})$

	$E^l(x) = 1 - \frac{x}{h}$				
h	l =1	l =2	l =3	l =4	l =5
0.3	1	0.44444	0.095712	0.030641	0.011123
0.4	1	0.0625	0.0088835	0.0016425	0.00032207
0.45	1	0.012346	0.001028	0.00010036	1.0005×10^{-5}
0.49	1	0.00041649	8.0094×10^{-6}	1.6001×10^{-7}	3.2×10^{-9}
0.5	≥ 0	0	0	0	0
0.51	0.009901	0.00038447	7.9902×10^{-6}	1.5999×10^{-7}	- - -
0.55	0.047619	0.0082645	0.00096838	9.9563×10^{-5}	- - -
0.6	0.090909	0.027778	0.0070298	0.0015416	- - -
0.7	0.16667	0.081633	0.040473	0.019341	- - -

TABLE 4. Computed degree of heterogeneity in exact arithmetic

For $l = 1$ and $h = \frac{1}{2}$, $E(x) = 1 - 2x > 0$ for $x < \frac{1}{2}$ and $E(x) = 0$ for $x = \frac{1}{2}$ only. For $l = 1$ and $-\frac{1}{2} < h < \frac{1}{2}$, the only solution is $x = 0$ such that $E(0) = 1$. As l increases, the approximation of the identity function $\mathbf{1}$ around $(\frac{1}{2}, \frac{1}{2})$ gets better. For $l = 5$ and $h > \frac{1}{2}$, we do not have access to the exact homogenisation path, thus the degree of heterogeneity cannot be computed explicitly in this case.

The quotient $\frac{x(h)}{h} = \tan\theta(h)$ is displayed for $l = 3, 4, 5$ on Figure 15, showing that $-\frac{\pi}{2} \leq \theta(h) \leq \frac{\pi}{4}$, with right equality for $h = \frac{1}{2}$ and left equality for $h = 0$.

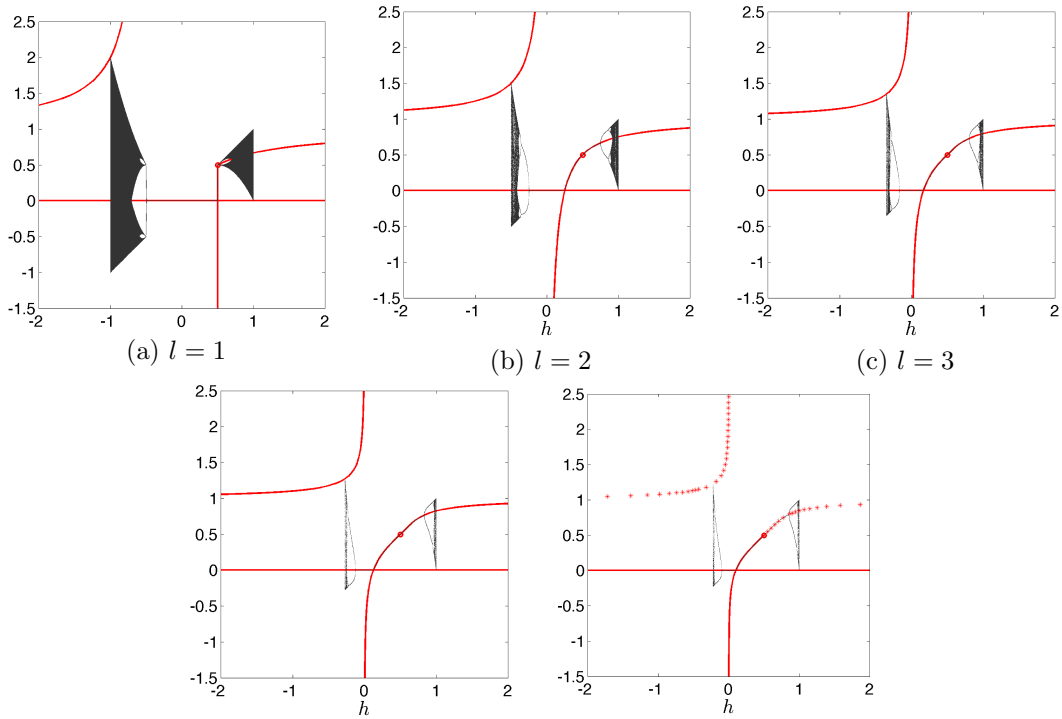


(a) $l = 3$ (b) $l = 4$ (c) $l = 5$

FIGURE 15. $\tan\theta(h) = \frac{x(h)}{h}$ versus h

When $l = 5$ and $x \geq \frac{1}{2}$, the root $x(h)$ is computed by inverting the map $x \mapsto h = \frac{x}{1-|2x-1|^l}$.

4. Numerical illustration of Picard iteration for $l = 1$ to 5



(a) $l = 1$ (b) $l = 2$ (c) $l = 3$ (d) $l = 4$ (e) $l = 5$

Figure 16. Orbit diagrams for fixed point iteration (black) and exact $x(h)$ (red)

On Fig. 16 (e) the red *dotted* curve is $x(h)$ computed as the inverse of $x \geq \frac{1}{2} \mapsto h = \frac{x}{1-(2x-1)^5}$. As l increases, $x(h)$ gets close to h (or $\theta(h) \sim \frac{\pi}{4}$) on an increasing interval around $h = \frac{1}{2}$. The corresponding plots for $\tan\theta(h)$ are displayed in Fig. 17.

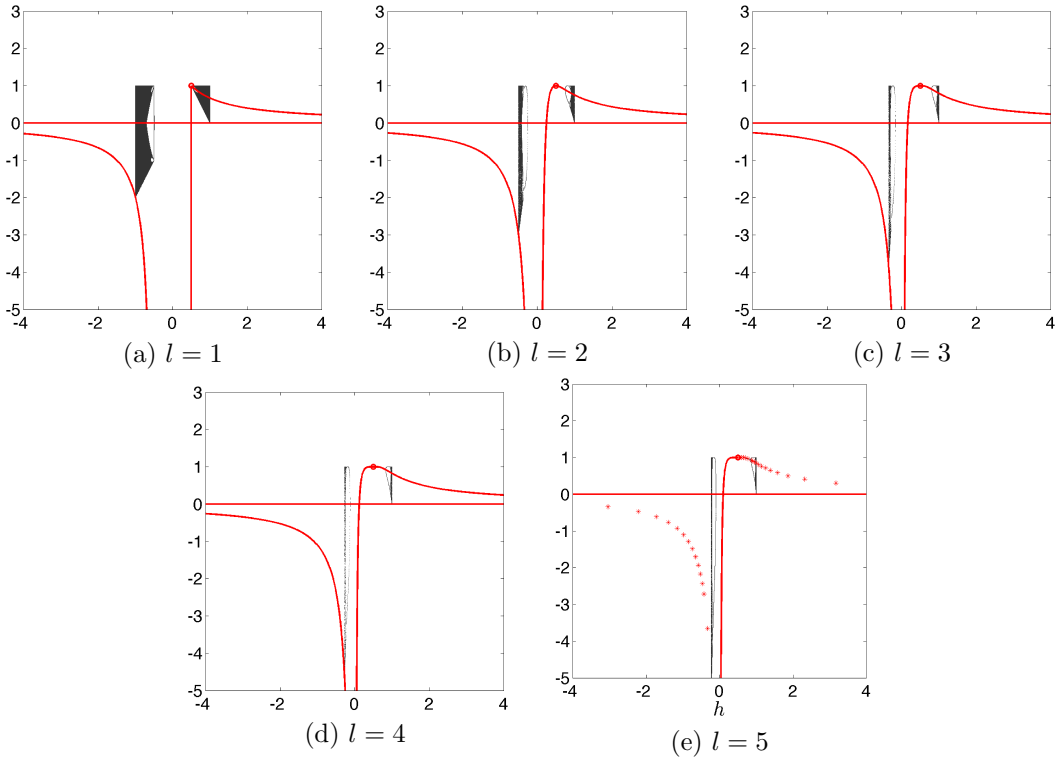
5. Numerical simulations on the Picard iteration for l positive

5.1. $0 < l < 1$. In Figure 18, we observe that the orbit diagrams present two *unexpected* phenomena for $0 < l \leq l_1 < 1$, $l_1 \approx \frac{1}{2}$:

1) the iterates do not escape to $\pm\infty$,

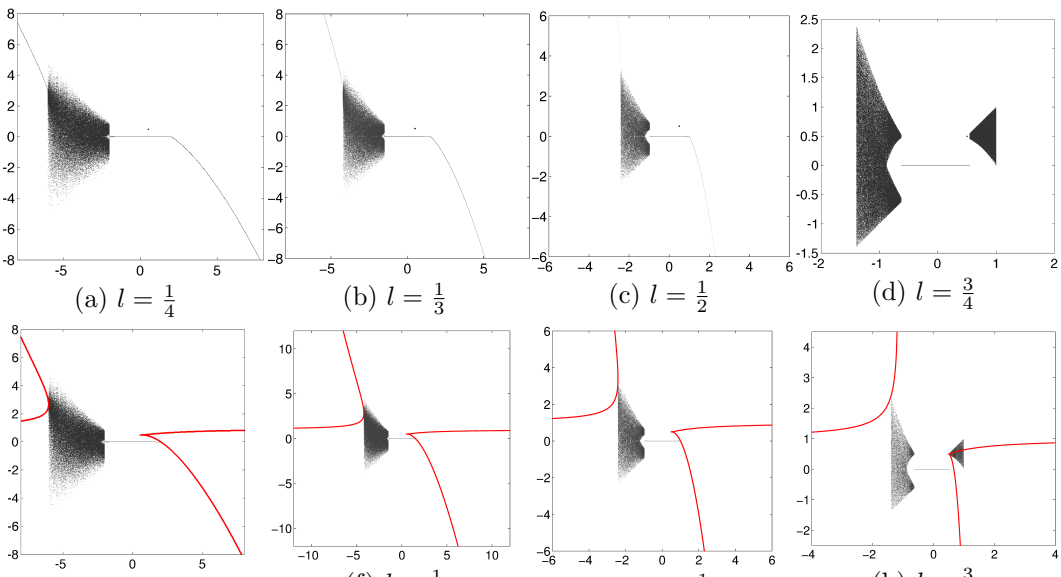
2) the forward chaos does not occur and is replaced by the single point $(\frac{1}{2}, \frac{1}{2})$ where the homogenisation path presents a cusp.

This shows also on Figure 19.



(a) $l = 1$ (b) $l = 2$ (c) $l = 3$ (d) $l = 4$ (e) $l = 5$

FIGURE 17. $\tan\theta(h) = \frac{x(h)}{h}$ versus h exact (red) and Picard (black)



(a) $l = \frac{1}{4}$ (b) $l = \frac{1}{3}$ (c) $l = \frac{1}{2}$ (d) $l = \frac{3}{4}$

(e) $l = \frac{1}{4}$ (f) $l = \frac{1}{3}$ (g) $l = \frac{1}{2}$ (h) $l = \frac{3}{4}$

FIGURE 18. Orbit diagrams for the Picard iteration (black) and inverse map (red)

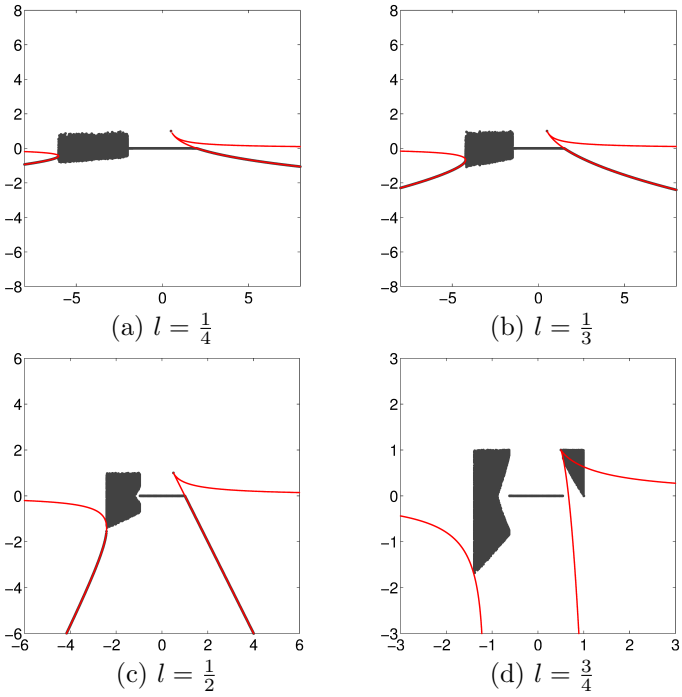
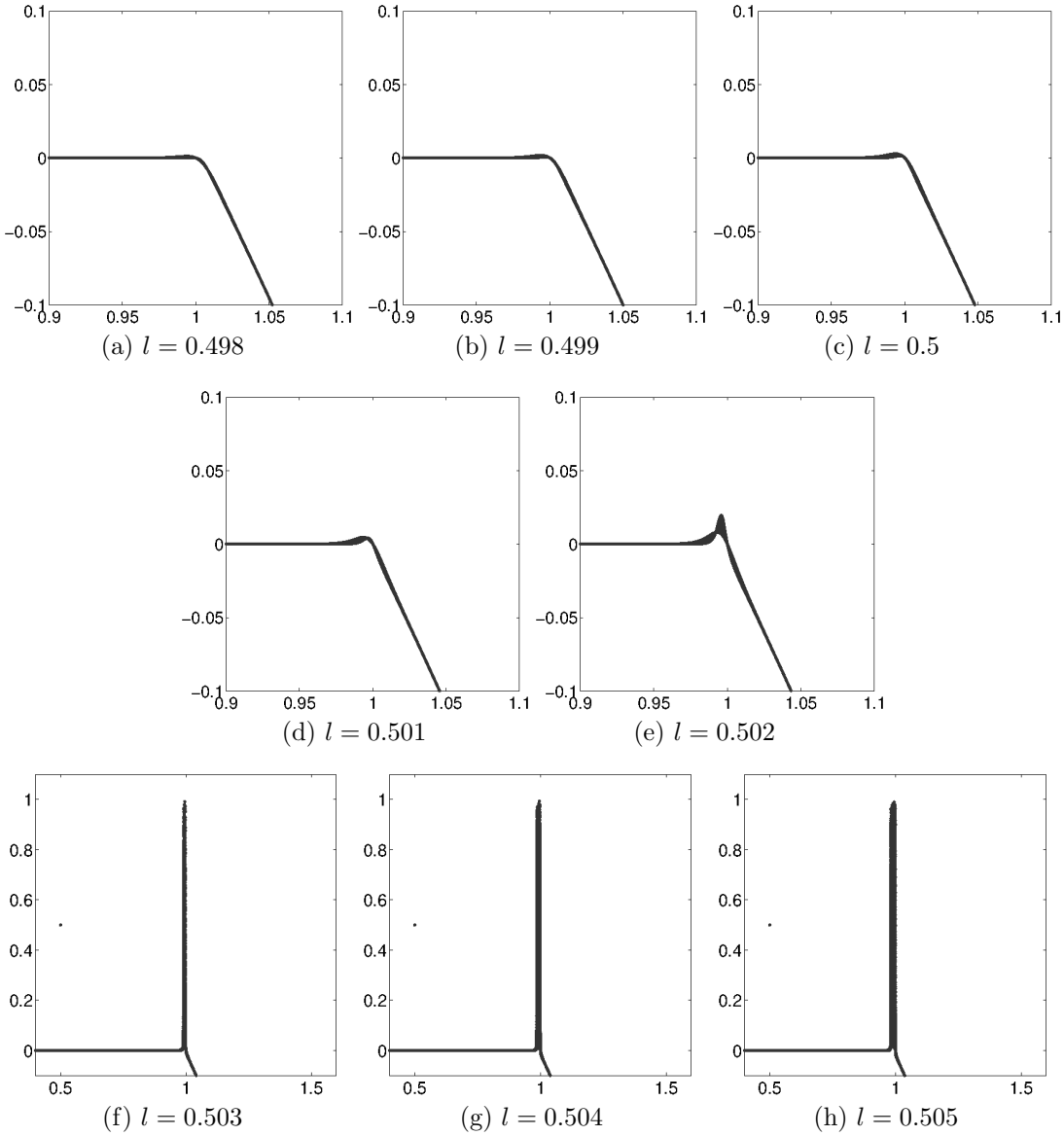


FIGURE 19. $\tan\theta(h) = \frac{x(h)}{h}$ versus h

5.2. **The emergence of the forward chaos near $l = \frac{1}{2}$ in $[0.498, 0.505]$ and h near 1.** Figure 20 illustrates the emergence of the forward chaos near $h = 1$ when l is around $\frac{1}{2}$. When $\frac{1}{2} < l \leq 1$ the forward chaos expands gradually for $h \in [\frac{1}{2}, 1]$.



(a) $l = 0.498$ (b) $l = 0.499$ (c) $l = 0.5$

(d) $l = 0.501$ (e) $l = 0.502$

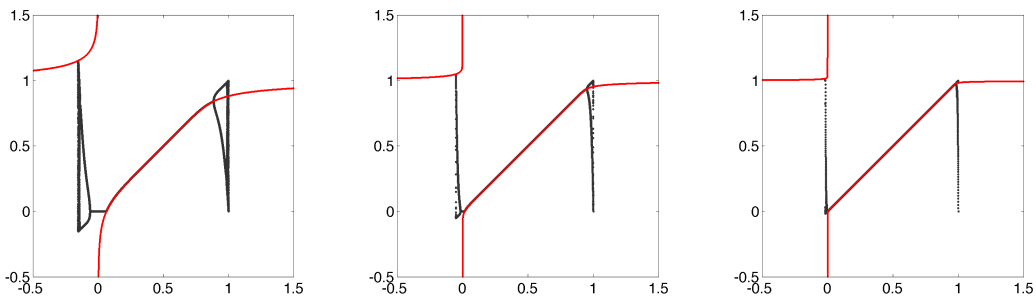
(f) $l = 0.503$ (g) $l = 0.504$ (h) $l = 0.505$

FIGURE 20. Forward chaos near $l = \frac{1}{2}$

5.3. l **large, display window** $[-0.5, 1.5] \times [-0.5, 1.5]$**.** When l increases the window of stability $[h_0, h_1]$ for $x(h)$, (h_1 being the first bifurcation from period 1 to 2), $h_0 = \frac{1}{2l}$, $h_1 < 1$, increases to cover the open interval $]0, 1[$ according to the table

l	2	3	4	64	∞
h_0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{128}$	0^+
h_1	$\frac{3}{4}$	≈ 0.8025	≈ 0.8275	≈ 0.9704	1^-

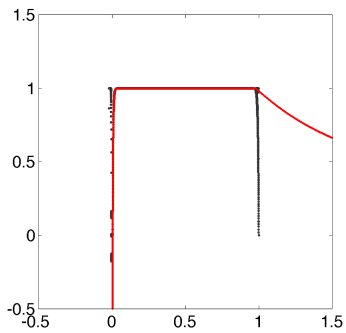
Section 6 will indicate why, for $l \to \infty$, Picard iteration provides the identity function in $]0, 1[$, that is $h_1 \to 1^-$.



(a) $l = 8$ (b) $l = 32$ (c) $l = 128$

FIGURE 21. Orbit diagrams for fixed point iteration (black) and $x(h)$ (red) $l \geq 8$

In the limit $l \to \infty$, the iterates do not escape to $\pm\infty$ iff $h \in [0, 1]$. They vary in $[0, 1]$ for $h = 0$ and 1, and describe the segment $x = h$ for $h \in]0, 1[$. The resulting symbiosis/assimilation shows on Figure 21 (c) ($l = 128$).



$l = 100$

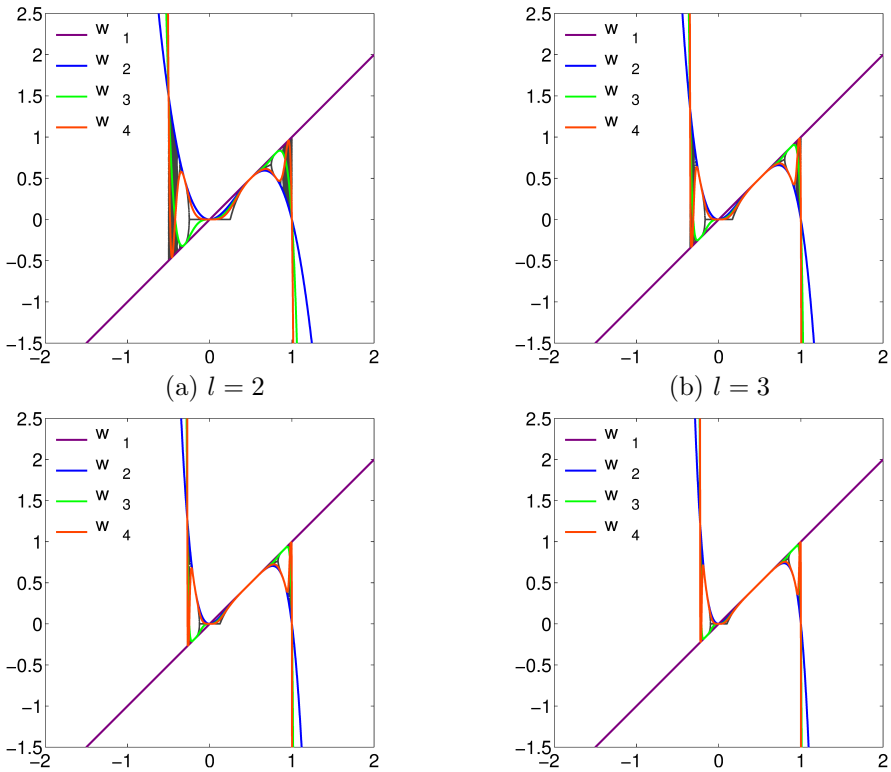
FIGURE 22. $\tan\theta(h) = \frac{x(h)}{h}$ versus h, Picard iteration (black), exact (red)

6. Wavetracks

The paper [Oblow, 1988] proved that the iterates defined by (1.2) ($l = 2$) are confined by the "supertrack" curves $h \mapsto w_k(h)$, $k \geq 1$ with

$$\begin{cases} w_1(h) & = h\Lambda_2(\frac{1}{2}) = h, \\ w_k(h) & = h\Lambda_2(w_{k-1}(h)), \quad k \geq 2, \end{cases} \tag{6.1}$$

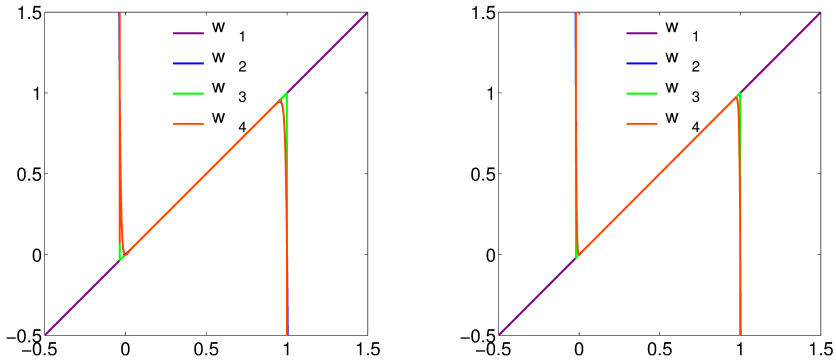
which are the successive iterates by $h\Lambda_2$ of the critical point $x = \frac{1}{2}$. The definition (6.1) can be extended to $l > 0$, yielding $w_1 = \mathbf{1}$ for all $l > 0$ and $w_2(h) = h(1 - |2h - 1|^l)$ which depends on l and satisfies $w_2(0) = w_2(1) = 0$, $w_2(\frac{1}{2}) = \frac{1}{2} = w_1(\frac{1}{2}) = x(\frac{1}{2})$. For $0 < h < 1$, $|2h - 1| < 1$ and $w_2(h) \to h = w_1(h)$ as $l \to \infty$. This establishes the claim that the identity function is provided by the Picard iteration. For all n, $h \in [\frac{1}{2}, 1]$, $w_2(h) \leq x_n(h) \leq w_1(h) = h$ and for $h \in [h_0, 0]$, $h \leq x_n(h) \leq w_2(h)$. Such more general curves are called *wavetracks* in [Chatelin, 2012, ch. 6]. The first four wavetracks are displayed on Figures 23 to 24, together with the Picard iterates.



(a) $l = 2$ (b) $l = 3$

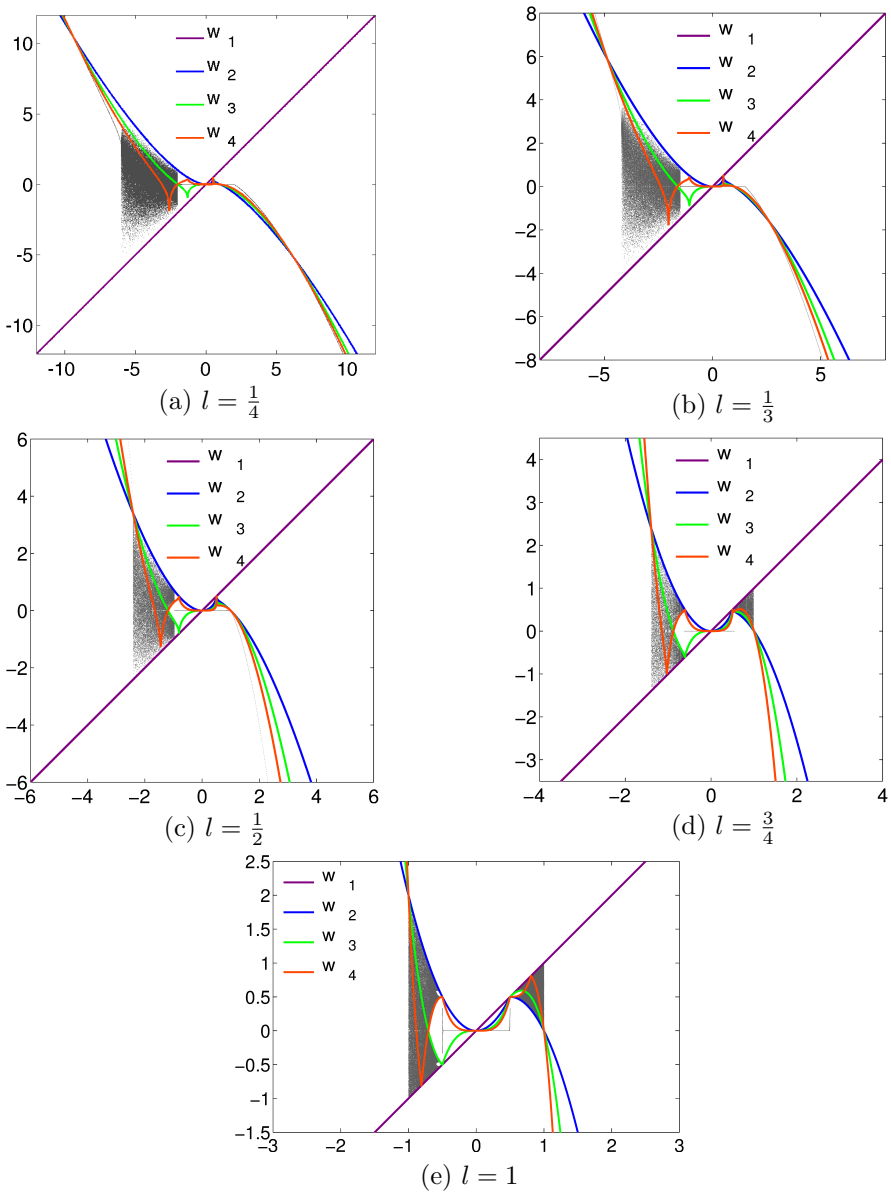
(c) $l = 4$ (d) $l = 5$

Figure 23. Smooth wavetracks: $l = 2$ to 5



(a) $l = 50$ (b) $l = 100$

FIGURE 24. Smooth wavetracks for l large



(a) $l = \frac{1}{4}$ (b) $l = \frac{1}{3}$ (c) $l = \frac{1}{2}$ (d) $l = \frac{3}{4}$ (e) $l = 1$

FIGURE 25. Continuous wavetracks for $0 < l \leq 1$

For $0 < l \leq l_1$, $l_1 \approx \frac{1}{2}$, the lower bound h is not achieved by x_n in the region of the backward chaos. The parameter l controls the smoothness of the wavetracks. For $l \leq 1$, the wavetracks present some cusps. By contrast they are everywhere smooth for $l > 1$ (observable periods).

7. Elements for future research

In order to substantiate the difference between (1.6) and (1.7), the fixed point equation (1.5) is studied in [Rincon-Camacho et al., 2014b]. In [Rincon-Camacho et al., 2014a] we relax the constraint that the solutions of (1.4) and (1.5) lie *on the real line* by using numbers which can be bireal, dual or bicomplex, enjoying a ring structure with zerodivisors. The epistemological

significance is discussed in [Chatelin and Rincon-Camacho, 2014]. Finally [Chatelin, 2014b] will use the Homotopic Deviation theory presented in [Chatelin, 2012, Chap. 7] to solve (1.4) and (1.5) when the closed form with radicals is not available ($l \geq 5$ is an integer), as an alternative to the inverse map of $x \mapsto h = \frac{x}{1-|2x-1|^l}$. The paper [Rincon-Camacho, 2014] summarizes the content of this report; it is the written version of the talk that Rincon-Camacho presented in Jaca (Spain).

Acknowledgment The work of M. M. Rincon-Camacho is supported by a grant from Total, Direction Scientifique.

Appendix

A. Cardano's method for a cubic equation

In order to solve a cubic equation given by

$$az^3 + bz^2 + cz + d = 0, \quad a, b, c, d \in \mathbb{R}, \quad a \neq 0, \tag{A.1}$$

we detail below Cardano's method as summarized in [Bronstein and Semendiaev, 1990]. The change of variable $y = z + \frac{b}{3a}$ transforms (A.1) into its *reduced form*

$$y^3 + 3py + 2q = 0 \tag{A.2}$$

where

$$2q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \tag{A.3}$$

$$3p = \frac{3ac - b^2}{3a^2}. \tag{A.4}$$

The discriminant D which determines the number of real roots of equations (A.2) is

$$D = p^3 + q^2. \tag{A.5}$$

Let us set

$$r = \text{sgn}\ (q)\sqrt{|p|}, \tag{A.6}$$

then

$$\frac{q}{r^3} = \frac{|q|}{|p|^{3/2}} \geq 0, \tag{A.7}$$

and $D = 0$ iff $q^2 = -p^3$. Either $p = q = 0$, (A.2) $\Leftrightarrow y^3 = 0$ or $p < 0$ and $D = 0 \Leftrightarrow \frac{q}{r^3} = 1$. Thus we distinguish the 7 following cases:

(i) If $D \leq 0$, $q^2 \leq -p^3$ and $p < 0$, thus $|q| \leq |p|^{3/2}$. There are three real roots:

$$y_1 = -2\, r \cos\frac{\varphi}{3},\ y_2 = 2\, r \cos\frac{\pi - \varphi}{3} \text{ and } y_3 = 2\, r \cos\frac{\pi + \varphi}{3}, \tag{A.8}$$

where φ is such that:

$$0 \leq \cos\varphi = \frac{q}{r^3} \leq 1,\ \varphi \in \left[0, \frac{\pi}{2}\right]. \tag{A.9}$$

The 3 angles $\theta_1 = \frac{\varphi}{3}$, $\theta_2 = \frac{\pi-\varphi}{3}$ and $\theta_3 = \frac{\pi+\varphi}{3}$ vary respectively in $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$ and $[\frac{\pi}{3}, \frac{\pi}{2}]$ according to Figure 26. We observe that $\cos\varphi = 1 \Leftrightarrow \varphi = 0 \Leftrightarrow D = 0$ and $\cos\varphi = 0 \Leftrightarrow \varphi = \frac{\pi}{2}$ $\Leftrightarrow q = 0$, $-\infty < D = p^3 < 0$ or $q \neq 0$, D unbounded at $-\infty$.

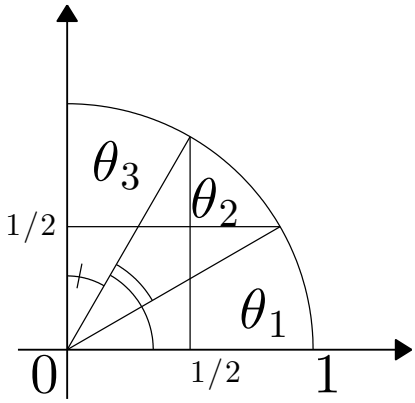


FIGURE 26. $D \leq 0$

The three real roots are distinct for $D < 0$.

(ii) If $D = p^3 < 0$, $q = 0$, $y_3 = 0$ and $y_{1,2} = \pm\sqrt[2]{-3p}$ where $\cos\theta_3 = 0$, $\cos\theta_{1,2} = \frac{\sqrt{3}}{2}$ and $\pm = \mathrm{sgn}(r)$ is indefinite, $|r| = \sqrt[2]{-p} \neq 0$.

(iii) If $D = 0$ and $-\infty < p < 0$, $q \neq 0$, $\varphi = 0$ hence $\theta_1 = 0$, $\theta_2 = \theta_3 = \frac{\pi}{3}$. Thus $y_1 = -2r$, $y_2 = y_3 = r$, $0 \neq r = \sqrt[3]{q}$.

(iv) If $D = 0$ and $p = q = 0$, $y_{1,2,3} = 0$.

(v) If $D > 0$ and $p < 0$, there is one real root and two complex conjugate roots

$$y_1 = -2\, r\cosh\frac{\varphi}{3},\ y_2 = r\cosh\frac{\varphi}{3} + i\sqrt{3}\, r\sinh\frac{\varphi}{3} \text{ and } y_3 = r\cosh\frac{\varphi}{3} - i\sqrt{3}\, r\sinh\frac{\varphi}{3}, \tag{A.10}$$

where

$$\cosh\varphi = \frac{q}{r^3} = \frac{|q|}{|p|^{3/2}} > 1, \tag{A.11}$$

since $D > 0 \Leftrightarrow q^2 > (-p)^3 \Leftrightarrow |q| > |p|^{3/2} > 0$.

(vi) If $D > 0$ and $p > 0$, there is also one real root and two complex conjugate roots given by

$$y_1 = -2\, r\sinh\frac{\varphi}{3},\ y_2 = r\left(\sinh\frac{\varphi}{3} + i\sqrt{3}\,\cosh\frac{\varphi}{3}\right) \text{ and } y_3 = r\left(\sinh\frac{\varphi}{3} - i\sqrt{3}\,\cosh\frac{\varphi}{3}\right), \tag{A.12}$$

where

$$\sinh\varphi = \frac{q}{r^3} > 0, \quad \text{for } \ q \neq 0 \text{ and } \ = 0 \text{ for } q = 0. \tag{A.13}$$

If $q = 0$, $y_1 = 0$, $y_2 = \bar{y}_3 = \pm i\sqrt{3}|r|$ satisfy $y(y^2 + 3p) = 0$ with $|r| = \sqrt{p}$. As before in case (ii) $\pm = \mathrm{sgn}(r)$ is indefinite.

(vii) If $p = 0$, $D = q^2 > 0$, $y_{k+1} = \sqrt[3]{-2q}e^{2i\frac{k\pi}{3}}$, $k = 0, 1, 2$.

When $p \to 0^-$ (resp. 0^+) in case (iv) (resp. (v)) with $q \neq 0$, $|r| \to 0$ and $\varphi \to \infty$. Thus $\lim_{p\to 0^-}\left(-2r\cosh\frac{\varphi}{3}\right)$ and $\lim_{p\to 0^+}(-2r\sinh\frac{\varphi}{3})$ are indeterminate forms $0 \times \infty$ which take the common value $\sqrt[3]{-2q}$.

Finally the roots of the equation (A.1) are given by

$$z_1 = y_1 - \frac{b}{3a}, \quad z_2 = y_2 - \frac{b}{3a}, \quad z_3 = y_3 - \frac{b}{3a}. \tag{A.14}$$

Formulae (A.6) to (A.14) summarise the modern approach based on the auxiliary variable φ and the circular and hyperbolic trigonometric functions $\cos\frac{\varphi}{3}$, $\cosh\frac{\varphi}{3}$ and $\sinh\frac{\varphi}{3}$. The original

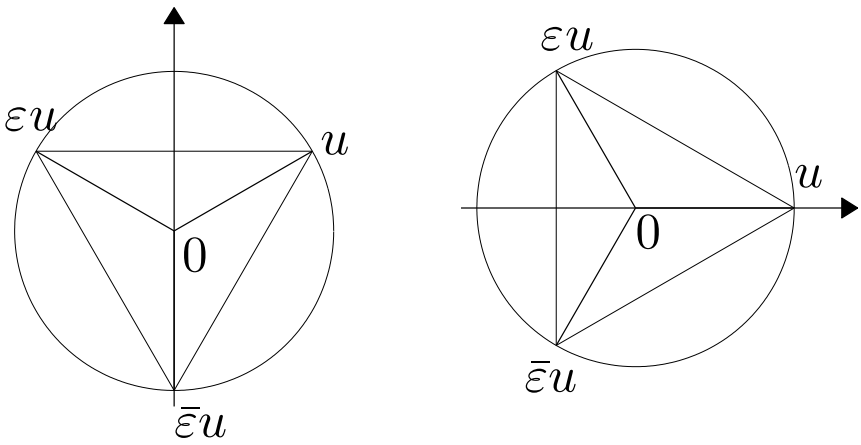
approach of Cardano is direct and purely algebraic, which is based solely on radicals. Set $u = (-q + D^{1/2})^{1/3}$, $v = (-q - D^{1/2})^{1/3}$ and $\varepsilon = e^{2i\frac{\pi}{3}}$, $\varepsilon + \bar{\varepsilon} = -1$. Then the 3-roots of (A.2) are given by

$$y_1 = u + v, \quad y_2 = \varepsilon u + \bar{\varepsilon} v, \quad y_3 = \bar{\varepsilon} u + \varepsilon v. \tag{A.15}$$

Their nature, real or complex, follows:

1) $D > 0$: $u \neq v \in \mathbb{R}$, hence $y_1 \in \mathbb{R}$, y_2 and $y_3 = \bar{y}_2 \in \mathbb{C}$. The case $D > 0$ has been divided above into the 3 possibilities: (v) $p < 0$, (vi) $p > 0$ and (vii) $p = 0$, $q \neq 0$.

2) $D < 0 \Leftrightarrow (i) \cup (ii)$: $u = (-q + i\sqrt{-D})^{1/3}$, $v = \bar{u}$ and $y_1 = 2\Re u$, $y_2 = 2\Re \varepsilon u$, $y_3 = 2\Re \bar{\varepsilon} u$. In other words, the solutions y_1, y_2, y_3 are the real parts of the 3 complex numbers u, εu and $\bar{\varepsilon} u$ which are the vertices of an *equilateral* triangle inscribed in the circle $\{z \in \mathbb{C} \ : \ |z| = 2|r|\}$. Figure 27 displays the two limit positions corresponding to (iii) $\varphi = 0 \Leftrightarrow D = 0 \Leftrightarrow u \in \mathbb{R}$ and (ii) $q = 0$ and $-\infty < D < 0 \Rightarrow \varphi = \frac{\pi}{2}$ and $y_3 = 0$.



(ii) $q = 0$, $D < 0 \Rightarrow \varphi = \frac{\pi}{2}$ (iii) $D = 0 = \varphi$

FIGURE 27. $\theta_1 = \frac{\varphi}{3} \in \{0, \frac{\pi}{6}\}$ when $D \leq 0$

In general, if the angle φ increases from 0 to $\frac{\pi}{2}$, the quotient $\frac{y_1}{\|u\|}$ (resp. $\frac{y_2}{\|u\|}$) decreases from 2 to $\sqrt{3}$ (resp. -1 to $-\sqrt{3}$) and $\frac{y_3}{\|u\|}$ increases from -1 to 0.

See Figure 28

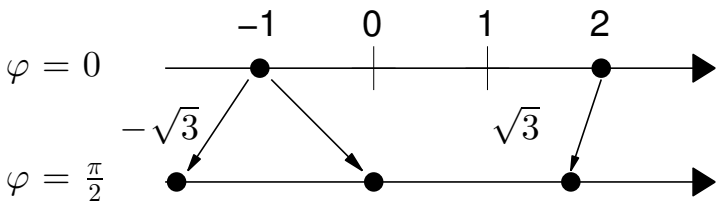


FIGURE 28. Variation of $\frac{y}{\|u\|}$

The corresponding rates of relative metric evolution are 0.268, 0.732 and 1 for y_1, y_2 and y_3 respectively.

3) $D = 0$, either (iii) $q \neq 0$, $u = v = \sqrt[3]{-q} = -r \in \mathbb{R}$ and $y_1 = 2u$, $y_2 = y_3 = -u$, or (iv) $q = p = 0$ and $y_{1,2,3} = 0$.

B. Ferrari's method for a quartic equation

The real solutions to the quartic equation

$$w^4 + bw^3 + cw^2 + dw + e = 0, \quad b, c, d, e \in \mathbb{R}, \tag{B.1}$$

are given by Ferrari's method presented in [Bronstein and Semendiaev, 1990]. We define $B = 8z + b^2 - 4c \in \mathbb{R}$, and set

$$A = \pm B^{1/2}, \tag{B.2}$$

where z is a real solution z of the auxiliary cubic equation

$$8z^3 - 4cz^2 + (2bd - 8e)z + e(4c - b^2) - d^2 = 0. \tag{B.3}$$

If $z > \frac{4c-b^2}{8}$, $B > 0$ and the square roots A are real $\neq 0$. Thus the solutions to the quartic equation (B.1) are the solutions of the following two quadratic equations with real coefficients:

$$w^2 + \frac{b + A}{2} w + (z + \frac{bz - d}{A}) = 0, \quad A \neq 0,$$

whose discriminants are

$$\Delta_- = \frac{(b - |A|)^2}{4} - 4\left(\frac{d + z(-b + |A|)}{|A|}\right), \quad \Delta_+ = \frac{(b + |A|)^2}{4} + 4\left(\frac{d - z(b + |A|)}{|A|}\right). \tag{B.4}$$

The real solutions to the equation (B.1) are:

$$w_1 = \frac{-b + |A|}{4} + \frac{1}{2}\sqrt{\Delta_-}, \quad w_2 = \frac{-b + |A|}{4} - \frac{1}{2}\sqrt{\Delta_-}, \quad \text{for } \Delta_- \geq 0, \tag{B.5}$$

$$w_3 = \frac{-b - |A|}{4} + \frac{1}{2}\sqrt{\Delta_+}, \quad w_4 = \frac{-b - |A|}{4} - \frac{1}{2}\sqrt{\Delta_+}, \quad \text{for } \frac{1}{2}\Delta_+ \geq 0. \tag{B.6}$$

In case $z = \frac{4c-b^2}{8} \neq 0$ is a *triple* solution of (B.3), Ferrari's method breaks down. This exceptional situation is treated elsewhere [Chatelin, 2014a].

We mention for future reference that the condition that z be a *real* solution of (B.3) can be relaxed in full generality when the solutions of (B.1) can be 1 or 2 pairs of complex conjugate numbers [Rincon-Camacho et al., 2014a].

C. $l = 3$

C.1. **Equation** (2.1)**.**

$$4x^3 - 6x^2 + (\frac{1}{2h} + 3)x - 1 = 0, \quad a = 4, \; b = -6; \; c = \frac{1}{2h} + 3, \; d = -1$$

$$\begin{aligned}
2q(h) &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \\
&= -\frac{2^4\cdot 3^3}{3^3\cdot 2^6} + \frac{2\cdot 3}{3\cdot 2^5 h} + \frac{2\cdot 3^2}{3\cdot 2^4} - \frac{1}{2} \\
&= -\frac{1}{2^2} + \frac{1}{2^4 h} + \frac{3}{2^3} - \frac{1}{2^2} \\
&= -\frac{1}{2} + \frac{3}{2^3} + \frac{1}{2^4 h} \\
&= -\frac{-2^2+3}{2^3} + \frac{1}{2^4 h} \\
&= -\frac{1}{2^3} + \frac{1}{2^4 h} \\
q(h) &= \frac{1}{2^5 h} - \frac{1}{2^4}
\end{aligned}$$

$$\begin{aligned}
3p(h) &= \frac{c}{a} - \frac{b^2}{3a^2} \\
&= \frac{1}{2^3 h} + \frac{3}{2^2} - \frac{2^2\cdot 3^3}{3\cdot 2^4} \\
&= \frac{1}{2^3 h} + \frac{3^2\cdot 2^2 - 2^2\cdot 3^3}{3\cdot 2^4} \\
&= \frac{1}{2^3 h} \\
p(h) &= \frac{1}{2^3 3 h}
\end{aligned}$$

$$\begin{aligned}
D(h) &= p(h)^3 + q(h)^2 \\
&= \frac{1}{2^9 3^3 h^3} + \left(\frac{1}{2^5 h} - \frac{1}{2^4}\right)^2 \\
&= \frac{1}{2^9\cdot 3^2 h^3} + \frac{1}{2^8}\left(\frac{1}{2h} - 1\right)^2 \\
&= \frac{1}{2^8}\left(\frac{1}{2\cdot 3^3 h^3} + \frac{1}{2^2 h^2} - \frac{1}{h} + 1\right) \\
&= \frac{1}{2^9 3^3}\left(\frac{1}{h^3} + \frac{3^3}{2h^2} - \frac{2\cdot 3^3}{h} + 2\cdot 3^3\right)
\end{aligned}$$

C.2. **Equation** (2.2)**.**

$$t^3 + \frac{3^3}{2}t^2 - 2\cdot 3^3 t + 2\cdot 3^3 = 0,\ \ a = 1,\ b = \frac{3^3}{2},\ c = -2\cdot 3^3,\ d = 2\cdot 3^3.$$

$$\begin{aligned}
2q &= \frac{2\cdot 3^9}{3^3 2^3} + \frac{3^3 2\cdot 3^3}{3\cdot 2} + \frac{2\cdot 3^3}{1} \\
&= 3^3\left(\frac{3^3}{2^2} + 3^2 + 2\right) \\
&= 3^3\left(\frac{3^3 + 3^2 2^2 + 2^3}{2^2}\right) \\
&= \frac{3^3\cdot 71}{2^2} \\
q &= \frac{3^3\cdot 71}{2^3}
\end{aligned}$$

$$\begin{aligned}
3p &= -2\cdot 3^3 - \frac{3^6}{3\cdot 2^3} \\
&= -2\cdot 3^3 - \frac{3^5}{2^2} \\
&= 3^3\left(-2 - \frac{3^2}{2^2}\right) \\
&= 3^3\left(\frac{-8-9}{2^2}\right) \\
&= -\frac{3^3\cdot 17}{2^2} \\
p &= -\frac{3^2\cdot 17}{2^2}
\end{aligned}$$

$$\begin{aligned}
D &= p^3 + q^2 \\
&= -\frac{3^6\cdot 17^3}{2^6} + \frac{3^6\cdot 71^2}{2^6} \\
&= \frac{3^6}{2^6}(71^2 - 17^3) \\
&= \frac{3^6\cdot 2^7}{2^6} \\
&= 2\cdot 3^6
\end{aligned}$$

D. $l = 4$

$$2x^3 - 4x^2 + 3x + \frac{1}{8h} - 1 = 0,\ \ a = 2,\ \ b = -2^2,\ \ c = 3\ \ d = \frac{1}{2^3 h} - 1$$

$$
\begin{aligned}
2q(h) &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \\
&= -\frac{2\cdot 2^6}{3^3 2^3} + \frac{2^2 3}{3\cdot 2^2} + \frac{1}{2^4 h} - \frac{1}{2} \\
&= -\frac{2^4}{3^3} + 1 + \frac{1}{2^4 h} - \frac{1}{2} \\
&= \frac{1}{2^4 h} + \frac{1}{2} - \frac{2^4}{3^3} \\
&= \frac{1}{2^4 h} + \frac{3^3 - 2^5}{2\cdot 3^3} \\
&= \frac{1}{2^4 h} + \frac{27-32}{2\cdot 3^3} \\
&= \frac{1}{2^4 h} - \frac{5}{2\cdot 3^3} \\
q(h) &= \frac{1}{2^5 h} - \frac{5}{2^2 3^3}
\end{aligned}
$$

$$
\begin{aligned}
3p &= \frac{c}{a} - \frac{b^2}{3a^2} \\
&= \frac{3}{2} - \frac{2^4}{3\cdot 2^2} \\
&= \frac{3}{2} - \frac{2^2}{3} \\
&= \frac{3^2 - 2^3}{2\cdot 3} \\
&= \frac{1}{2\cdot 3} \\
p &= \frac{1}{2\cdot 3^2}
\end{aligned}
$$

$$
\begin{aligned}
D(h) = p^3 + q(h)^2 &= \frac{1}{2^3 3^6} + \left(\frac{1}{2^5 h} - \frac{5}{2^2 3^3}\right)^2 \\
&= \frac{1}{2^3 3^6} + \frac{1}{2^1 0 h^2} - \frac{5}{2^6 3^3 h} + \frac{5^2}{2^4 3^6} = \frac{1}{2^{10} h^2} - \frac{5}{2^6 3^3 h} + \frac{2+5^2}{2^4 3^6} \\
&= \frac{1}{2^{10} h^2} - \frac{5}{2^6 3^3 h} + \frac{3^3}{2^4 3^6} \\
&= \frac{1}{2^{10} 3^3}\left(\frac{3^3}{h^2} - \frac{5\cdot 2^4}{h} + 2^6\right)
\end{aligned}
$$

E. $l = 5$

E.1. **Equation** (2.6)**.**

$$
\begin{aligned}
2q(h) &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \\
&= -\frac{2^4\cdot 5^3}{3^3\cdot 2^9} + \frac{2\cdot 5}{3\cdot 2^6}\left(\frac{3\cdot 5}{2^2} + \frac{1}{2^2 h}\right) \\
&\quad - \frac{5^2}{2^9} - \frac{3\cdot 5}{2^{10} h} \\
&= -\frac{5^3}{2^5\cdot 3^3} + \frac{5^2}{2^7} + \frac{5}{2^7 3h} - \frac{5^2}{2^9} - \frac{3\cdot 5}{2^{10} h} \\
&= \frac{5^2}{2^9\cdot 3^3} - \frac{5}{2^{10}\cdot 3h} \\
&= \frac{-2^4 5^3 + 2^2 5^2 3^3 - 3^3 5^2}{2^9 3^3} + \frac{2^3 5 - 3^2 5}{2^{10} 3h} \\
&= \frac{5^2}{2^9 3^3} - \frac{5}{2^{10} 3h} \\
q(h) &= \frac{5^2}{2^{10}\cdot 3^3} - \frac{5}{2^{11}\cdot 3h}
\end{aligned}
$$

$$
\begin{aligned}
3p(h) &= \frac{c}{a} - \frac{b^2}{3a^2} \\
&= \frac{3\cdot 5}{2^5} + \frac{1}{2^5 h} - \frac{2^2\cdot 5^2}{3\cdot 2^6} \\
&= \frac{3\cdot 5}{2^5} - \frac{5^2}{3\cdot 2^4} + \frac{1}{2^5 h} \\
&= \frac{3^2\cdot 5 - 2\cdot 5^2}{3\cdot 2^5} + \frac{1}{2^5 h} \\
&= -\frac{5}{3\cdot 2^5} + \frac{1}{2^5 h} \\
p(h) &= -\frac{5}{3^2\cdot 2^5} + \frac{1}{2^5\cdot 3h}
\end{aligned}
$$

$$
\begin{aligned}
D(h) &= p(h)^3 + q(h)^2 \\
&= \left(-\frac{5}{3^2\cdot 2^5} + \frac{1}{2^5\cdot 3h}\right)^3 + \left(\frac{5^2}{2^{10}\cdot 3^3} - \frac{5}{2^{11}\cdot 3h}\right)^2 \\
&= \left(-\frac{5}{3^2 2^5}\right)^3 + 3\left(-\frac{5}{3^2 2^5}\right)^2\left(\frac{1}{2^5 3h}\right) + 3\left(-\frac{5}{3^2 2^5}\right)\left(\frac{1}{2^5 3h}\right)^2 + \left(\frac{1}{2^5 3h}\right)^3 \\
&\quad + \left(\frac{5^2}{2^{10}3^3}\right)^2 - 2\left(\frac{5^2}{2^{10}3^3}\right)\left(\frac{5}{2^{11}3h}\right) + \left(\frac{5}{2^{11}3h}\right)^2 \\
&= -\frac{5^3}{3^6 2^{15}} + \frac{3\cdot 5^2}{2^{15}3^5 h} - \frac{3\cdot 5}{2^{15}3^4 h^2} + \frac{1}{2^{15}3^3 h^3} + \frac{5^4}{2^{20}3^6} - \frac{2\cdot 5^3}{2^{21}3^4 h} + \frac{5^2}{2^{22}3^2 h} \\
&= \frac{1}{2^{15}3^3 h^3} + \left(\frac{5^2}{2^{22}3^2} - \frac{5}{3^3 2^{15}}\right)\frac{1}{h^2} + \left(\frac{5^2}{3^4 2^1 5} - \frac{5^3}{2^{20}3^4}\right)\frac{1}{h} + \frac{5^4}{2^{20}3^6} - \frac{5^3}{3^6 2^{15}} \\
&= \frac{1}{2^{15}3^3 h^3} - \frac{5\cdot 113}{2^{22}3^3 h^2} + \frac{5^2}{2^{20}3h} - \frac{5^3}{2^{20}3^3} \\
&= \frac{1}{2^{15}\cdot 3^3}\left(\frac{1}{h^3} - \frac{5\cdot 113}{2^7 h^2} + \frac{5^2\cdot 3^2}{2^5 h} - \frac{5^3}{2^5}\right)
\end{aligned}
$$

E.2. **Equation** (2.7)**.**

$$
t^3 - \frac{5\cdot 113}{2^7}t^2 + \frac{5^2\cdot 3^2}{2^5}t - \frac{5^3}{2^5} = 0, \quad a = 1,\ b = -\frac{5\cdot 113}{27},\ c = \frac{5^2\cdot 3^2}{2^5},\ d = -\frac{5^3}{2^5}
$$

$$
\begin{aligned}
2q &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \\
&= -\frac{2\cdot 5^3\cdot 113^3}{3^3\cdot 2^{21}} + \frac{3^2 5^3\cdot 113}{3\cdot 2^{12}} - \frac{5^3}{2^5} \\
&= -\frac{5^3 113^3}{3^3 2^{20}} + \frac{3\cdot 5^3\cdot 113}{2^{12}} - \frac{5^3}{2^5} \\
&= \frac{-5^3 113^3 + 3^4 2^8 5^3\cdot 113 - 3^3 5^3 2^{15}}{3^3 2^{20}} \\
&= \frac{5^3 113(-113^2 + 3^4 2^8) - 3^3 5^3 2^{15}}{3^3 2^{20}} \\
&= \frac{5^3 113\cdot 31\cdot 257 - 3^3\cdot 5^3\cdot 2^{15}}{3^3 2^{20}} \\
&= \frac{5^3(113\cdot 31\cdot 257 - 3^3\cdot 2^{15})}{3^3\cdot 2^{20}} \\
&= \frac{5^4\cdot 13\cdot 239}{3^3\cdot 2^{20}} \\
q &= \frac{5^4\cdot 13\cdot 239}{3^3\cdot 2^{21}}
\end{aligned}
$$

$$
\begin{aligned}
3p &= \frac{c}{a} - \frac{b^2}{3a^2} \\
&= \frac{5^2 3^2}{2^5} - \frac{5^2 113^2}{3\cdot 2^{14}} \\
&= \frac{2^9 3^3 5^2 - 5^2 113^2}{3\cdot 2^{14}} \\
&= \frac{5^2(2^9 3^3 - 113^2)}{3\cdot 2^{14}} \\
&= \frac{5^3\cdot 211}{3\cdot 2^{14}} \\
p &= \frac{5^3\cdot 211}{3^2\cdot 2^{14}}
\end{aligned}
$$

$$
\begin{aligned}
D &= p^3 + q^2 \\
&= \frac{5^9 211^3}{3^6 2^{42}} + \frac{5^8 13^2 239^2}{3^6 2^{42}} \\
&= \frac{5^8}{3^6 2^{42}}(5\cdot 211^3 + 13^2 239^2) \\
&= \frac{5^8 2^{21} 3^3}{3^6 2^{42}} \\
&= \frac{5^8}{3^3\cdot 2^{21}}
\end{aligned}
$$

E.3. **Details from equation** (2.6)**.**

$$
p(h) = -\frac{5}{3^2\cdot 2^5} + \frac{1}{2^5\cdot 3h}, \quad \text{set } t = \frac{1}{h}
$$

$$
p(t) \geq 0 \ \text{ if } \ \frac{t}{2^5\cdot 3} \geq \frac{5}{3^2\cdot 2^5}, \quad t \geq \frac{5}{3}
$$

$$p(h) \geq 0 \ \text{ if } \ h \in]0, 0.6]$$
$$p(h) < 0 \ \text{ if } \ h \in]-\infty, 0[\cup]0.6, \infty[$$

$$q(h) = \frac{5^2}{2^{10} \cdot 3^3} - \frac{5}{2^{11} \cdot 3h}, \ \text{ set } \ t = \frac{1}{h}$$
$$q(t) \geq 0 \ \text{ if} \frac{5^2}{2^{10} \cdot 3^3} \geq \frac{5t}{2^{11} \cdot 3}, \ \ t \leq \frac{5 \cdot 2}{3^2}$$
$$q(h) > 0 \ \text{ if } \ h \in]-\infty, 0[\cup]0.9, \infty[$$
$$q(h) \leq 0 \ \text{ if } \ h \in]0, 0.9]$$

$$r(h) = \operatorname{sgn}\left(\frac{5^2}{2^{10} \cdot 3^3} - \frac{5}{2^{11} \cdot 3h}\right)\sqrt{\left|-\frac{5}{3^2 \cdot 2^5} + \frac{1}{2^5 \cdot 3h}\right|}$$

$$\frac{q(h)}{r(h)^3} = \frac{|q(h)|}{|p(h)|^{3/2}} = \frac{5\sqrt{|h|}|5 \cdot 2h - 3^2|}{2^3\sqrt{2}|3 - 5h|^{3/2}}$$

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