About relative and absolute logic evolutions under laws of degree 2 and 4 and in 4 dimensions at most

M. Monserrat Rincon-Camacho, Françoise Chatelin and Philippe Ricoux

Technical Report TR/PA/14/75

Publications of the Parallel Algorithms Team
http://www.cerfacs.fr/algor/publications/
ABOUT RELATIVE AND ABSOLUTE LOGIC EVOLUTIONS UNDER LAWS OF DEGREE 2 AND 4 AND IN 4 DIMENSIONS AT MOST

M. MONSERRAT RINCON-CAMACHO (1),(2), FRANÇOISE CHATELIN (1),(2), AND PHILIPPE RICOUX (3)

Abstract. This report is a sequel for TR/PA/14/55 and 14/68 for \( l = 2 \) and 4. It considers both real and complex solutions as the case may be. Then, a coupling of the real solutions provides new non real solutions. This extends the logical evolutions, relative and absolute, beyond \( \mathbb{R} \) into a pencil of three numerical planes whose nature can be complex, bireal or dual.

Keywords: Logic evolution beyond \( \mathbb{R} \), coupling of real roots, complex, dual, bireal numbers, triple multiplicative nature of \( \mathbb{R}^2 \).

1. Introduction

1.1. An overview. In the reports [Rincon-Camacho et al., 2014a, Rincon-Camacho et al., 2014b] we have looked at the real solutions of the respective equations

\[
(1.1) \quad x = h(1 - |2x - 1|^l) = h\Lambda_l(x), \ h \in \mathbb{R}, \text{ and }
\]

\[
(1.2) \quad y = 1 - m|y|^l = G_l(m, y), \ m \in \mathbb{R}.
\]

In this work, we specialise \( l \) to be 2 and 4 and consider all roots (2 and 4) provided by the Fundamental Theorem of Algebra over the fields \( \mathbb{R} \) or \( \mathbb{C} \). However this theorem does not consider the roots in \( \mathbb{R}^2 \) which may belong to a ring with zerodivisors, such as the bireals in \( 2\mathbb{R} = \mathbb{R} \oplus \mathbb{R}u, \ u^2 = 1, \ u \neq \pm 1 \), and the duals in \( \mathbb{D} = \mathbb{R} \oplus \mathbb{R}n, \ n^2 = 0, \ n \neq 0 \). See [Rincon-Camacho and Latre, 2013, Chatelin, 2016, chapter 2] for more.

By allowing the coupling of the real roots, we extend the real logic evolutions defined by the laws \( h\Lambda_l \) and \( G_l \), which were presented earlier. When real and complex roots coexist, the logic evolution becomes trine: it takes values in the 4D-space \( \mathbb{R}^4 \) spanned by \( \{1, i, n, u\} \) where the squares of the non real vectors are respectively \(-1, 0, 1\). The 2D-solutions of (1.1) or (1.2) belong to the pencil of three orthogonal planes \( \mathbb{C} \cup \mathbb{D} \cup 2\mathbb{R} \) for \( l = 4 \). The pencil reduces to two planes for (1.1) when \( l = 2 \).

1.2. Quantitative representation in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). The newly introduced 2D-solutions are denoted \( \hat{x}(h) \) and \( \hat{y}(m) \) and are written generically as \( \alpha + \omega\beta \) where \( \omega \in \{i, n, u\} \). If the triple nature of \( \omega \) is not taken into account, one may represent the maps \( \hat{x} \) and \( \hat{y} \) in \( \mathbb{R}^3 \), as well as the two component maps \( \alpha \) and \( \beta \) in \( \mathbb{R}^2 \).

2. Quadratic evolutions, \( l = 2 \)

2.1. \( x = h\Lambda_2(x), \ h \in \mathbb{R} \).
The equation

\[ x = 4hx(1 - x) \Leftrightarrow x \left( x - \frac{1}{4h} \right) = 0, \ h \neq 0 \]

has the two real roots \( x_1 = 0 \) and \( x_2 = 1 - \frac{1}{4h}, \ h \neq 0 \) which are distinct for \( h \neq \frac{1}{4} \).

The coupling yields the following new roots outside \( \mathbb{R} \):

\[ h \notin \{0, \frac{1}{4}\}, \ \hat{x}(h) = \frac{1}{2}(1 - \frac{1}{4h})(1 \pm u) = (1 - \frac{1}{4h})e_{\pm} \ \text{in} \ 2\mathbb{R}, \ \text{that is} \ \alpha = \pm \beta = \frac{1}{2} - \frac{1}{8h}, \]

\[ h = \frac{1}{4}, \ \hat{x}(\frac{1}{4}) \in n\mathbb{R}^* \ \text{in} \ \mathbb{D}, \ \text{that is} \ \alpha = 0, \ \beta \ \text{arbitrary}. \]

The new solutions for (2.1), \( h \neq 0 \), are zerodivisors, being proportional to an idempotent \( e_+ \) or \( e_- \), or to a nilpotent \( n \). The logic is dual in \( 2\mathbb{R} \cup \mathbb{D} \).

2.2. \( y = G_2(m, y), \ m \in \mathbb{R} \).

The equation

\[ y = 1 - my^2 \Leftrightarrow y^2 + \frac{1}{m}y - \frac{1}{m} = 0, \ m \neq 0 \]

has 2 roots whose nature depends on \( \Delta = 1 + 4m \). There are three cases, leading to a trine logic:

\[ m < -\frac{1}{4}, \ \hat{y}_\pm(m) = \frac{1}{2m}(1 \pm i\sqrt{-\Delta}) \in \mathbb{C}, \]

\[ m = -\frac{1}{4}, \ \hat{y}(\frac{1}{4}) \in 2 + n\mathbb{R} \ \text{in} \ \mathbb{D}, \]

\[ m > -\frac{1}{4}, \ \hat{y}_\pm(m) = \frac{1}{2m}(1 \pm u\sqrt{\Delta}) \in 2\mathbb{R} \setminus \mathbb{R}. \]

For \( m \neq \frac{1}{4} \), \( \alpha = \frac{1}{2m}, \ \beta = \pm \frac{1}{2m}\sqrt{|\Delta|}, \ \text{and for} \ m = \frac{1}{4}, \ \alpha = 2, \ \beta \ \text{arbitrary}. \)

Lemma 2.1. For \( m \neq 0 \), \( \mu(\hat{y}) = -\frac{1}{m} \) and \( \mu(\hat{y} - 1) = 1. \)

Proof. Use \( \mu(\hat{y}) = \hat{y}\hat{y}^* \). For example, in \( 2\mathbb{R} \ \hat{y} - 1 = \frac{1}{2m}(1 - 2m + u\sqrt{\Delta}) \) and \( \hat{y}^* - 1 = \frac{1}{2m}(1 - 2m - u\sqrt{\Delta}) \), then \( \mu(\hat{y} - 1) = \frac{1}{2m^2}|(1 - 2m)^2 - 1 - 4m| = 1. \)

2.3. Quantitative representations.

See Figure 1.
3. Quartic evolutions, $l = 4$

3.1. $x = h\Lambda_4(x), \ h \in \mathbb{R}$.

The roots of equation

\[(3.1) \quad x = h(1 - (2x - 1)^4) \iff x \left( x^3 - 2x^2 + \frac{3}{2}x + \frac{1}{16h} - \frac{1}{2}\right) = 0, \ h \neq 0 \]

have been studied in Section 2.4 of [Rincon-Camacho et al., 2014a], see (2.3) therein. Beside $x = 0$ (simple for $h \neq \frac{1}{8}$ and double for $h = \frac{1}{8}$) there exists another real root $x_1(h) = -2r(h)\sinh\frac{\varphi}{3} + \frac{\sqrt{3}}{3}$ where $\varphi = \varphi(h) \geq 0, \ h \neq 0, \ r(h) = (\text{sgn } q)\frac{1}{3\sqrt{2}}$. The associated pair of complex conjugate roots is provided by (A.12): $\hat{x}_\pm(h) = \frac{2}{3} + r(h)(\sinh\frac{\varphi}{3} \pm i\sqrt{3}\cosh\frac{\varphi}{3})$ with $\varphi \geq 0, \ h \neq 0$.

The coupling of the two real roots $x = 0$ and $x_1(h)$ yields new unreal roots:

- $h \not\in \{0, \frac{1}{8}\}$, $\hat{x}_\pm(h) = \frac{1}{2}(-r(h)\sinh\frac{\varphi}{3} + \frac{1}{3})(1 \pm u) = (-2r(h)\sinh\frac{\varphi}{3} + \frac{1}{3})e_{\pm} \in \mathbb{R} \setminus \mathbb{R}$,
- $h = \frac{1}{8}$, $\hat{x}(\frac{1}{8}) \in \mathbb{R}^*$ in $\mathbb{D}$.

As for the quadratic evolution, the new solutions are zerodivisors.
Lemma 3.1. For \( h \neq 0 \), the complex conjugate pair \( \hat{x}_\pm = \alpha \pm i\beta \) describes the hyperbola

\[
\beta^2 - 3\left(\alpha - \frac{2}{3}\right)^2 = \frac{1}{6}.
\]

Proof. \( \sinh \varphi = \frac{1}{r}(\alpha - \frac{2}{3}) = (\text{sgn} q)\sqrt[3]{2}(\alpha - \frac{2}{3}), \cos \varphi = \frac{1}{r}\sqrt[3]{\beta} = (\text{sgn})\sqrt[6]{\beta} \) for \( q \neq 0 \Leftrightarrow \varphi \neq 0 \); \( \varphi = 0 \) for \( h = h' \). Then \( 6\beta^2 - 18(\alpha - \frac{2}{3})^2 = 1 \) establishes (3.2). The centre of the hyperbola is \( \left(\frac{2}{3}, 0\right) \) and the slopes of the asymptotes are \( \pm \sqrt{3} \). The tips of the branches occur for \( h = h' = 0.675 \) with values \( \left(\frac{2}{3}, \pm \frac{1}{\sqrt{6}}\right) \). \( \square \)

3.2. \( y = G_4(m, y), m \in \mathbb{R}. \)

The real roots of the equation

\[
y = 1 - my^4 \iff y^4 + \frac{1}{m}y - \frac{1}{m} = 0, \quad m \neq 0
\]

have been studied in Section 2.4 of [Rincon-Camacho et al., 2014b]. For \( l = 4 \), \( m_0 = -\frac{3}{4} \approx -0.1055 \) and \( y(m_0) = \frac{4}{3} \). Using \( z \) as the unique positive solution of the auxiliary equation (2.2) \( z^3 + \frac{1}{m}z - \frac{1}{8m^2} = 0 \), the roots of (3.3) satisfy the pair of quadratic equations

\[
(Q_{\pm}) \quad y^2 \pm \sqrt{2z} \ y + z \mp \frac{1}{2m\sqrt{2z}} = 0
\]

with discriminant \( \Delta_{\pm} = -2z \pm \frac{1}{m}\sqrt{\frac{2}{z}} \).

We recall the table

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>( m_0 )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta_- )</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( \Delta_+ )</td>
<td>-</td>
<td>-</td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>nature of roots</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(3)</td>
</tr>
</tbody>
</table>

where

(1) refers to \( 2 \) pairs of complex conjugate (c.c.) numbers,

(2) " \( 1 \) double root \( \frac{4}{3} \) and \( 1 \) c.c. pair,

(3) " \( 2 \) real numbers \( s \neq s' \) and \( 1 \) c.c. pair.

The real roots occur for \( m \geq m_0 \). Their coupling yields either \( \hat{y}(m_0) \in \frac{4}{3} + n\mathbb{R} \) in \( D \) or for \( m > m_0 \) the 4 numbers \( s, s', se_+ \pm se_- = \pm \frac{2}{\sqrt{3}} \pm \frac{2}{\sqrt{3}} \) in \( 2\mathbb{R}. \)

Setting again \( \hat{y}_\pm = (\alpha, \pm \beta) \), it follows that \( \alpha = \mp \sqrt{\frac{2}{3}} \) and \( |\beta| = \frac{1}{2}|\Delta_{\pm}(m)|^{1/2} \). Let \( \tau \) denote any trigonometric function \( \cos, \cosh, \sinh \) then with \( \varphi(m) \) defined by \( \tau(\varphi(m)) = \frac{3\sqrt{3}}{16|m|} z = \frac{2}{\sqrt{|m|}} \tau\left(\frac{\varphi(m)}{3}\right) \). Hence there is no algebraic expression for \( z \) in terms of an arbitrary \( m \). However if \( m = m_0 \), \( D(m_0) = 0, \cos \varphi = \cosh \varphi = 1 \Leftrightarrow \varphi = 0 \), \( z(m_0) = 2r(m_0) = \frac{2}{\sqrt{3}m_0} = \frac{2}{\sqrt{3}} \approx \frac{32}{3}, \sqrt{2z} = \frac{8}{3} \). The evolutions \( m \mapsto (\alpha, \beta) \) will be experimentally studied in Section 4.3.

3.3. Quantitative representations.

See Figure 2
As the parameter $h$ (resp. $m$) varies in $\mathbb{R}$, there are two exceptional values: $0$ where $x$ (resp. $y$) is not defined and $h_0 = \frac{1}{l}$ (resp. $m_0 = -\frac{(l-1)^{l-1}}{l}$) where $x = 0$ (resp. $y = \frac{l}{l-1}$) is a double root. Both values signal a singularity.

These 2 values delineate 3 intervals for the variation of the parameter to which are assigned the three colours red-yellow-blue of decreasing hue as the parameter increases. To the transition value $h_0 > 0$ (resp. $m_0 < 0$) is attributed the colour green (resp. orange).

The next two sections represent the parametric evolutions of the real and 2D-solutions in the real plane of triple nature by means of the corresponding colour chart for the parameter variation.

4.1. $x = h\Lambda_l(x)$, $l = 2$ and 4, $x(h_0) = 0$.

![Colour chart for $h \in \mathbb{R}$](image)

Figure 3 (resp. 4) corresponds to $l = 2$ (resp. 4).
Figure 3. \( l = 2 \): Solutions to \( x = h\Lambda_2(x) \) in 1 and 2 dimensions

Figure 4. \( l = 4 \): Solutions to \( x = h\Lambda_4(x) \) in 1 and 2 dimensions
We observe that the evolution in \((2\mathbb{R} \setminus \mathbb{R}) \cup \mathbb{D}\) is unchanged from \(l = 2\) to 4. The only change is the complex hyperbola for \(l = 4\). The discontinuity in \(h\) from \(+\infty\) to \(-\infty\) (blue - red), which shows for \(\alpha = |\beta| = \frac{1}{2}\), corresponds to \(\hat{x}_\pm = e_\pm\).

4.2. \(y = G_l(m, y), \ l = 2 \text{ and } 4, \ y(m_0) = \frac{l}{l-1}\).

Figure 5 (resp. 6) corresponds to \(l = 2\) (resp. 4).

**Figure 5.** \(l = 2\): Solutions to \(y = G_2(m, y)\) in 1 and 2 dimensions,
\[
m_0 = -\frac{1}{4}, \ y(-\frac{1}{4}) = 2
\]
When \( l = 2 \), the behaviour is such that \( \mu(\hat{y} - 1) = 1 \). With \( \hat{y} = (\alpha, \beta) \) this leads to the hyperbola \((\alpha - 1)^2 - \beta^2 = 1\) for \( m > m_0 \), the dual line \( \alpha = 2 \) for \( m_0 = -\frac{3}{4} \) and the circle \((\alpha - 1)^2 + \beta^2 = 1\) for \( m < m_0 \).

The red partial curve \( m < m_0 \) correspond to \(|\alpha| < \frac{1}{2}\sqrt{2z(m_0)} = \frac{3}{4}, \ |\beta| < |\Delta_+(m_0)|^{1/2} = \frac{3}{4}\sqrt{2} \approx 1.8856 \). We study experimentally in more detail the case \( l = 4 \) in the next section.

4.3. \( y = G_4(m, y) \).

Figure 7 displays the asymptotes \( \beta = \pm(\alpha - 1) \) for the bireal behaviour, corresponding to \( m \to 0 \).
Figure 7. The asymptotes $\beta = \pm (\alpha - 1)$ as $m \to 0$ in $\mathbb{R}^2$.

A zoom near $m = \infty$ in the bireal plane is displayed by Figure 8. The left blue curve points at 0 to a computational artefact discussed in Section 6.4.

Figure 8. $\lim_{m \to \infty} y(m) = (0, 0)$ and $\lim_{m \to m_0} y(m) = (\frac{4}{3}, 0)$ in $\mathbb{R}^2$.

We turn to the complex plane. See Figure 9 for $m \to 0$ and Figure 10 for a zoom around 0 when $|m| \to \infty$. 
Figure 9. The asymptotes 
\[ \beta = \pm (\sqrt{3} \alpha + \frac{1}{\sqrt{3}}) \] as \( m \to 0 \) in \( \mathbb{C} \)

Figure 10. The slopes are \( \pm 1, \infty \) at \((0,0)\) in \( \mathbb{C} \).

Figure 11 provides two zooms on the loop obtained in \( \mathbb{C} \) for \( m < m_0 \). Two computed inscribed circles are displayed in black. In Figure 11 (a) the circle passes through \((\frac{4}{3},0)\), and in Figure 11 (b) the circle is tangent to the loop horizontally.

A careful inspection of Figure 11 suggests a jump from \( m = +\infty \) to \( m = -\infty \) near 0, a phenomenon already encountered on Figure 10 and clearly visible on Figure 8. We propose a partial explanation in Section 6.4 below.

5. When the parameter vanishes

5.1. Over \( \mathbb{R}^2 \). When \( h \) or \( m \) tends to 0, the pair of real solutions undergoes a discontinuity:
- for \( \Delta u, (0, x(h)) \) tends to \((0, +\infty)\) as \( h \to 0^- \) and to \((0, -\infty)\) as \( h \to 0^+ \),
• for $G_l$, $(y_1(h), y_2(h))$ tends to $(1, +\infty)$ as $m \to 0^-$ and to $(1, -\infty)$ as $m \to 0^+$. 

This shows in the bireal plane on the lines $\beta = \pm \alpha$ for $\Lambda$ and on the asymptotes $\beta = \pm (\alpha - 1)$ for $G$ which meet respectively at $(0, 0)$ and $(1, 0)$.

5.2. Over $\mathbb{C}$. The picture gets more contrasted between $l = 2$ and $4$ in the complex plane. On the one hand, there is no complex roots for $\Lambda_2$ and, when they exist for $G_2$, they do not escape to $\infty$ but rather describe a unit circle centered at $(1, 0)$. On the other hand, the complex roots describe a hyperbola for $\Lambda_4$ centered at $(\frac{2}{3}, 0)$ and 2 distinct curves for $G_4$. The yellow-red curve is better considered as consisting of two parts according to the 2 colours.

On the red curve corresponding to $m < m_0$, $|\alpha| \leq \frac{4}{3}$, no complex roots can escape to $\infty$. This red curve is the analogue for $G_4$ of the red circle for $G_2$.

By contrast, on the yellow curve ($m_0 < m < 0$), complex pairs do escape to $\infty$ in modulus as $m \to 0^-$. The same is true on the blue curve ($m > 0$) as $m \to 0^+$. The asymptotes have slopes $\pm \sqrt{3}$, as for $\Lambda_4$, but they meet at $(-\frac{1}{3}, 0) = (\frac{2}{3}, 0) - (1, 0)$.

6. When the parameters escape to $\pm \infty$

6.1. Over $\mathbb{R}$. As $|h| \to \infty$, the pair of real roots $(0, x(h))$ for $\Lambda$ converges to $(0, 1)$ where $0 \neq 1$. By contrast, for $G$, the real roots $y_1(m)$ and $y_2(m)$ converge each to 0 which becomes double ($l = 2$) or quadruple ($l = 4$) for $|m| \to \infty$.

6.2. Over $2\mathbb{R}$. The coupling of 0 and 1 for $\Lambda$ yields the idempotent pair $\frac{1}{2}(1 \pm u) = e_{\pm}$. It shows on Figures 3 (b) and 4 (b) by a jump for $h$ from blue $(h = +\infty)$ to red $(h = -\infty)$ at $\alpha = |\beta| = \frac{1}{2}$ ($\hat{x} = e_{\pm}$). As for $G$, since 0 appears multiple at $\infty$, the coupling yields the dual axis $n\mathbb{R}$ for $m = +\infty$, represented as the light blue dual axis on Figure 12.

6.3. Over $\mathbb{C}$. There is nothing more to add for $\Lambda$, hence we turn to $G_l$. If $m$ is allowed to vary in the extended line $\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$, then the dual axis is obtained for $m = -\infty$ if $l = 2$ and for $m = \pm \infty$ if $l = 4$. It is represented in the later case as a violet (=red+blue) line on Figure 13.

![Figure 12. $m \in \mathbb{R} \cup \{+\infty\}$](image-url)
6.4. About numerical artefacts for $G_4$ around 0 ($|m|$ large enough). We can explain the artefact on Figure 8 as a result of the coupling for $m > 0$ very large of the two real variables $y_1(m) > 0$ and $y_2(m) < 0$. They both tend to 0 as $m \to \infty$ with different speeds: $y_1$ tends to $0^+$ faster than $y_2$ to $0^-$. Therefore, for $m$ large enough, computation sees $y_1 = 0$ and $0 \neq y_2 < 0$.

The coupling of 0 and $y_1$ yields the bireal numbers $y_1(\frac{1 \pm u}{2}) = y_1 e_\pm$ for $y_1 < 0$ small enough.

The explanation of the superposition of a blue line ($m \to \infty$) and red near the origin on the top of the red loop ($m \to -\infty$) requires to consider the coupling of two numbers of a different nature, one real and one complex, both converging to 0 as $m \to \infty$.

Such a coupling ($\mathbb{R}$ and $\mathbb{C}$) as well as $\mathbb{C}$ with $\mathbb{C}$ is beyond the limited scope of this report. We only mention that this can be done in the framework of the ring of bicomplexes $\mathbb{C} = \mathbb{C} \oplus \mathbb{C}u$ spanned by the 4 vectors 1, $i$, $u$, $iu$ where $u$ is a unipotent number ($u^2 = 1$) and $i$, $iu$ are complex numbers (square = −1), all numbers with 4 real dimensions [Chatelin, 2016, Rincon-Camacho and Latre, 2013].

7. Conclusion

The quantitative representations of real roots couplings given by Figures 14 and 15 are displayed below in the three colours red-yellow-blue, where the readability of yellow curves is enhanced by means of a gray shade.
(a) $h \mapsto (\alpha, \beta)$  
green line at ($\frac{1}{4}$, 0)  

(b) $m \mapsto (\alpha, \beta)$  
orange line at ($-\frac{1}{4}$, 2)  

Figure 14. $l = 2$
The philosophically inclined reader is referred to the report [Chatelin and Rincon-Camacho, 2014] which presents some epistemological views on the comparative evolutions of the numbers 0 and 1 as numbers with 1 and 2 dimensions under the absolute and relative laws 1.2 and 1.1 respectively.

**Acknowledgement** M. M. Rincon-Camacho is grateful for the financial support from the Scientific Direction TOTAL through the contract DS-2755-B with CERFACS (Qualitative Computing group).

**References**


