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Qualitative Computing

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DIFFERENTIAL INFORMATION PROCESSING IN THE LIGHT OF QUATERNIONS

FRANÇOISE CHATELIN [(1),(2)]

Abstract. In our computerised society, Information Processing (IP) is increasingly viewed as a booming branch of computer science. It is much too early to know if "Big Data" will live up to the expectations of its proponents. But clearly the overwhelming emphasis on Big Data casts a dark shadow on any non-mechanical IP. Time will tell if machine-based IP can compete with nature-based IP, which gave us electric light, radio, television among many other practical commodities.

The goal of this report is to further our understanding of nature-based IP. By extracting from electromagnetism its bare quaternionic architecture, it reveals the *computational* aspect underlying the physical phenomenon. This more abstract point of view enables us to propose a general 4-fold differential theory of information based on 4 real variables. The theory applies to living organisms. It offers alternative ways to think about pressing societal issues such as natural resources in a physically limited world and natural vs. artificial intelligence.

Keywords: Information Processing, Big Data, Electromagnetism, Maxwell's equations, organic potential, semantic field, syntactic field, source of intelligence, creation, symbolisation, organic sense, quaternions, nabla, living organisms.

The oracle who dwells at Delphi neither
reveals nor conceals, but signifies.
Heraclitus of Ephesus
(ca. 535-475 BCE)

1. A quaternionic frame

1.1. **Definition.** Let be given the variable quaternion $x = x_0 + x_1 i + x_2 \jmath + x_3 k = x_0 + X \in \mathbb{H}$ and the map $p : x \mapsto p = p_0 + P$ from $\mathbb{H}$ to itself representing a smooth *information potential*. We assume that each component p_j, $j = 0$ to 3 has continuous partial derivatives of order two at least with respect to x_i: we set $\partial_i p_j = \frac{\partial}{\partial x_i} p_j(x)$. Using the hamiltonian symbol "nabla": $\nabla = \partial_1 i + \partial_2 \jmath + \partial_3 k$, we define the 4D-gradient in $\mathbb{H}$: $D = \partial_0 + \nabla$. We recall that $\langle \nabla, \nabla \rangle = \Delta = -\nabla \times \nabla = -\nabla^2$.

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1.2. **Connection with classical vector calculus in $\mathbb{R}\oplus\mathbb{R}^3$.** Let $q = q_0+Q \in \mathbb{H}$, where $\vec{Q} = (q_1\ q_2\ q_3)^T \in \mathbb{R}^3$. We use the mathematical notation $\langle\cdot,\cdot\rangle$ (resp. $_\wedge$) for the scalar product in $\mathbb{R}^3$ or $\mathbb{R}^4$ (resp. the vector product in $\mathbb{R}^3$) which goes back to Clifford. These products are widely known as dot and cross products. We spare the $\times$ sign for multiplication in $\mathbb{H}$. Then

$$\begin{aligned} \nabla q_0 &= \vec{\text{grad}}\ q_0, \\ \langle\nabla, Q\rangle &= \text{div}\ \vec{Q}, \\ \nabla_\wedge Q &= \vec{\text{rot}}\ \vec{Q} \\ \nabla\times Q &= -\text{div}\ \vec{Q} + \vec{\text{rot}}\ \vec{Q}. \end{aligned} \tag{1.1}$$

As a direct consequence

$$\begin{aligned} \langle\nabla, \nabla_\wedge Q\rangle &= \text{div}\ \vec{\text{rot}}\ \vec{Q} = 0, \\ \nabla_\wedge\nabla q_0 &= \vec{\text{rot}}\ \vec{\text{grad}}\ q_0 = 0, \\ \langle\nabla, \nabla q_0\rangle &= \Delta q_0, \\ \nabla\langle\nabla, Q\rangle - \nabla_\wedge(\nabla_\wedge Q) &= \Delta Q. \end{aligned} \tag{1.2}$$

Finally, if $r = R \in \mathfrak{I}\mathbb{H}$, $\vec{R} \in \mathbb{R}^3$,

$$\begin{aligned} &\langle Q, \nabla_\wedge R\rangle = \langle\nabla, R_\wedge Q\rangle + \langle R, \nabla_\wedge Q\rangle \\ \Leftrightarrow\ &\langle\vec{Q}, \vec{\text{rot}}\ \vec{R}\rangle - \langle\vec{R}, \vec{\text{rot}}\ \vec{Q}\rangle = \langle\nabla, R_\wedge Q\rangle. \end{aligned} \tag{1.3}$$

This last formula is established by direct check.

2. Two information fields derived from p by the gradient D

We shall use the following notation, where $q,\ q' \in \mathbb{H}$:

$$\begin{aligned} [\![q, q']\!] &= \tfrac{1}{2}[q, q'] = \tfrac{1}{2}(q\times q' - q'\times q) = Q_\wedge Q' \in \mathfrak{I}\mathbb{H} \\ \{q, q'\} &= \tfrac{1}{2}(q\times q' + q'\times q) = q_0q_0' - \langle Q, Q'\rangle + q_0Q + q_0'Q \in \mathbb{H}, \end{aligned}$$

so that $q\times q' = \{q, q'\} + [\![q, q']\!]$ is divided into its symmetric part $\{q, q'\}$ in $\mathbb{H}$ and its antisymmetric one $[\![q, q']\!]$ in $\mathfrak{I}\mathbb{H}$. We recall that if $q = q_0 + Q$, then $\bar{q} = q_0 - Q$.

Definition 2.1. The gradient quaternion D and the information potential p induce the following two fields:
(i) $e = \{\bar{D}, \bar{p}\} \in \mathbb{H}$ is the semantic field,
(ii) $B = [\![\bar{D}, \bar{p}]\!] = \nabla_\wedge P =\in \mathfrak{I}\mathbb{H}$ is the syntactic field.

Lemma 2.1. *The induced semantic field satisfies the identities:*
$\mathfrak{R}e = \partial_0 p_0 - \text{div}\ \vec{P} = \langle\bar{D}, p\rangle$, $\mathfrak{I}e = -\partial_0 P - \nabla p_0 \cong \vec{E}$,
$\text{div}\ \vec{E} = -\text{div}\ \partial_0 P - \Delta p_0$, $\vec{\text{rot}}\ \vec{E} = -\partial_0 B = -\partial_0(\nabla_\wedge P)$.

Proof. Direct calculation: div $= \langle\nabla,\cdot\rangle$ and $\vec{\text{rot}}$ $= (\nabla_\wedge\cdot)$. □

Lemma 2.2. *The syntactic field* $B = \nabla_\wedge P$ *is equivalent to* $\vec{B} = \vec{\text{rot}}\ \vec{P}$, *hence* $\text{div}\ \vec{B} = 0$ *and* $\vec{\text{rot}}\ \vec{B} = \nabla(\text{div}\ \vec{P}) - \Delta\vec{P}$.

Proof. Clear. Use (1.2). □

We shall first assume that the potential p satisfies $\mathfrak{R}(\bar{D} \times \bar{p}) = 0 \Leftrightarrow e_0 = 0$. When the semantic field is pure, the potential p is called *organic*.

3. Maxwell's equations with a 3D-semantic field

The semantic field e becomes a 3D-vector under the assumption

$$\partial_0 p_0 = \text{div}\ \vec{P} \tag{3.1}$$

which means that the rate of change $\partial_0 p_0$ equals the sum $\text{div}\ \vec{P} = \sum_{\jmath=1}^{3} \partial_\jmath p_\jmath$ of the individual partial derivatives $\partial_\jmath p_\jmath$, $\jmath \neq 0$. In other words (3.1) enforces *equality* between the change of p_0 relative to x_0 and the global change $\text{div}\ \vec{P}$. Loosely speaking, the infinitesimal effect on p_0 of the first variable $x_0 = \mathfrak{R}x$ synthetises that on P of the three variables in X. Alternatively (3.1) means that $\bar{p}$ and D are *orthogonal* in $\mathbb{R}^4$, that is $\langle D, \bar{p}\rangle = 0$. It implies that $\text{div}\ \vec{E} = -\partial_0^2 p_0 - \Delta p_0 = -\Delta_4 p_0$.

When semantics and syntax are coupled, the map $p \mapsto \Omega = (E, B)$ defines the *organic informative* function with value in the 6D-space $\mathbb{R}^6$.

Definition 3.1. The coupled evolution of the components E and B of the organic informative function Ω takes place in $\mathbb{H}$ and is called organic.

Proposition 3.1. *Under* (3.1), *the organic evolution satisfies the two equations:*

$$(i) \quad \{D, B\} + [\![D, E]\!] = 0 \tag{3.2}$$

$$(ii) \quad [\![D, B]\!] - \{D, E\} = s \tag{3.3}$$

where $s \in \mathbb{H}$ *is a source of organic intelligence.*

Proof. With $D = (\partial_0, \nabla)$, $E = -\partial_0 P - \nabla p_0$, $B = \nabla_\wedge P$, we get
$\{D, B\} = -\langle \nabla, B\rangle + \partial_0 B$, $[\![D, B]\!] = \nabla_\wedge B$,
$\{D, E\} = -\langle \nabla, E\rangle + \partial_0 E$, $[\![D, E]\!] = \nabla_\wedge E = [\![D, e]\!]$.
(i) The organic evolution of the informative function Ω in $\cong \mathbb{R}^6$ entails the identity (i) $\Leftrightarrow$ (3.2):

$$\{D, B\} + [\![D, E]\!] = -\langle \nabla, B\rangle + \partial_0 B + \nabla_\wedge E =$$

$$\partial_0 B + \vec{\text{rot}}\ \vec{E} = \partial_0(\nabla_\wedge P) + \nabla_\wedge(-\partial_0 P) = 0$$

Since the lefthand side is independent of e_0 (3.2) is valid for an arbitrary potential. This follows from $\text{div}\ B = 0$ (Lemma 2.2) and $\vec{\text{rot}}\ E = -\partial_0 B$ (Lemma 2.1).

(ii) Let $\mathcal{C}$ be the differential creation map defined by (3.3) : $p \mapsto \mathcal{C}(p) = s$: it measures the difference between the antisymmetric part of $D \times B$ and the symmetric part of $D \times E$. The two fields E and B in $\mathbb{R}^3$ are coupled through the relation $s = \text{div } E - \partial_0 E + \vec{\text{rot}}\ B$, where

$$s_0 \ = \text{div } E, \ \ S = -\partial_0 E + \vec{\text{rot}}\ B.$$

□

We find that (i) is a tautology whereas in (ii) the map $\mathcal{C}$ produces the quaternion *intelligence* s when p is organic.

We recall that $D\bar{D} = \langle D, D\rangle = \|D\|^2 = \sum_{i=0}^{3} \partial_i^2 = \Delta_4$ is the 4D-Laplacian. Now $D \times D = (\partial_0, \nabla) \times (\partial_0, \nabla) = (\partial_0^2 - \Delta, 2\nabla\partial_0 + \nabla_\wedge \nabla)$. The real component of D^2 is the d'Alembertian $\Box = \partial_0^2 - \Delta$. We also denote $\langle D, \cdot\rangle = \sum_{i=0}^{4} \partial_i = \text{Div}_4$.

Theorem 3.2. *The intelligence s satisfies*

$$s_0 = -\Delta_4 p_0, \quad S = \Box P + 2\nabla\partial_0 p_0.$$

Proof. $s_0 = \langle \nabla, E\rangle = -\langle \nabla, \partial_0 P + \nabla p_0\rangle = -(\partial_0^2 p_0 + \Delta p_0) - \Delta_4 p_0$ by (3.1).

$S = -\partial_0 E + \vec{\text{rot}}\ (\nabla_\wedge P) = \partial_0(\partial_0 P + \nabla p_0) + \nabla_\wedge(\nabla_\wedge P)$ with $\nabla_\wedge(\nabla_\wedge P) = \nabla\langle \nabla, P\rangle - \Delta P$ by (1.2). Hence $S = \partial_0^2 P - \Delta P + 2\nabla\partial_0 p_0$. □

If the 4 variables x are chosen such that $x_0 = t$ represents *time* and $x_\jmath$, $\jmath \neq 0$ represent the 3 *space* variables, then the equations (3.2) and (3.3) reveal the two forms of the celebrated Maxwell's equations for electromagnetism. The 3D-semantic (resp. syntactic) field becomes the *electric* field $\vec{E}$ (resp. *magnetic* field $\vec{B}$) and (3.2) is Faraday's induction law $\partial_t \vec{B} + \vec{\text{rot}}\ \vec{E} = 0$. In (3.3), one recognises Gauss' equation in $\mathfrak{R}s = \text{div } E = 4\pi\rho$ which defines the electric *charge* density ρ in $\mathbb{R}$. In $\mathfrak{I}s = \vec{\text{rot}}\ B - \partial_t E = 4\pi\vec{J}$, we see Ampère's law with Maxwell's correction to define $\vec{J}$, the electric current density in $\mathbb{R}^3$.

Remark 3.1. In the general (purely mathematical) approach of Proposition 3.1, the output $s = \mathcal{C}(p)$ in (3.3) is the image of p by the creation map $\mathcal{C}$ derived from p. In classical electromagnetism, the inhomogeneous form (3.3) is interpreted as a definition of ρ and J resulting from an existing electric source.

Remark 3.2. The route we have followed above starts from the organic potential to derive the fields E and B. When physicists present Maxwell's equations in $\mathfrak{I}\mathbb{H}$, they usually follow the reverse route: from E and B in $\mathbb{H}$ they infer $p = \varphi + A$ (physics notation) which −from a mathematical point of view− should verify $\partial_t\varphi - \text{div } A = 0$ (Lorenz gauge condition on $\bar{p} = \varphi - A \Leftrightarrow \text{Div}_4\bar{p} = 0$).

4. Conservative aspects of an organic evolution in $\mathbb{H}$

It is clear that the conservative aspect of organic evolution should take into account the assumption (3.1) which guarantees that the semantic field remains *tri-dimensional*. The condition actually means that $\mathrm{Div}_4\bar{p} = 0$: the conjugate potential $\bar{p}$ is divergence-free.

Lemma 4.1. *The fields E and B are invariant under the transformation: $p \mapsto p'$, defined by*

$$p'_0 = p_0 - \partial_0 f, \quad P' = P + \nabla f,$$

where f is a real harmonic function of x in $\mathbb{R}^4$.

Proof. We check that the transformation $p - p' = \bar{D}f$ leaves E and B invariant. $E' = E - \partial_0\nabla f + \nabla\partial_0 f = E$, $B' = B + \nabla_\wedge\nabla f = B$. The assumption (3.1) imposes that $\langle\bar{D}, \bar{D}f\rangle = 0 \Leftrightarrow \partial_0^2 f + \Delta f = \Delta_4 f = 0$, where $\Delta_4 = \sum_{i=0}^3 \partial_i^2$ is the 4D-Laplacian. □

The transformation $p \mapsto p'$ is the gauge symmetry of classical electromagnetism where f is smooth and arbitrary. We observe that Maxwell's theory of electromagnetism −as it is currently presented in physics− does *not* require the assumption $\partial_t p_0 = \mathrm{div}\ P$: the real and vector parts of the electromagnetic potential are assumed to be independent at (t, X). It follows that, in what is known as *gauge* theory, the function f should be smooth enough, but otherwise arbitrary. Physicists have devised a selection of gauge conditions connecting p_0 and P (Lorenz, Coulomb, ...). It is remarkable that, in a quaternionic frame, f should necessarily be *harmonic* if B and a *tri-dimensional* E are to be invariant.

Two distinct conservation laws for organic evolution are derived from the non homogeneous form (3.3) $\Leftrightarrow s = \mathcal{C}(p)$.

Lemma 4.2. $\mathfrak{R}(\bar{D} \times \mathcal{C}(p)) = \mathfrak{R}(\bar{D} \times s) \Leftrightarrow \mathrm{Div}_4 s = \partial_0 s_0 + \mathrm{div}\ S = 0$ *where* $\mathrm{Div}_4 = \sum_{i=0}^3 \partial_i$ *is the 4D-divergence.*

Proof. $\mathfrak{R}(\bar{D} \times \mathcal{C}(p)) = \partial_0\mathrm{div}\ E - \mathrm{div}\ \partial_0 E + \langle\nabla, \nabla_\wedge B\rangle = 0$. Hence $\mathfrak{R}(\bar{D} \times s) = \partial_0 s_0 + \langle\nabla, S\rangle = \partial_0 s_0 + \mathrm{div}\ S = 0$. □

Lemma 4.2 expresses that *intelligence* (in the form of s) is divergence-free in $\mathbb{R}^4$: it is conserved under organic evolution. The physical analogue in electromagnetism is *electric charge* conservation.

Lemma 4.3. $\mathfrak{R}(\bar{E}\times\mathcal{C}(p)) = \mathfrak{R}(\bar{E}\times s) \Leftrightarrow \langle E, S\rangle + \langle\nabla, E_\wedge B\rangle = -\frac{1}{2}\partial_0(\|E\|^2 + \|B\|^2)$.

Proof. $\mathfrak{R}(\bar{E} \times s) = \langle E, S\rangle$ and $\mathfrak{R}(\bar{E} \times \mathcal{C}(p)) = \langle E, \nabla_\wedge B\rangle - \langle E, \partial_0 E\rangle$.

Use the auxiliary formulae:

- $\langle E, \nabla_\wedge B\rangle = \langle\nabla, B_\wedge E\rangle + \langle B, \nabla_\wedge E\rangle$ by (1.3),
- $\nabla_\wedge E = \vec{\mathrm{rot}}\ \vec{E} = -\partial_0 B$ by Lemma 2.1,
- $\langle E, \partial_0 E\rangle = \sum_{j=1}^3 e_j\partial_0 e_j = \frac{1}{2}\partial_0 \sum_{j=1}^3 e_j^2 = \frac{1}{2}\partial_0\|E\|^2$.

Therefore $\mathfrak{R}(\bar{E} \times \mathcal{C}(p)) = \langle \nabla, B_\wedge E\rangle + \langle B, \vec{\text{rot}}\ \vec{E}\rangle - \langle E, \partial_0 E\rangle = \langle \nabla, B_\wedge E\rangle - \frac{1}{2}\partial_0(\|B\|^2 + \|E\|^2) = \langle E, S\rangle$. Alternatively $\frac{1}{2}\partial_0(\|E\|^2 + \|B\|^2) = \langle \nabla, B_\wedge E\rangle - \langle E, S\rangle$. □

The quantity $\|E\|^2 + \|B\|^2$ is the square euclidean norm $\|\Omega\|^2$ of $\Omega = (E, B)$, the semantico-syntactic, or organic informative function in $\mathbb{R}^6$. Its evolution is ruled by Lemma 4.3. We shall elaborate on this in [Chatelin, 2016, Chapter 6].

In electromagnetism, the vector $N = E_\wedge B$ is Poynting's vector, and the quantity $\langle E, S\rangle + \langle \nabla, N\rangle = -\frac{1}{2}\partial_t(\|E\|^2 + \|B\|^2)$ is interpreted as $-\partial_t u$, where u is the energy density. The result is known as Poynting's theorem. The alternative geometric interpretation of the energy density u as $\frac{1}{2}\|\Omega\|^2$, half the square euclidean norm of the electromagnetic field Ω is a direct by-product of the quaternionic frame of computation (assuming that $\varepsilon_0 = \mu_0 = 1$).

5. Electromagnetism in the vacuum

In most physics textbooks, Maxwell's equations for electromagnetism (EM) are presented in the mathematical language of *vector calculus*. Even though the frame of evolution is $\mathbb{R}^4$, its linear decomposition into $\mathbb{R} \oplus \mathbb{R}^3$ is strictly maintained with $\mathbb{R}^3$ as a vector product algebra.

But there are inconspicuous consequences for this choice: for example by positing a priori that the electric field is 3D, the experimental perspective misses the key assumption (3.1) which *connects* p_0 and P in the potential $p = p_0 + P$ concealing the fact that the function f in the gauge theory for electromagnetism is not arbitrary but *harmonic* when $\mathbb{H}$ replaces $\mathbb{R} \oplus \mathbb{R}^3$.

This lends weight to the disillusioned remark by Maxwell: "The limited use which has up to the present time been made of Quaternions must be attributed to the repugnance of most mature minds to new methods involving the expenditure of thought" [Maxwell, 1870]. It seems that Maxwell's statement remains valid even today despite the more than centenary revolution of Special Relativity in Physics...

Maxwell went on to assume $s = 0$ in (3.3) to describe the electromagnetic behaviour in the "vacuum" (no electric charge or current). Therefore the equations become

$$\text{div}\ \vec{E} = \text{div}\ \vec{B} = 0, \quad \vec{\text{rot}}\ \vec{E} = -\partial_t \vec{B} \ \text{ and } \ \vec{\text{rot}}\ \vec{B} = \partial_t \vec{E}, \tag{5.1}$$

or more compactly: $\nabla \times B = \partial_t E$ and $\nabla \times E = -\partial_t B$ in $\mathbb{H}$.

An easy consequence of Theorem 3.2 is given by

Proposition 5.1. *The electromagnetic potential $p = p_0 + P$ leading to* (5.1) *satisfies the three conditions:*

(i) $\partial_t p_0 = \text{div}\ P \Leftrightarrow e_0 = 0$,

(ii) $\Delta_4 p_0 = 0 \Leftrightarrow \text{div}\ \vec{E} = 0$,

(iii) $\partial_t \nabla p_0 = -\frac{1}{2}\Box P$ *with* $\Box = \partial_t^2 - \Delta \Leftrightarrow \vec{\text{rot}}\ \vec{B} = \partial_t \vec{E}$.

Proof. (3.1) $\Leftrightarrow$ (i) $\Rightarrow \partial_t^2 p_0 = \partial_t \mathrm{div}\ P$.

(ii) div $E = \langle \nabla, E\rangle = -\partial_t \mathrm{div}\ P - \Delta p_0 = -\partial_t^2 p_0 - \Delta p_0$ by (i), hence $\Delta_4 p_0 = 0$.

(iii) $\vec{\mathrm{rot}}\ \vec{B} = \partial_t E \Leftrightarrow \nabla_\wedge(\nabla_\wedge P) + \partial_t^2 P + \partial_t \nabla p_0 = 0$ and $\nabla_\wedge(\nabla_\wedge P) = \nabla\langle\nabla, P\rangle - \Delta P = \partial_t \nabla p_0 - \Delta P$ by (1.2) and (i). Therefore $2\partial_t \nabla p_0 + \partial_t^2 P - \Delta P = \Box P + 2\partial_t \nabla p_0 = 0$, where $\Box$ is the d'Alembertian.

$\Box$

The EM potential is manifested by the electromagnetic waves $\Omega = (E, B)$. Depending on the frequency, some of them can be seen as light; but the majority remains invisible to us. The mathematical elegance of the above quaternionic derivation is a belated vindication of the visionary insights of Hamilton and Maxwell. This is also a tribute to the extraordinary skills of the experimental physicists who succeeded to infer these laws from Nature only, without any help from the quaternions!

6. Three particular organic potentials

Corollary 6.1. *An organic potential $p = p_0 + P$ produces the imaginary organic intelligence S iff*

$$\Delta_4 p_0 = 0. \tag{6.1}$$

It produces the real organic intelligence s_0 iff

$$\Box P = -2\partial_0 \nabla p_0. \tag{6.2}$$

Proof. Follows from Proposition 5.1 written with the general variable $x = x_0 + X \in \mathbb{H}$. The potential p is organic iff (3.1) $\Leftrightarrow$ (i). The source of organic intelligence $s = s_0 + S$ is such that $s_0 = 0 \Leftrightarrow s = S \Leftrightarrow$ (6.1) $\Leftrightarrow$ (ii) $\cup$ (i), and such that $S = 0 \Leftrightarrow s = s_0 \Leftrightarrow$ (6.2) $\Leftrightarrow$ (iii) $\cup$ (i). $\Box$

Definition 6.1. An organic potential which produces a real (resp. imaginary) organic intelligence is called actual (resp. surd). It is a potential of knowledge when the output is zero.

Corollary 6.2. *An organic potential p such that the two conditions* (6.1) *and* (6.2) *hold is a potential of knowledge.*

Proof. Clear. $\Box$

Following Remark 3.1 we consider in mathematics that $s = \mathcal{C}(p) = 0$ is a property inherent to any $p = p_0 + P$ characterised in Corollary 6.2. Given any harmonic scalar component p_0, $\Delta_4 p_0 = 0$, the vector part $\vec{P}$ should satisfy

$$\Box \vec{P} = -2\partial_0 \nabla p_0 = -2\nabla \mathrm{div}\ \vec{P},\ \mathrm{div}\ \vec{P} = \partial_0 p_0$$

The capacity to carry "knowledge" attributed to such a potential should be understood as a mathematical interpretation of the physical power of the electromagnetic potential p characterised in Proposition 5.1. This particular potential (which

produces electromagnetic waves) led to the revolution in physics brought by the understanding that the electromagnetic waves are carriers of specific information naturally contained in time and space.

This suggests other possibilities in the continuum time-space represented by $\mathbb{R}^4$. Let be given $x = (x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$, there exists *four* ways to single out the *one* real axis necessary to specify a quaternion field. In other words, the real axis in $\mathbb{H}$ can be chosen at will from a choice of 4 orthogonal directions in $\mathbb{R}^4$. An indication of the actual anisotropy of space is that its manifestation on earth appears with *colour* to our eyes. We shall say more in Section 9.

Lemma 4.1 has told us that the gauge function f should be harmonic when p is organic. What are the further consequences on f of a potential of knowledge?

Lemma 6.3. *When p carries knowledge, not only f but also the 4D-gradient $Df = (\partial_0 f, \nabla f)$ are harmonic functions with values in $\mathbb{R}$ and $\mathbb{R}^4$ respectively.*

Proof. (3.1) $\Leftrightarrow \Delta_4 f = 0$, (6.1) $\Leftrightarrow \Delta_4(\partial_0 f) = 0$, (6.2) $\Leftrightarrow 2\partial_0\nabla(-\partial_0 f) = -(\partial_0^2 - \Delta)(\nabla f)$, then $-\partial_0^2\nabla f = \Delta(\nabla f) \Leftrightarrow \Delta_4(\nabla f) = 0$. □

7. When the semantic field is four-dimensional

So far we have assumed that the potential p is organic leading to the source of intelligence $\mathcal{C}(p) = s$ by (3.3). When p is arbitrary in $\mathbb{H}$, we set $\hat{\mathcal{C}} : p \mapsto [\![D, B]\!] - \{D, e\} = \hat{s}$.

When we relax the assumption (3.1) there exists a non organic signification $\mathfrak{R}\{\bar{D}, \bar{p}\} = \mathfrak{R}(\bar{D} \times \bar{p}) \neq 0$. Equivalently $e = e_0 + E$ with $e_0 = \langle \bar{D}, p\rangle = \langle D, \bar{p}\rangle \neq 0$: the conjugate potential $\bar{p}$ has a nonzero divergence. Then $[\![D, e]\!] = [\![D, E]\!]$ is unchanged in (3.2) whereas in general $\{D, e\} = \{D, E\} + \partial_0 e_0 + \nabla e_0$. This modifies $s = \mathcal{C}(p)$ into $\hat{s} = s - (\partial_0 e_0 + \nabla e_0)$ in (3.3).

Lemma 7.1. *When $e_0 \neq 0$, the source of intelligence $s = s_0 + S$ becomes $\hat{s} = \hat{s}_0 + \hat{S}$ where $\sigma = \hat{s} - s = -De_0$ is non organic.*

Proof. Clear: $De_0 = (\partial_0 e_0, \nabla e_0)$ and D is the 4-gradient in $\mathbb{H}$. □

When $e_0 \neq 0$, the evolution creates a *new* (non organic) form of intelligence $\sigma \in \mathbb{H}$ which derives from the scalar potential e_0, the real component of the 4D-semantic field $e = e_0 + E$.

The vector $\Theta = e_0 + \Omega = (e_0, E, 0, B)$ defines the *significant informative* function with value in the 7D-space $\mathbb{R}^7$.

Definition 7.1. The coupled evolution of the components e and B of the significant informative function Θ takes place in $\mathbb{H}$ and is called significant evolution. It creates a non organic form of intelligence σ called symbolic.

We set $\sigma = \sigma_0 + \Sigma$ in $\mathbb{H}$. When the variable $x = (t, x_1, x_2, x_3)$ represents time and space, the *symbolic* intelligence σ is called *signal*.

Lemma 7.2. *$\bar{D} \times \sigma = -\Delta_4 e_0$ is real.*

Proof. $\bar{D}\times\sigma = -\bar{D}(De_0) = -(\partial_0 - \nabla)(\partial_0 + \nabla)e_0 = -(\partial_0^2 - \nabla\times\nabla)e_0 = -\Delta_4 e_0$. □

Proposition 7.3. *The symbolic intelligence σ satisfies the equation*

$$\mathrm{Div}_4\sigma = -\Delta_4 e_0. \tag{7.1}$$

Proof. $\mathrm{Div}_4\sigma = \langle D, \sigma\rangle = \langle 1, -\bar{D}De_0\rangle = -\Delta_4 e_0$, to be compared with Lemma 4.2. □

Since Lemma 7.2 guarantees that $\bar{D}\times\sigma$ is real, this means that $\mathfrak{R}(\bar{D}\times\sigma) = \mathrm{Div}_4\sigma$ and

$$\mathfrak{I}(\bar{D} \times \sigma) = \partial_0\Sigma - \nabla\sigma_0 - \nabla_\wedge\Sigma = 0. \tag{7.2}$$

Corollary 7.4. *The symbolic intelligence $\sigma = \sigma_0 + \Sigma$ satisfies the relations*

$$\langle \nabla\sigma_0, \Sigma\rangle = \frac{1}{2}\partial_0\|\Sigma\|^2,\ \mathrm{div}\ \partial_0\Sigma = \Delta\sigma_0,\ \vec{\mathrm{rot}}\ (\partial_0\Sigma - \vec{\mathrm{rot}}\ \Sigma) = 0$$

Proof. (7.2) $\Leftrightarrow$ $\nabla\sigma_0 = \partial_0\Sigma - \nabla_\wedge\Sigma$. Hence $\langle\nabla\sigma_0, \Sigma\rangle = \langle\partial_0\Sigma, \Sigma\rangle$, $\langle\nabla, \partial_0\Sigma\rangle = \mathrm{div}\ \partial_0\Sigma = \Delta\sigma_0$, $\nabla_\wedge\nabla\sigma_0 = \nabla_\wedge\partial_0\Sigma - \nabla_\wedge(\nabla_\wedge\Sigma) = 0$. □

Remark 7.1. There is conservation of *total* intelligence $s+\sigma$ in $\mathbb{H}$ iff $\mathrm{Div}_4(s+\sigma) = \mathrm{Div}_4\sigma = 0$, that is iff e_0 is harmonic $\Leftrightarrow$ $\Delta_4 e_0 = 0$.

Remark 7.2. The necessary and sufficient condition for $\|\Sigma\|^2$ to be independent of x_0 is that $\langle\nabla\sigma_0, \Sigma\rangle = 0 \Leftrightarrow \Sigma$ and $\vec{\mathrm{grad}}\ \sigma_0$ are orthogonal in $\mathbb{R}^3$. As a particular case we get $\nabla\sigma_0 = 0$, that is σ_0 does not depend on $x_\jmath$, $\jmath \neq 0$, but on x_0 only.

8. An example: the significant potential is the source of organic intelligence

In this example, we assume that p is smooth enough with continuous partial derivatives of order 3 at least. And we choose for second potential the source of intelligence s derived from the original organic potential p satisfying (3.1). We recall that $s = s_0 + S$ with $s_0 = \mathrm{div}\ E$ and $S = \vec{\mathrm{rot}}\ B - \partial_0 E$.

Lemma 8.1. *The source of intelligence s is an organic potential iff the potential p satisfies the conditions*

$$\mathrm{div}\ P = \partial_0 p_0, \tag{3.1}$$

$$\Delta_4\partial_0 p_0 = 0. \tag{8.1}$$

Proof. Apply (3.1) to s: $\partial_0 s_0 = \mathrm{div}\ S$ should hold together with $\partial_0 p_0 = \mathrm{div}\ P$.

$\partial_0 s_0 = \partial_0\mathrm{div}\ E = -(\partial_0^2\mathrm{div}\ P + \Delta\partial_0 p_0) = -(\partial_0^2 + \Delta)\partial_0 p_0 = -\Delta_4\partial_0 p_0$. And $\mathrm{div}\ S = \langle\nabla, \nabla_\wedge B - \partial_0 E\rangle = \langle\nabla, \partial_0^2 P + \nabla\partial_0 p_0\rangle = \Delta_4\partial_0 p_0$. Hence $\partial_0 s_0 = \mathrm{div}\ S \Leftrightarrow$ (8.1) $\cup$ (3.1). □

Under (3.1) and (8.1), the quaternions p and $s = \mathcal{C}(p)$ are both organic potentials. The intelligence potential s produces $u = \mathcal{C}(s) = \mathcal{C}^2(p)$ that we call *organic sense*. Three cases are remarkable when the organic sense is real, imaginary or zero:

(a) s is an *actual* potential if $u = u_0 \in \mathbb{R} \Leftrightarrow U = 0 \Leftrightarrow \Box S = -2\nabla s_0 \Leftrightarrow$ (6.2)
(b) s is a *surd* potential if $u = U \in \mathfrak{I}\mathbb{H} \Leftrightarrow u_0 = 0 \Leftrightarrow \Delta_4 s_0 = 0 \Leftrightarrow$ (6.1).
(c) s is a *knowledge* potential if $u = 0$.

When $\partial_0 p_0$ is *not* harmonic in $\mathbb{R}^4$, the potential s is significant. Therefore it produces $\hat{u} = \hat{\mathcal{C}}(\mathcal{C}(p))$. The map $(\hat{\mathcal{C}} - \mathcal{C}) \circ \mathcal{C} = \mathcal{S}$ is called *symbolisation*: its output $\hat{u} - u = \mathcal{S}(p)$ is the non organic, or symbolic sense derived from p.

9. The 4-fold quaternionic nature of the time-space continuum $\mathbb{R}^4$

We mentioned earlier that when $\mathbb{R}^4$ represents the *time-space continuum*, that is 1 variable for time t and 3 variables for space, then there exist four inequivalent ways to endow $\mathbb{R}^4$ with a quaternionic structure. They are listed in the table below according to the indices $i = 0$ to 3 for the variables x_i and canonical basis vectors $\mathbf{e}_i$:

$\mathbb{H}$	$\mathbb{R}$	$\mathfrak{I}\mathbb{H}$		
$\mathbb{H}_0$	0	1	2	3
$\mathbb{H}_1$	1	2	3	0
$\mathbb{H}_2$	2	3	0	1
$\mathbb{H}_3$	3	0	1	2

The field $\mathbb{H}_0$ corresponds to the case treated above by physics: the scalar component x_0 is temporal and the vector (or imaginary) part X belongs to the 3D-space $\mathbb{R}^3 \cong \mathfrak{I}\mathbb{H}_0$. In this choice, no distinction is made between the 3 space directions. It follows that there is no preferred direction in $\mathbb{R}^3$ for physical light, a fact which is required by objectivity in physics: we all inhabit a common outer world.

The remaining three fields $\mathbb{H}_j$, $j = 1$ to 3, use the *space* variable x_j and canonical basis vector $\mathbf{e}_j$ to define the real axis $\mathbb{R}\mathbf{e}_j$. This introduces a spatial discrimination which violates the isotropy of space inherent to physics. Of course, since $\mathbb{H}_j = \mathbb{R}\mathbf{e}_j \oplus \mathfrak{I}\mathbb{H}_j$, the imaginary space $\mathfrak{I}\mathbb{H}_j \cong \mathbb{R}^3$ which *mixes* time and space variables x_i, $i \neq j$, is itself isotropic with respect to $\mathbf{e}_i$, $i \neq j$ for calculation.

In addition to its *objective* (or physical) structure $\mathbb{H}_0$, $\mathbb{R}^4$ is endowed with a triple *subjective* nature by $\mathbb{H}_j$, $j = 1$ to 3, manifested through the existence of *a living organism* at location (x_1, x_2, x_3) in space. The 3 self-referential directions of a sentient body reveal the *anisotropy of space* which is concealed when life is excluded by a purely material point of view. But four dimensions allow us to perceive only the tip of the iceberg in differential information theory. The use of 8

and 16 dimensions will reveal much more, see [Chatelin, 2016, Chapter 6, Sections 6.5.2 and 6.7.3].

10. A reassessment of $\mathbb{H}$

Despite the valiant efforts of Maxwell, the quaternions were never fully accepted by physicists. They were superseded by the easier-to-grasp vector calculus developed by Gibbs and Heaviside which seemed technically sufficient at the time. The modern consensus about Hamilton's quaternions shows in the following opinion expressed by a historian of science: "There is certainly something tragic in the thought of the brilliant Hamilton devoting the last twenty-two years of his life to quaternions which are *now* of little interest". [Crowe, 1967, p. 42, italics in original]. Today's mathematicians look upon real quaternions as a special case of complex matrices of order 2. Indeed, $\mathbb{H} \cong \{A = \begin{pmatrix} w & -z \\ \bar{z} & \bar{w} \end{pmatrix}, w, z \in \mathbb{C}\}$ with tr $A = 2\Re w$ and det $A = |w|^2 + |z|^2$. And complex quaternions [Hamilton, 1850] are viewed as an instantiation of $\mathbb{C}^{2\times 2}$.

However quaternions made an inconspicuous come back at the end of last century in engineering domains which make use of spatial 3D-rotations (orbital mechanics, computer graphics). Quaternions are computationally *more efficient* than their matrix counterparts in $\mathbb{C}^{2\times 2}$. This better spatial efficiency is but an example of the reward for the expenditure of thought (from $\mathbb{R}\oplus\mathbb{R}^3$ to $\mathbb{H}$) indicated by Maxwell in the quote from [Maxwell, 1870] in Section 5. Quaternions are better at multiplication than orthogonal matrices. But the revival will gain its full momentum in all domains concerned with *differential* processing of information stemming from 4 real variables. And the candidate domains are many, which include primarily biology and all sciences of living organisms as well as neuro- and cognitive sciences. [Chatelin and Rincon-Camacho, 2015].

Two main reasons sustain this reassessment of the role of quaternions for life sciences. First, there is no epistemological reasons to restrict x_0 to represent time, and X to represent 3D-space. Second, the quaternionic gradient $D = \partial_0 + \nabla$ is the key ingredient to write the differential treatment of the information field p in $\mathbb{H}$. In the light of a quaternionic differential calculus, EM appears as a physical manifestation of a *more general* theory of differential information in 4D inherent to sentient bodies in space-time.

It may not be too far-fetched to find a (subconscious) reason for Hamilton's fascination by quaternion differential calculus in its generic information-theoretic power.

Very much ahead of his time, he sensed that this calculus belonged to the "mathematics of the future" in his own words. And time has revealed how right he was: this calculus is a language tuned to express the elusive ability naturally exhibited

by living organisms which consists in naturally "intelligent" information processing.

11. Future directions

The pivotal role of the field structure $\mathbb{H}$ for $\mathbb{R}^4$ suggests that the informative functions Ω and Θ in $\mathbb{R}^6$ and $\mathbb{R}^7$ respectively may benefit from being thought of as vectors in $\mathbb{R}^8$. There are several plausible multiplicative 8D-structures, such as the complex quaternions $\mathbb{H}(\mathbb{C})$ (Hamilton), the dual ones $\mathbb{H}(\mathbb{D})$ (Clifford), the octonions $\mathbb{G}$ (Graves) and the split-octonions $\not{\mathbb{G}}$. These ideas are developed in a forthcoming report [Chatelin, 2015].

As a final remark we mention that $\mathbb{H}(\mathbb{C}) \cong \mathbb{C}^{2\times 2}$ plays a role in Quantum Mechanics (Pauli matrices) and that $\mathbb{H}(\mathbb{D})$ is an important structure in the study of rigid body kinematics, and its applications to robotics and computer graphics [Chatelin et al., 2014].

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