Using Overlapping and Filtering Techniques for Parallel Preconditioners

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Introduction

- Schwarz type preconditioner ≈ Block Jacobi + overlap
- \(\Rightarrow\) overlap helps convergence

Objectif

use the overlapping technique from DDM in incomplete LU type preconditioners
Plan

Overlapping techniques

Parallel preconditioners

Numerical results

Conclusion
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Overlapping in Domain Decomposition Method

K-way partition

\[ \Omega_1 \quad \Omega_2 \quad \Omega_3 \quad \Omega_4 \]

iLU type preconditioners are based on nested dissection

- Block incomplete LDU preconditioner
- Nested preconditioners
Overlapping in Domain Decomposition Method

K-way partition

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iLU type preconditioners are based on nested dissection

- Block incomplete LDU preconditioner
- Nested preconditioners
Basic Overlapping Strategy

- Given a domain
- Choose a separator
- Split into 3 parts
- Find connections
- Extend each part
- Rearrange the matrix
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Multi-level Overlapping Strategy

- A Parallel Multi-Level Overlapping Strategy !!
- How much we have to pay? How many vertices we have to add?
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- How much we have to pay? How many vertices we have to add?
On a $n \times n$ 2D grid, for $p$ parts, $V_{add} \approx 8(\sqrt{p} - 1)n$

Example: $p = 4 \times 4 = 16$, $V_{add} \approx 24n$

For each part, $v_{add} \approx \frac{4(\sqrt{p} - 1)p}{n^2}$, it's affordable!
Analysis of Overlapping on a Regular Grid

- On a $n \times n$ 2D grid, for $p$ parts, $V_{\text{add}} \approx 8(\sqrt{p} - 1)n$
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On a $n \times n \times n$ 3D grid, for $p$ parts, $V_{\text{add}} \approx 12(\sqrt[3]{p} - 1)n^2$

example: $p = 8$, $V_{\text{add}} \approx 12n^2$

for each part, $v_{\text{add}} \approx \frac{6(\sqrt[3]{p} - 1)p}{n^2}$, it’s affordable!
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Block iLDU Preconditioner based on Nested Dissection

\[ A = \begin{pmatrix}
A_{11} & A_{13} & A_{17} \\
A_{22} & A_{23} & A_{27} \\
A_{31} & A_{33} & A_{37} \\
A_{44} & A_{46} & A_{47} \\
A_{55} & A_{56} & A_{57} \\
A_{64} & A_{65} & A_{66} & A_{67} \\
A_{71} & A_{72} & A_{73} & A_{74} & A_{77}
\end{pmatrix} = (L + D)D^{-1}(D + U) \]

- Approximation on off-diagonal blocks.
Nested Exact Factorization [Grigori et al., 2010]

\[
A = \begin{pmatrix}
T_1^1 & T_1^2 & U_1^2 \\
L_1^1 & L_1^2 & S_1^1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
L_1^1 & L_1^2 & 0
\end{pmatrix} + \begin{pmatrix}
T_1^1 & T_1^2 \\
L_1^1 & D & S_1^1
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & U_1^2
\end{pmatrix}
\]

\[
= (L_1 + D)D^{-1}(D + U_1) - L_1D^{-1}U_1
= (L_1 + F_1)F_1^{-1}(F_1 + U_1)
\]

where

\[
F_1 = D - L_1F_1^{-1}U_1
= \begin{pmatrix}
T_1^1 & T_1^2 \\
S_1^1 - L_1^1(T_1^1)^{-1}U_1^1 - L_1^2(T_1^2)^{-1}U_1^2
\end{pmatrix}
\]

Parallism in each recursion level!
Nested Preconditioners [Grigori et al., 2010]

- Nested exact factorization

\[
\begin{align*}
A &= F_0 \\
F_k &= (L_{k+1} + F_{k+1})F_{k+1}^{-1}(F_{k+1} + U_{k+1}), \text{ for } k = 0 \ldots K - 1 \\
F_K &= D - \sum_{k=1}^{K} L_k F_k^{-1} U_k
\end{align*}
\]

- NSSOR Preconditioner

\[
F_K = D
\]

- NMILUR Preconditioner

\[
F_K = D - \sum_{k=1}^{K} \text{rowsum}(L_k F_k^{-1} U_k) = D - \sum_{k=1}^{K} \text{Diag}(L_k F_k^{-1} U_k 1)
\]

\[
A1 = P_{NMILUR} 1 \text{ (filtering on vector } 1.)
\]
Nested Preconditioners [Grigori et al., 2010]

- Nested exact factorization

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F_k = (L_{k+1} + F_{k+1})F_{k+1}^{-1}(F_{k+1} + U_{k+1}), \text{ for } k = 0 \ldots K - 1 \\
F_K = D - \sum_{k=1}^{K} L_k F_k^{-1} U_k
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- NSSOR Preconditioner

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F_K = D - \sum_{k=1}^{K} \text{rowsum}(L_k F_k^{-1} U_k) = D - \sum_{k=1}^{K} \text{Diag}(L_k F_k^{-1} U_k \mathbf{1}) \\
A\mathbf{1} = P_{\text{NMILUR}}\mathbf{1} \text{ (filtering on vector } \mathbf{1}.)
\]
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**Numerical results**

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Boundary value problem
(provided by Achdou, Nataf)

- Advection-diffusion
- Non-homogeneous
- Convective Skyscraper
- Anisotropic

\[ \text{div}(a(x)u) - \text{div}(\kappa(x)\nabla u) = f \quad \text{in} \ \Omega \]
\[ u = 0 \quad \text{on} \ \partial\Omega_D \]
\[ \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \partial\Omega_N \]

discretized on a cartesian grid

Simulation of Black Oil Model
(provided by R. Masson IFP)

- BO60x60x32
  - compositional triphase Darcy flow simulator (oil, water and gas).

Tim Davis Collection

- Dubcova2 (PDE solver)
- thermomech (FEM problem)
Impact of Overlap on matrix size and nnz

<table>
<thead>
<tr>
<th>matrix</th>
<th>$n(A)$</th>
<th>$n(A_o)$</th>
<th>increase</th>
<th>$nnz(A)$</th>
<th>$nnz(A_o)$</th>
<th>increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>2dANI400</td>
<td>160000</td>
<td>169355</td>
<td>5%</td>
<td>798400</td>
<td>845131</td>
<td>5%</td>
</tr>
<tr>
<td>2dNH400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2dAD400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dubcova2</td>
<td>65025</td>
<td>74913</td>
<td>15%</td>
<td>1030225</td>
<td>1200940</td>
<td>16%</td>
</tr>
<tr>
<td>thermomech</td>
<td>102158</td>
<td>108984</td>
<td>6%</td>
<td>711558</td>
<td>759895</td>
<td>6%</td>
</tr>
<tr>
<td>3dCSKY40</td>
<td>64000</td>
<td>90643</td>
<td>41%</td>
<td>438400</td>
<td>622410</td>
<td>41%</td>
</tr>
<tr>
<td>3dANI40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BO60×60×32</td>
<td>115200</td>
<td>156109</td>
<td>35%</td>
<td>791482</td>
<td>1074100</td>
<td>36%</td>
</tr>
</tbody>
</table>

**Table**: analyse on a $4 \times 4$ partition

- On a $400 \times 400$ grid
  \[
  V_{add} \approx 8n(\sqrt{p} - 1) = 8 \times 400(\sqrt{16} - 1) = 9600 \implies 6\%
  \]
GMRES (no restart, \( \text{maxiter} = 200, \text{tol} \leq 10^{-8}, \text{err} = \frac{\|x-x_{\text{exact}}\|}{\|x\|} \))

<table>
<thead>
<tr>
<th>matrix</th>
<th>without overlap</th>
<th>with overlap</th>
<th>improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>2dANI400</td>
<td>200 e-4</td>
<td>200 e-4</td>
<td></td>
</tr>
<tr>
<td>2dNHI400</td>
<td>114 e-7</td>
<td>82 e-7</td>
<td>-28%</td>
</tr>
<tr>
<td>2dAD400</td>
<td>114 e-7</td>
<td>82 e-7</td>
<td>-28%</td>
</tr>
<tr>
<td>dubcova2</td>
<td>39 e-7</td>
<td>35 e-8</td>
<td>-10%</td>
</tr>
<tr>
<td>thermo</td>
<td>5 e-10</td>
<td>4 e-11</td>
<td>-20%</td>
</tr>
<tr>
<td>3dCSKY40</td>
<td>145 e-6</td>
<td>93 e-7</td>
<td>-36%</td>
</tr>
<tr>
<td>3dANI40</td>
<td>200 e-8</td>
<td>46 e-8</td>
<td>-75%</td>
</tr>
</tbody>
</table>

**Table**: ilu(0)-like approximation on 4x4 partition
convergence rate for Nested Preconditioners

GMRES (restart = 200, maxiter = 1000, tol \leq 10^{-8}, err = \frac{\|x - x_{\text{exact}}\|}{\|x\|})

<table>
<thead>
<tr>
<th>matrix</th>
<th>4 x 4</th>
<th></th>
<th>8 x 8</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>iter</td>
<td>err</td>
<td>iter</td>
<td>err</td>
</tr>
<tr>
<td>2dANI400</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
<tr>
<td>2dNH400</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
<tr>
<td>2dAD400</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
<tr>
<td>dubcova2</td>
<td>46</td>
<td>e-7</td>
<td>64</td>
<td>e-7</td>
</tr>
<tr>
<td>thermo</td>
<td>6</td>
<td>e-9</td>
<td>6</td>
<td>e-9</td>
</tr>
<tr>
<td>3dCSKY40</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
<tr>
<td>3dANI40</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
<tr>
<td>BO</td>
<td>fail</td>
<td></td>
<td>fail</td>
<td></td>
</tr>
</tbody>
</table>
convergence rate for Nested Preconditioners

GMRES \((restart = 200, \text{maxiter} = 1000, tol \leq 10^{-8}, \text{err} = \frac{\|x - x_{\text{exact}}\|}{\|x\|})\)

<table>
<thead>
<tr>
<th>Matrix</th>
<th>NSSOR</th>
<th>NSSOR + overlap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 x 4</td>
<td>8 x 8</td>
</tr>
<tr>
<td></td>
<td>4 x 4</td>
<td>8 x 8</td>
</tr>
<tr>
<td>matrix</td>
<td>iter</td>
<td>err</td>
</tr>
<tr>
<td>2dANI400</td>
<td>fail</td>
<td>fail</td>
</tr>
<tr>
<td>2dNH400</td>
<td>fail</td>
<td>fail</td>
</tr>
<tr>
<td>2dAD400</td>
<td>fail</td>
<td>fail</td>
</tr>
<tr>
<td>dubcova2</td>
<td>46</td>
<td>e-7</td>
</tr>
<tr>
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<td>6</td>
<td>e-9</td>
</tr>
<tr>
<td>3dCSKY40</td>
<td>fail</td>
<td>fail</td>
</tr>
<tr>
<td>3dANI40</td>
<td>fail</td>
<td>fail</td>
</tr>
<tr>
<td>BO</td>
<td>fail</td>
<td>fail</td>
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</tbody>
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convergence rate for Nested Preconditioners

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<th>Matrix</th>
<th>NSSOR</th>
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<th>NMILUR + overlap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 × 4</td>
<td>8 × 8</td>
<td>4 × 4</td>
</tr>
<tr>
<td>2dANI400</td>
<td>fail</td>
<td>fail</td>
<td>301 e-6</td>
</tr>
<tr>
<td>2dNH400</td>
<td>fail</td>
<td>fail</td>
<td>84 e-7</td>
</tr>
<tr>
<td>2dAD400</td>
<td>fail</td>
<td>fail</td>
<td>81 e-7</td>
</tr>
<tr>
<td>dubcova2</td>
<td>46 e-7</td>
<td>64 e-7</td>
<td>33 e-8</td>
</tr>
<tr>
<td>thermo</td>
<td>6 e-9</td>
<td>6 e-9</td>
<td>3 e-9</td>
</tr>
<tr>
<td>3dCSKY40</td>
<td>fail</td>
<td>fail</td>
<td>93 e-7</td>
</tr>
<tr>
<td>3dANI40</td>
<td>fail</td>
<td>fail</td>
<td>40 e-8</td>
</tr>
<tr>
<td>BO</td>
<td>fail</td>
<td>fail</td>
<td>97 e-7</td>
</tr>
</tbody>
</table>

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Plan

Overlapping techniques
Parallel preconditioners
Numerical results

Conclusion
Conclusion and Future Work

Conclusion

- **Overlap**
  - nested dissection, parallel
  - reasonable storage cost

- **Convergence rate**
  - BILDU
  - NSSOR and NMILUR

Future work

- parallel implementation