Wave propagation in vortices and vortex bursting

H. Moet*, F. Laporte†, G. Chevalier* and T. Poinsot**

* CERFACS, 42 Avenue G. Coriolis, 31057 Toulouse, France
† Airbus Deutschland, Huenefeldstr. 1-5, D-28183 Bremen, Germany
** IMFT, Allée du Professeur Camille Soula, 31400 Toulouse, France

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Abstract

The propagation of pressure waves in a Lamb-Oseen vortex has been investigated using three-dimensional Direct Numerical Simulations as well as a set of Large-Eddy Simulations. The pressure wave is initiated by locally increasing the core radius of a Lamb-Oseen vortex at its edge. This wave travels along the vortex axis. Behind the wave the axial velocity increases so that sufficient swirl may trigger the helical instability. An abrupt change of flow structure in the vortex core is observed in the case of intersecting pressure waves: this phenomenon is known as vortex bursting. The occurrence of helical instability and vortex bursting is also evidenced in Large-Eddy Simulations of a vortex system similar to that of an aircraft wake.
I. INTRODUCTION

The behaviour of aircraft trailing vortices plays an essential role in current air traffic control and related aspects such as capacity and safety at airports. Numerous studies have been conducted which provided profound insight in the formation, transport, decay and stability of trailing vortices contributing to the understanding of the dynamics of aircraft wakes. The analysis of smoke visualization of aircraft wake vortices by Tombach\textsuperscript{1}, laboratory tests by Sarpkaya \textit{et al.}\textsuperscript{2} and flight test campaigns conducted by NASA 1995 at Wallops Flight Facility showed catastrophic events abruptly changing the vortex core. The phenomenon is called vortex bursting and was seen to occur in one or both of the vortices of the vortex system studied in the tests. A notable observation is the continuous motion of the remaining vortex after the occurrence of vortex bursting and the subsequent vanishing of the other vortex. This suggests that the second vortex was not destroyed but purely vanished from the visualization due to the dispersion of the marker from the vortex core. Tombach mentioned the burst moved along the vortex and also showed that the bursts manifested conical parcels of smoke travelling along the vortex cores. These portions of the vortices occasionally collided leaving behind a 'disk-like' parcel of smoke. Spalart\textsuperscript{3} made similar comments by describing the bursting phenomenon as travelling portions of smoke marking regions of contraction and expansion sometimes making 'pancakes' (illustrated in Fig. 1).

Theoretical and experimental studies\textsuperscript{4–7} have shown that waves may travel along longitudinal vortex cores. The collision of two waves propagating in opposite directions may be responsible for the formation of the pancake, then called vortex bursting. The intention of this paper is to give further evidence of this interpretation.

Leibovich and Kribus\textsuperscript{7} found that large-amplitude axisymmetric waves may exist for inviscid flows and suspected these waves responsible for flow structures observed in vortex breakdown. Hopfinger \textit{et al.}\textsuperscript{5} conducted experiments in a rotating tank with an oscillating grid responsible for generating turbulence and they observed that concentrated vortices generated by the experimental set-up support waves of helical distortions. The nonlinear waves
have the property of transporting mass, momentum and energy from the turbulent region to the rotation-dominated flow. Interaction between waves resulted in a local disruption of the vortex core and small turbulence production (although reformation of the vortex cores was observed afterwards). An interesting dynamical aspect recovered, was that the waves travelled as single entities and over distances many times larger than their characteristic length without appreciable change.

The vortex breakdown phenomenon is known to occur in different flow conditions. For example, leading edge vortices generated by a delta wing placed in a uniform flow may experience such a phenomenon. Spalart\cite{3} pointed out that the identification of vortex bursting with vortex breakdown must be avoided, however the present authors would like to note that the two catastrophic events affecting the vortex core might be related.

Vortex breakdown occurs in presence of combined swirling and axial flows. It manifests itself by a brutal disorganization of the flow structure first observed by Werle\cite{8} and by Peckham and Atkinson\cite{9} for leading edge vortices above a delta wing. The phenomenon described by Sarpkaya\cite{10} as an abrupt change in the core of a swirling flow is characterized by a rapid dilatation of the vortex core generally breaking up into a three-dimensional unsteady flow which transitions to turbulence. Leibovich\cite{11} in turn defined vortex breakdown as a disturbance characterized by the formation of an internal stagnation point on the vortex axis followed by a reversed flow in a region of limited extent. Vortex breakdown is a spectacular phenomenon affecting a range of different types of flows. It results in the abrupt changes in lift and drag, unsteadiness and poor control for flows over delta wings at a high angle of attack. A beneficial feature can be recognized in combustion processes in which breakdown leads to the stabilization of flames in swirl combustors as evidenced by Selle et al.\cite{12} but is also observed to affect geophysical flows such as tornadoes. Breakdown is provoked by a destabilizing source namely an adverse pressure gradient and seems to occur when some critical value is reached for the amount of swirl. Investigators observed different features in the flows subjected to vortex breakdown which might indicate different breakdown types or they might be due to different visualization behaviour of the used tracers for the same
phenomenon. The different features/types are the bubble, spiral and the double helix (discovered by Sarpkaya$^{10,13}$). Billant et al.$^{14}$, who studied experimentally vortex breakdown in swirling jets, questioned the stability of the flow to helical disturbances in the dynamics of vortex breakdown. They came up with the points of view that breakdown might result from the ultimate development of helical disturbances or that breakdown is an independent phenomenon over which helical disturbances may grow.

During the past 50 years many studies have focussed on vortex breakdown and are reviewed in the publications by Leibovich$^{11}$, Althaus et al.$^{15}$ and Deleri$^{16}$. Furthermore, in this framework valuable information can be found in the work of Mayer et al.$^{17}$ who conducted a theoretical study on the linear stability of the Batchelor or q-vortex which covered all azimuthal wave numbers. Their study predicted the amplification of linearly unstable helical modes for a range of the swirl intensity and also determined the amplification growth rates of the most unstable modes.

The present authors have the opinion that a wave propagating at a constant speed along a vortex core, can be seen as a stationary wave in a frame of reference moving at the wave propagation speed, that might be interpreted as a vortex breakdown of bubble type, for example. This change in frame of reference might be the key to relate wave propagation in a vortex surrounded by a flow at rest, and stationary vortex breakdown in a uniform flow. In this framework the propagation of isolated pressure waves on a single vortex is studied in a simple configuration, which is discussed in Section III. The stability of the vortex affected by such a pressure wave is investigated in Section IV as well as the interaction of intersecting waves which is discussed in Section V. In Section VI a more realistic configuration consisting of a vortex pair with axial flow in the core representing an aircraft wake subjected to Crow instability is presented to show that the Crow instability can generate pressure waves travelling along the vortex cores, causing local helical instabilities and vortex burstings.
II. GOVERNING EQUATIONS, NUMERICAL METHODS AND MODELING

A. Governing equations

The nondimensional formulation of the 3D compressible Navier-Stokes equations reads

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0 \quad (1)
\]

\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{1}{Re} \frac{\partial \tau_{ij}}{\partial x_j} \quad (2)
\]

\[
\frac{\partial (\rho E)}{\partial t} + \frac{\partial \left((\rho E + p)u_j \right)}{\partial x_j} = \frac{1}{Re} \frac{\partial (u_i \tau_{ij})}{\partial x_j} - \frac{1}{Re Pr} C_p \frac{\partial q_j}{\partial x_j} \quad (3)
\]

where the nondimensional variables are: the density \(\rho\), the velocity vector \((u_1, u_2, u_3)^T\), the pressure \(p\), the total energy \(E\), the heat flux vector \((q_1, q_2, q_3)^T\) given by Fourier’s law, and the shear stress tensor \(\tau_{ij}\). The \(C_p\) coefficient is the specific heat at constant pressure.

The nondimensional variables are defined as the local dimensional variables divided by a reference variable or a combination of the reference variables. The reference variables are: the density \(\rho_{ref}\), the velocity \(a_{ref}\), the pressure \(p_{ref}\), the length \(l_{ref}\), the temperature \(T_{ref}\), the dynamic viscosity \(\mu_{ref}\) and the specific heat \(C_{p,ref}\). The reference Reynolds number is

\[Re = \frac{\rho_{ref} a_{ref} l_{ref}}{\mu_{ref}}\]

while \(Pr\) stands for the Prandtl number (in the present simulation \(Pr = 0.75\)).

B. LES approach and subgrid scale model description

In the Large-Eddy Simulation (LES) approach the previous equations are filtered spatially, so that any variable \(\phi(x)\) may be decomposed into a resolved (or large scale) part \(\phi(x)\) and a nonresolved (or subgrid scale) part \(\phi^\prime (x)\), with \(\phi(x) = \phi(x) + \phi^\prime (x)\). This procedure may be obtained by a convolution integral of the variable with any filter function depending on a filter width \(\Delta\). Practically, the filter width is simply given by the computational mesh cell size \(\Delta x\). For compressible flows, the Favre-filtered variables defined as \(\phi(x) = \phi(x) + \phi^\prime (x)\) are used, with \(\phi = \phi \bar{\phi} / \bar{p}\). The dimensionless Favre-filtered equations are:
where the subgrid scale (SGS) stress tensor \( \sigma_{ij} = -\langle \mu \tilde{u}_i \tilde{u}_j \rangle \) and the SGS heat flux \( Q_j = \tilde{\rho} C_p \tilde{T} \tilde{u}_j - \tilde{\rho} C_p \tilde{\tilde{u}}_j \) are to be modeled and where the following classical approximations (see Erlebacher et al.) have been made:

- The Favre-filtered shear stress tensor is identified with the filtered shear stress tensor
- The Favre-filtered heat flux is identified with the filtered heat flux
- The filtered kinetic energy term \( \tilde{K} \tilde{u}_j \) in the energy equation is approximated by \( \tilde{\rho} \tilde{K} \tilde{u}_j - \sigma_{ij} \tilde{u}_j \), where \( K = 1/2 \mu_i \mu_i \) is the kinetic energy. The SGS momentum \( \sigma_{ij} \) and the SGS heat flux \( Q_j \) are modeled through the subgrid-scale eddy-viscosity concept:

\[
\sigma_{ij} = \frac{1}{3} \sigma_{kk} \delta_{ij} = -2 \mu_{sgs} \left( \tilde{S}_{ij} - \frac{1}{3} \delta_{ij} \tilde{S}_{kk} \right) \tag{7}
\]

\[
Q_j = -\frac{\mu_{sgs} C_p}{Pr_t} \frac{\partial \Theta}{\partial x_j} \tag{8}
\]

where \( \mu_{sgs} \) is the SGS dynamic viscosity and \( \tilde{S}_{ij} \) is the large scale strain rate tensor; while \( Pr_t \) is the turbulent Prandtl number, defining the modified temperature \( \Theta = \tilde{T} - 1/(2 \rho C_v) \sigma_{kk} \), where \( C_v \) is the specific heat at constant volume.

The SGS viscosity model is based on the Structure Function model (Métais & Lesieur) initially developed in spectral space (effective viscosity model) and then transposed into the physical space. The expression of the Structure Function is

\[
F_2(\tilde{x}, \Delta, t) = \langle || \tilde{u}(\tilde{x} + \tilde{r}, t) - \tilde{u}(\tilde{x}, t) || ||\tilde{r}|| = \Delta \rangle \tag{9}
\]

where \( \Delta \) is the cutoff length and where \( \langle \rangle \) denotes spatial averaging, here over the sphere of radius \( \Delta \). As the information brought by the model is local in space it leads to a poor estimation of the kinetic energy at the cutoff which may be improved by a suitable filtering in order to remove the influence of the large scales on the SGS viscosity. The procedure
defined by Ducros et al.\textsuperscript{20} is to apply (possibly \( n \) times) a discrete Laplacian high-pass filter to the velocity field before calculating the Structure Function. The optimum value of \( n \) found by Ducros et al.\textsuperscript{20} for their simulations is \( n = 3 \). This value has also been used here. Finally, the Filtered Structure Function model reads:

\[
\nu_{\text{sgs}} = \bar{p} \mu_{\text{sgs}} = \nu_{\text{sgs}}(\bar{x}, \Delta, t) = \alpha^{(n)} \Delta \sqrt{F_2^{(n)}(\bar{x}, \Delta, t)}
\]

where the superscript \((n)\) indicates that the filter has been applied \( n \) times. The value of \( \alpha \) used here is \( \alpha^{(3)} = 0.00084 \). The Structure Function model formulation of Métais & Lesieur\textsuperscript{19} in spectral space insures that the SGS viscosity vanishes when there is no energy at the cutoff wavelength. This property is particularly important for the simulation of transitional flows as those of interest in the present paper.

C. Numerical method

The numerical code used for this study is a parallel, three-dimensional, finite differences Navier-Stokes solver using both regular meshes and irregular meshes\textsuperscript{21}. For the regular mesh formulation space discretization is performed by a sixth-order compact scheme with spectral-like resolution\textsuperscript{22} for both convective and viscous terms. A special treatment is applied for the nonuniform mesh formulation (see section IID). Time integration is performed by means of a three-stage third-order Runge-Kutta method.

Temporal Direct Numerical Simulations (DNS) as well as Large-Eddy Simulations (LES) have been performed in order to compute the unsteady behaviour of the vortex. The nonreflecting boundary conditions are treated according to the characteristic approach developed by Poinsot and Lele\textsuperscript{23} with a pressure relaxation term on the incoming waves in order to maintain the pressure close to the free-stream value.
Part of the simulations has been conducted with a formulation of the compact finite difference schemes on nonuniform meshes, consisting of a uniform mesh region surrounded by a coarsening mesh. Consequently, the precision of the compact scheme reduces to a fourth- (respectively third-) order compact scheme for the approximation of the first (respectively second) derivatives on nonuniform meshes. This has been accomplished by a full inclusion of the metrics for calculating the coefficients of the compact scheme (Gamet et al.\textsuperscript{21}). This technique, called fully integrated metrics, consists of computing the derivatives directly on the irregular mesh. For this purpose the compact scheme used on regular meshes is adapted. This has been accomplished by respecting the constraint which imposes that the obtained scheme for nonuniform meshes must reduce exactly to the scheme for uniform meshes. Gamet et al. discussed the extension to irregular meshes in terms of theory, linear stability, analysis and precision for the kind of compact schemes used for our study. Note that the nonuniform meshes used here are such that the gridpoints in the region which contains both the vortex and the region where instability and transition processes take place are uniformly spaced. This implies that the finite difference scheme provides 6\textsuperscript{th} order accuracy in the uniform region. The use of the LES model on the kind of nonuniform mesh used in this study is acceptable since the local isotropy assumption (used for LES modeling) is satisfied in the uniform mesh region where the cut-off length is constant. Moreover, the boundary between the uniform mesh and the coarsened mesh was chosen sufficiently far from the region where instability processes govern the flow field. In addition, the vortex induced velocity field in the coarsened mesh region is representative of a very weak, uniform flow and may be assumed laminar.
III. WAVE PROPAGATION IN A VORTEX

A. Initial vortex flow field

For the study of wave propagation in a vortex, the initial condition consists of a Lamb-Oseen vortex characterized by solid rotation in the core and constant circulation at infinity. The two-dimensional tangential velocity profile and the pressure field can be given in cylindrical coordinates by:

\[ v_\theta(r, t) = V_{\theta_{\text{max}}} \frac{r_c}{r} (1 - e^{-\beta(r^2/r_c^2)}) = \frac{\Gamma_0}{2\pi r} (1 - e^{-\beta(r^2/r_c^2)}) \]  
\[ \frac{dP}{dr} = \frac{\rho v_\theta^2(r, t)}{r} \]

with \( \alpha = 1.40 \) and \( \beta = 1.2564 \). The vorticity distribution can be expressed as follows:

\[ \omega_z(r, t) = \frac{\Gamma_0}{4\pi \nu t} e^{-\beta(r^2/r_c^2)} \text{ with } r_c(t) = \sqrt{4\nu/\beta t} \]

The initial condition for the three-dimensional computations has been obtained by extrusion of the two-dimensional vortex flow field. Table I sums up the parameters that determine the computational domain such as the domain lengths \( L_{x,y,z} \), the number of points in each direction \( n_{x,y,z} \) and the parameters fixing the uniform mesh region \( L_{\text{uniform}} \) and \( n_{p_{\text{uniform}}} \). The boundary conditions are shown in Fig. 2. The choice of the boundary conditions in the crossplane was made following the approach proposed by Sreedhar et al. who applied symmetry boundary conditions in the two directions of the cross plane. This implies the presence of an infinite number of image vortices in both directions, although primary focus was concentrated on a single vortex. The domain size of the cross plane has been taken sufficiently large with respect to the core radius in order to have negligible effects of the image vortices on the overall dynamics of the vortex.

For these simulations a pressure wave was generated by imposing a variation of the core radius along the vortex axis as sketched in Fig. 3. Two parts of the vortex core can be distinguished with constant radius \( r_{c_1} \) and \( r_{c_2} \) connected by a part where the core radius
follows a sinusoidal distribution. Consequently, this vortex shape leads to a variation in
the minimum underpressure (determined by integrating Eq. (12)) in the vortex core along
the axis. Note that this is a simple configuration for which a smooth sinusoidal variation
is applied to the vortex core radius along the axis. This configuration is used to extract
the physical dynamics of the phenomena caused by the propagation of a pressure wave.
Although the initial condition is not an exact solution of the Navier-Stokes equations it is
somewhat representative because very similar dynamics are recovered in the simulation of a
more realistic configuration (see Section VI). Several simulations (see Table II) have been
performed to study the properties of waves and the effects they may have on the vortex
stability.

In this section a set of LES simulations has been performed without any white noise, to
retain a flow without background perturbations to study the properties of the pressure wave
in stable conditions.

Reynolds number The Reynolds number based on the circulation of the vortex has been
fixed to a value of $Re = O(10^7)$ for these LES simulations. It corresponds to the realistic
flight Reynolds number (based on a circulation of $\Gamma = 600 m^2 s^{-1}$ and the kinematic viscosity
$\nu = 1.5 \cdot 10^{-5}$).

B. Global dynamics and evolution of flow properties

The variation of the vortex size leads to a variation in the minimum underpressure
reached in the vortex core. Concurrently, the adverse pressure gradient ($dp/dz < 0$) in the
core results in the generation of a pressure wave travelling as a single entity along the vortex
axis in the positive $z$-direction that transports fluid from the region with the larger vortex
core size to the smaller vortex core region. This is illustrated by the profiles of the normalized
pressure evolving in time (Fig. 4), which also give the impression that the pressure wave
travels at approximately constant speed. The propagation speed has been determined for
the complete set of runs and takes on a slightly larger value if $r_{c2}$ is larger. Fig. 5 displays
the average propagation speeds as a function of the ratio $r_{c2}/r_{c1}$ ($r_{c1}$ is kept constant).

Isosurfaces of vorticity magnitude show a surface wrinkle travelling in positive $z$-direction along the vortex core which appears to be a small annular structure (see Fig. 6 for Run 1 with $r_{c2}/r_{c1} = 2.0$), the internal core structure has been visualized by taking a sufficiently high value of $|\omega|$). This structure marks the region located just behind the front of the pressure wave. Note that the annular structure becomes less visible with a smaller initial value of $r_{c2}$. The generation of azimuthal vorticity may be explained by considering the equation for the vorticity transport in azimuthal direction:

$$\frac{\partial \omega_\theta}{\partial t} = \omega_z \frac{\partial v_\theta}{\partial z}$$

(14)

Eq. (14) shows that the axial vorticity $\omega_z$ is redistributed into azimuthal vorticity $\omega_\theta$. The initial condition was generated such that the circulation is constant along the $z$-coordinate and as $r_{c2} > r_{c1}$, which yields $\partial v_\theta/\partial z > 0$. For the azimuthal component of vorticity this implies $\partial \omega_\theta/\partial t > 0$ as $\omega_z > 0$ in the initial condition. Due to the axial symmetry of the flow an annular structure of vorticity is formed (see Fig. 7 showing isocontours and an isosurfaces of azimuthal vorticity in the beginning of the simulation when the pressure wave is generated).

Multiple effects have been identified due to the pressure wave travelling through the vortex core. At a particular location along the vortex axis, the core radius ($r_c = r(V_{\theta max})$) increased with at least 30% while the peak tangential velocity ($V_{\theta max}$) decreased by approximately 20% immediately after the pressure wave has passed. The circulation ($\Gamma = \int_\Omega \omega_z d\Omega$) did not experience any decay and maintained a constant value.

Another essential effect is the generation of the axial velocity in the vortex core, which is directed in positive $z$-direction ($w > 0$). This increase of axial velocity can be translated to the amount of swirl by using the swirl parameter, which is defined as:

$$q = 1.12 \alpha \frac{V_{\theta max}}{W_{max}}$$

(15)

Several remarks can be made for the evolution of the axial velocity in the vortex core related to the theoretical study conducted by Mayer et al. They studied the linear stability of
a Batchelor vortex (also known as q-vortex) and predicted the occurrence of the helical instability when the swirl parameter ranges from $0 \lesssim q \lesssim 1.5$. The Batchelor vortex has the same tangential velocity distribution as the Lamb-Oseen vortex (11) but has a nonzero axial velocity component which is expressed as:

$$v_z(r, \theta, z) = W_{max} e^{-\beta(r^2/r_c^2)} = 1.12\alpha \frac{V_{\max}}{q} e^{-\beta(r^2/r_c^2)} \tag{16}$$

Fig. 8 presents the history of the swirl parameter $q$ extracted at three stations, which are placed equidistantly in the $z$-direction along the vortex axis. It shows that $q$ reaches values ($\min_t q \approx 1.2$) that may trigger the helical instability for Batchelor vortices. Section IV will show that the amount of swirl (defined by Eq. (15)) is sufficiently high for the deformed vortex core to amplify unstable helical modes.

**IV. HELICAL INSTABILITY DUE TO WAVE PROPAGATION**

In section III a significant increase in the axial velocity was observed as one of the essential consequences of the propagation of the pressure wave. This aspect plays an important role in the development of the helical instability. Contrary to the approach followed in the previous section where the propagation of pressure waves was studied in stable conditions, the simulations in this section are initialized with a random white noise in order to provoke the intrinsic dynamics related to the helical instability. The random white noise has been superimposed on the transverse Cartesian velocity components of the base flow as $u = u_{base}(1 + \epsilon \tilde{u})$, with $\epsilon \tilde{u}$ the random perturbation with maximum amplitude $1\%$ of the local base flow, i.e. $\epsilon = 1 \cdot 10^{-2}$ and $||\tilde{u}|| \leq 1$. The configuration considered in Section III which is shown in Fig. 3 with $r_{c2}/r_{c1} = 2.0$ is used in the present section as the initial condition for a DNS and an LES simulation called Run DNS$_1$ and Run LES$_1$ respectively (the initial conditions are summarized in Table IV). The increase of the axial velocity in the core caused by the pressure wave generated with this configuration reaches a value that results in helical instability and motivates the choice for this configuration.
The DNS has been performed for a Reynolds number based on the circulation and the kinematic viscosity at a value of $Re_\Gamma = 10^4$. For the LES simulation the filtered structure function model as described in Section II B is used and a Reynolds number of $Re_\Gamma = 10^7$ is chosen. Furthermore, the computational domain and the boundary conditions for this configuration and for both approaches are shown in Fig. 2.

The following section describes the results concerning the global dynamics of the single vortex subjected to wave propagation governed by the development of helical instability.

A. Global dynamics and evolution of flow properties

The global dynamics of the DNS$_1$ are illustrated in Fig. 9 and show initially the creation and the propagation of the pressure wave, which is visible on the isosurface of vorticity magnitude at $t/T_{rot} = 2.03$. The unstable helical modes of the vortex are amplified by the substantial increase of the axial velocity in the vortex core (see $t/T_{rot} = 4.06 - 6.01$). The helical instability develops and a range of azimuthal modes are observed. A distinction can be made between large helical structures and somewhat finer structures near the location where the pressure wave is initially generated. The various helical modes can be distinguished in Fig. 10 where isosurfaces and isocontours of azimuthal vorticity $\omega_\theta$ and the radial velocity perturbation are shown for simulation DNS$_1$ at $t/T_{rot} = 4.06$.

Note that it is sometimes difficult to distinguish one particularly amplified mode in a plane cut through the vortex when comparing the isocontours of azimuthal vorticity and the radial velocity perturbation. This may be explained by the changing character of the (axial) flow in the plane resulting in the amplification of different modes at different times.

The propagation of the pressure wave in unstable conditions has led to similar behaviour of the characteristic flow parameters as revealed in Section III B. The normalized pressure increases immediately after the wave has passed. The vortex core size increases to approximately $1.5 - 1.6r_{c_0}$, the peak tangential velocity decreases substantially, while the circulation remains constant.
In Fig. 9 and 10 the front of the pressure wave can be distinguished by the axisymmetric annular structure of vorticity as observed in Fig. 6. A qualitative similarity can be found compared to the vortex ring structure encountered in the axisymmetric bubble of stationary vortex breakdown. This ring-like structure was observed by numerous investigators such as Faler and Leibovich\textsuperscript{26} and Br"ucker and Althaus\textsuperscript{27}. Sarpkaya\textsuperscript{10,28} stated that the flow in the interior of the bubble is unsteady due to gyrations of a tilted toroidal ring. Novak and Sarpkaya\textsuperscript{29} mentioned a breakdown featuring a (two-celled) bubble preceding a spiral, also illustrated recently through numerical results of Ruith \textit{et al.}\textsuperscript{30}, claiming that the flow behind the axisymmetric bubble might become unstable developing an azimuthal instability and resulting in helical breakdown. In the framework of the propagative phenomena Maxworthy \textit{et al.}\textsuperscript{31} stated that it is plausible to conclude that the waves considered in the present study may be responsible for vortex breakdown in the sense that for stationary breakdown the strong axial flow counters the forward propagation of the wave(s), effectively bringing the process to rest. Furthermore, Maxworthy \textit{et al.} also explain that when the waves reach a certain critical amplitude, they become unstable and when the spiral disturbances have grown they cause turbulent breakdown. The stationary breakdown phenomenon and helical instability triggered by wave propagation seem to be related because certain structural characteristics found in breakdown are also encountered in the global dynamics of the simulated propagative phenomena.

\textbf{Large-Eddy Simulation} \hspace{1em} The LES approach has been followed in order to be able to simulate the flow at a significantly higher Reynolds number ($Re_r = 10^7$) which corresponds to a more realistic condition in the framework of aircraft wake vortices. The global dynamics are shown in Fig. 11 and shows that the initial evolution for the Run LES\textsubscript{1} is similar to that observed for Run DNS\textsubscript{1}. At $t/T_{rot} = 2.03$ similar structures are observed during the generation and propagation of the pressure wave as in Fig. 9. However, from $t/T_{rot} = 4.06$ to 6.01 the helical instability does not seem to develop in the same manner and the helical structures of vorticity reveal to be significantly finer than observed in the DNS simulation. The dynamics of the vortex at a substantially lower Reynolds number yielding higher viscosity used for
the DNS results in the damping of the higher azimuthal modes and explains why the LES simulation recovers finer structures. This is illustrated by comparing the DNS (Fig. 10) and the LES (Fig. 12) at the same instant in time \( t/T_{\text{rot}} = 4.06 \). The various helical modes are illustrated in Fig. 12 where isosurfaces and isocontours of azimuthal vorticity \( \omega_\theta \) and the radial velocity perturbation are shown for simulation LES\(_1\) at \( t/T_{\text{rot}} = 4.06 \). One may now distinguish very fine helical structures from relatively large structures.

Note that the helical instability results in the amplification of radial velocity modes. It leads to outward directed radial velocity, which leads to the outward directed radial transfer of a passive scalar or smoke. Consequently, the wave propagation in vortices and the resulting helical instability have an effect on mixing/dispersion of smoke, which explains the underlying mechanisms responsible for the observations made by Tombach\(^1\), Sarpkaya\(^2\) and the smoke visualisations made by NASA at Wallops Flight Facility.

The fine helical structures are found to be concentrated at an approximately fixed radial distance from the vortex centre at approximately \( r/r_{c_1} \approx 1.5 \). This was also mentioned by Mayer et al.\(^{17}\) where the higher modes, also called ‘ring modes’ were seen to be localized in a small radial range (although they predicted the azimuthal modes to be located at a smaller radial location, namely at \( r/a \approx 0.75 \) in our notations this corresponds to \( r/r_{c_1} \approx 0.67 \) the difference between the results is not yet clear). Eventually, in the region where the helical instability has developed into an advanced stage, saturation of the helical instability follows and the region defining the vortex core has become larger (see \( t/T_{\text{rot}} = 6.01 \)).

**V. VORTEX BURSTING**

In this section a different configuration is considered to simulate two pressure waves propagating in opposite direction and intersecting each other. The configuration used is depicted in Fig. 13 and is determined by the ratios \( r_{c_2}/r_{c_1} = 2.0 \) and \( r_{c_3}/r_{c_1} = 2.0 \). This configuration is used as initial condition for DNS and LES, denoted by Run DNS\(_2\) and Run LES\(_2\) respectively (parameters concerning the initial conditions are summed up in
A random white noise is superimposed on the vortex base flow (as described in Section IV) to trigger helical instabilities. The Reynolds number based on the circulation is taken \( Re_T = 10^4 \) for the DNS and \( Re_T = 10^7 \) for the LES. The computational domain and the boundary conditions are identical as used for Run DNS\(_1\) and LES\(_1\) except for the use of symmetry boundary conditions at both axial boundaries.

The following sections describe the global dynamics of the single vortex subjected to multiple wave propagation resulting in vortex bursting (Section VA).

A. Global dynamics and evolution of flow properties

The initial stages of the simulation are governed by the generation and the propagation of the pressure waves in opposite directions (see Fig. 14 at \( t/T_{rot} = 1.02 \) and 2.03, note that Fig. 14 gives a perspective view and that the phenomenon is symmetric). The high viscosity is suspected of damping the amplification of the helical modes and therefore the helical instability is not seen to develop in this configuration.

An essential event takes place after \( t/T_{rot} = 3.05 \) (see Fig. 14): the two pressure waves intersect and lead to a drastic change in the structure of the vortex core. The change consists in a complete loss of coherence of the vortex at the location where the waves have intersected and in the creation of an expanding annular structure. This vortex bursting phenomenon is characterized by an abrupt change of the internal structure of the vortex accompanied by a drastic increase of the vortex core radius. The intersection of the pressure waves seems to be the main event responsible for the vortex bursting.

Fig. 16 illustrates the evolution of the profiles of normalized pressure \( \left( (p-p_{min})/(p_\infty-p_{min}) \right) \).

It shows the waves travelling at an approximately constant speed until they eventually intersect at the midpoint. The normalized pressure increases significantly more at the intersection location than other locations along the vortex axis where the waves have passed and results in an important enlargement of the vortex core, approximately \( 2.5 \times r_{c0} \). The circulation remains constant during the entire simulation.
Large-Eddy Simulation  The LES of the same configuration exhibits similar dynamics. Two waves travelling in opposite direction intersect (see Fig. 15 $t/T_{rot} = 1.02$ and $2.03$). At $t/T_{rot} = 3.05$ the helical instability starts to develop at the two locations near the point where the pressure waves have originally been generated. The two waves intersect, which results in a drastic event leading to an abrupt change in the vortex flow. It also leads to the loss of coherence and the creation of an expanding annular structure ($t/T_{rot} > 3.05$) at the location where the waves have intersected. The helical instability amplifies in the region along the vortex axis where the axial velocity has significantly increased and fine helical structures can be distinguished at $t/T_{rot} = 4.06$ and $5.08$. Eventually, the helical modes saturate and the flow field locally transitions, which may be seen at $t/T_{rot} = 6.01$.

VI. CROW INSTABILITY AND VORTEX BURSTING

A different flow configuration has been simulated to study the helical instability and the phenomenon of vortex bursting in a more realistic framework using the LES approach. In this study attention is paid to the combination of the reconnection phenomenon and the presence of axial velocity in the vortices. The configuration consists of two counter-rotating vortices with axial velocity in the core (left over from the roll-up process) subjected to the development of the Crow instability which is a natural sinusoidal instability developing in a counter-rotating vortex pair (see also Crow\textsuperscript{32}). The instability amplifies until saturation and reconnection of the two vortices takes place leading to the generation of a vortex ring. The reconnection process and the combination of the axial velocity in the vortex cores is suspected in some occasions to be the source of helical instability and vortex bursting.

The initial condition for the more realistic simulation is a pair of counter-rotating q-vortices (Eq. (16)). They are characterized by an initial nondimensional core size of $(a/b)_{0} = 0.15$, with $a$ the dispersion radius of vorticity and $b$ the initial horizontal separation between the two vortices. The generated initial condition is a two-dimensional vortex flow which is a quasi-steady solution of the Euler equations. The two-dimensional flow is then extruded in
the axial direction to obtain the three-dimensional initial condition for the LES simulations. The simulations were carried out with a very weak peak axial velocity $W_{max}$ in the vortex core and thus with a relatively large value of $q$. In general, trailing vortices are observed to have an axial velocity deficit in the core (wake type) directly after roll-up of the vorticity sheet trailing of the trailing edge of the wing. This deficit substantially decreases downstream (see Jacquin et al.\textsuperscript{34}) and therefore two values of $q$ have been considered, namely $q \approx 5.23$ and $10.45$, which correspond to values found in aircraft wake vortices in the far field.

The axial domain size scales with the wavelength of the Crow instability (see Widnall\textsuperscript{35}) such that only one wavelength is simulated. For $a/b = 0.15$, this means a wavelength of $\lambda_{Crow} \approx 7.8b$. Note that during the extrusion a weak sinusoidal forcing has been imposed in the axial direction in order to trigger the Crow instability.

The LES simulations have been performed with a Reynolds number based on the circulation of $Re_\Gamma = 10^7$.

A. Global dynamics

This section discusses the global dynamics of the vortex pair with axial velocity and which is subjected to the Crow instability and the observation of the helical instability and vortex bursting. For the first value ($q \approx 5.23$) a grid convergence study has been conducted and the second larger value ($q \approx 10.45$) was chosen to determine the influence of the axial velocity. Table VI sums up the characteristic parameters of the simulation and the computational domain. The transverse grid resolution was taken equal for all simulations, namely $n_x \times n_y = 257 \times 211$, while the axial grid resolution was varied, for the largest axial resolution this led to a computational domain counting a number of $\approx 17 \cdot 10^6$ grid points ($n_z = 313$).

The global dynamics are shown using the isosurfaces of vorticity magnitude as a function of the nondimensional descent time defined as

$$t^* = \frac{t V_d}{b_0} \quad (17)$$

18
with $b_0$ the initial vortex spacing and $V_d = \Gamma/(2\pi b_0)$ the descent velocity of the vortex pair due to mutual induction.

The first run ($\text{Crow}_{1a}$) of the grid convergence study was conducted with $n_z = 40$ and the global dynamics (shown in Fig. 17) shows the linear phase of the Crow instability characterized by the growing sinusoidal deformation of the vortex axis. Eventually reconnection takes place and a vortex ring is formed. In the vortex ring at $t^* = 6.37$ a clear difference is found in the size of the vortex core which shows the existence and the propagation of a wave along the annular vortex axis.

The reconnection process is the probable source of the generation of the pressure wave which is generated at both sides of the vortex ring where reconnection took place. As was mentioned in Section III B the pressure wave causes the generation of axial velocity in the vortex core and might therefore promote the development of the helical instability.

The axial resolution for Run $\text{Crow}_{1a}$ was not sufficient to capture the instability and was therefore doubled several times (see Table VI). The sensitivity of the computation to the axial resolution with respect to the development of the helical instability is evidenced in Fig. 18 where the results of the simulations with refined grids are illustrated. The figure shows the structure of the vortex ring at time $t^* = 6.37$. For the simulation $\text{Crow}_{1a}$ and $\text{Crow}_{1b}$ one may recognize structures observed in Section IV A related to the helical instability. It was observed considering isosurfaces of vorticity magnitude that the helical structures surround a concentrated core (see Fig. 11 at $t^* = 6.01$). A similar structure can be recognized by looking at the left half of the vortex ring where a concentrated core structure is located near to a somewhat larger hollow core. A surface wrinkle can also be observed for $\text{Crow}_{1b}$ on the right half of the vortex ring. It marks the location where the vortex core has increased in size as a result of the pressure wave generated during reconnection at the right side of the vortex ring. Between the location where a hollow core can be distinguished and the location where an increased vortex core is observed (surface wrinkle) a striated vortex core surface can be seen for simulation $\text{Crow}_{1c}$, evidencing the helical instability. The axial resolution does not allow to capture properly the instability. Run $\text{Crow}_{1d}$ however, shows that with
$n_z = 313$, sufficient axial resolution is attained to capture the helical instability developing clearly from both sides of the vortex ring. The significant increase in core size ($t^* = 6.37$) is due to the well-developed helical instability and the intersection of the two pressure waves generated at both reconnection locations causes vortex bursting.

Note that the spatial structure of the vortex ring does not appear to be symmetric although the pressure waves generated at the two reconnection locations are expected to travel at the same speed. This is probably due to the remnants of the axial velocity in the vortex core introduced at the initialization.

In Fig. 19, the pursuit of this simulation shows the abrupt change in vortex core structure which proves that reconnection of the vortices may lead to the generation of pressure waves that intersect resulting in helical instability and vortex bursting.

The results found with Run Crow$_2$ (not shown here) are similar to the results of Run Crow$_{1d}$ and revealed the generation of pressure waves travelling in the core of the vortex ring. They are responsible for the occurrence of the helical instability and show the drastic change of flow characteristics at the location of intersection of the waves, indicating the occurrence of vortex bursting. This leads to the conclusion that even with a very weak initial axial velocity the reconnection process is the main phenomenon responsible for the generation of pressure waves and the consequent vortex bursting. Consequently, one may state that the wave propagation is a robust phenomenon which is responsible for the development of the helical instability and subsequent vortex bursting as is repeatedly observed in the smoke visualizations of aircraft wakes during flight tests.

**VII. CONCLUSION**

The present paper presents DNS and LES simulations of wave propagation and vortex bursting in aircraft wake vortices considering a simple configuration of a single vortex without background perturbations. A pressure wave is generated which propagates at an approximately constant speed along the vortex causing the production of axial velocity in
the core. The propagation speed is weakly dependent of the initial condition. However, the magnitude of the induced axial velocity in the core is strongly related to the initial condition. Consequently, depending on the configuration, the amount of swirl, which is directly related to the maximum of axial velocity in the vortex core, may reach values that lie in a region resulting in the amplification of modes corresponding to the helical instability. Other consequences of the generation and propagation of pressures wave is the increase of the vortex core size and decrease of tangential velocity while maintaining a constant circulation.

Direct Numerical and Large-Eddy simulations are conducted to study the effect of the propagation of pressure waves in the vortex core while considering two different vortex configurations. Background perturbations were added in order to trigger three-dimensional instabilities. The configuration resulting in the generation of a single pressure wave showed that the axial velocity generated in the wake of the wave reaches values leading to the amplification of the helical instability. The global dynamics confirm the amplification of the azimuthal modes. Further analysis showed that the vortex experienced an increase of the core and a decrease of tangential velocity while maintaining a constant circulation.

For the configuration with two intersecting pressure waves the vortex bursting phenomenon occurs. At the intersection location an abrupt change in flow topology was observed in the vortex core, namely a significant core growth. However, during vortex bursting the circulation calculated at the bursting location remained constant. The results present an explanation for observations done during some flight tests where vortex bursting was visualized by smoke in the form of so-called ‘pancakes’.

An additional set of computations was made for a realistic aircraft wake vortex pair with axial velocity subjected to Crow instability. The helical instability and vortex bursting were observed and attributed to the reconnection process causing the generation of multiple pressure waves. Even though, the presence of the axial velocity in the vortex core contributes to the phenomena it is not a necessary condition. Finally, a turbulent vortex ring is formed with a substantially enlarged core at two locations on the vortex ring.

The present study showed that a simple model based on a single vortex is sufficient to ex-
tract the particular physical dynamics as well as deducing the evolution of characteristic flow properties caused by the propagation of pressure waves. The results obtained with the simple configuration are representative because the simulation of a more realistic configuration governed by the development of the Crow instability in a vortex pair recovered very similar dynamics.
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TABLE III. Average propagation speed of the solitary pressure wave.

TABLE IV. Parameters of the simulations.

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TABLE VI. Parameters of the simulations and the computational domain.
<table>
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<tr>
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<th>$L_{\text{uniform}}/r_c$</th>
<th>$n_{p_{\text{uniform}}}$</th>
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PHYS. FLUIDS – Moet – table II
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PHYS. FLUIDS – Moet – table VI
FIG. 1. Schematic sketch of the vortex bursting phenomenon (courtesy of Spalart\(^3\)).

FIG. 2. Computational domain with initial condition and boundary conditions.

FIG. 3. Schematic sketch of the variation of the vortex radius along the axis.

FIG. 4. Temporal evolution of the profile of minimum pressure in the vortex core.

FIG. 5. Propagation speed of the solitary pressure wave.

FIG. 6. Isosurfaces of vorticity magnitude shown for time instants \(t/T_{rot} = 1.62, 3.05, 4.47 \& 6.1\) with \(T_{rot} = \frac{2\pi r_c}{V_{\theta_{max}}}\). The annular structure indicating the front of the pressure wave is marked with an arrow.

FIG. 7. The generation of azimuthal vorticity \(\omega_\theta\) visualized by isocontours and an isosurface of \(\omega_\theta\).

FIG. 8. History of the swirl parameter in the vortex core for three stations located at \(z_{\text{station}}^* = 20.625, 27.5 \& 34.375\) as a function of \(T_{rot} = \frac{2\pi r_c}{V_{\theta_{max}}}\). Note that: horizontal dashed line denotes \(q = 1.5\).

FIG. 9. Isosurfaces of vorticity magnitude \(|\omega|\) and azimuthal vorticity \(\omega_\theta\) of run DNS\(_1\) \((T_{rot} = \frac{2\pi r_c}{V_{\theta_{max}}}\).

FIG. 10. Different azimuthal modes \((k_\theta)\) visualized by isosurfaces (below) and isocontours (above) of azimuthal vorticity \(\omega_\theta\) and the radial velocity perturbation for DNS\(_1\) at \(t/T_{rot} = 4.06\).

FIG. 11. Isosurfaces of vorticity magnitude \(|\omega|\) and azimuthal vorticity \(\omega_\theta\) of run LES\(_1\) \((T_{rot} = \frac{2\pi r_c}{V_{\theta_{max}}}\).

FIG. 12. Different azimuthal modes \((k_\theta)\) visualized by isosurfaces (below) and isocontours (above) of azimuthal vorticity \(\omega_\theta\) and the radial velocity perturbation for LES\(_1\) at \(t/T_{rot} = 4.06\).

FIG. 13. Sketch of the variation of the vortex radius along the axis.

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FIG. 16. Temporal evolution of the profile of minimum pressure in the vortex core for simulation DNS$_2$.

FIG. 17. Isosurfaces of vorticity magnitude showing the development of the Crow instability (Run Crow$_{1a}$) as a function of the nondimensional descent time, $t^* = tV_d/b_0 = 2.23, 3.61, 4.99$ & $6.37$.

FIG. 18. Isosurfaces of vorticity magnitude showing the different features captured by varying the axial grid resolution for the simulations Crow$_{1a-1d}$, at $t^* = tV_d/b_0 = 6.37$.

FIG. 19. Isosurfaces of vorticity magnitude for simulation Crow$_{1d}$, at $t^* = tV_d/b_0 = 6.48$ and $6.59$. 
swirl parameter $q$

$z^* = 20.625$
$z^* = 27.5$
$z^* = 34.375$

stable
unstable

PHY. FLUIDS – Moet – figure 8
swirl parameter $q$

run 1: $r_c = 2.0$
run 2: $r_c = 1.75$
run 3: $r_c = 1.5$
run 4: $r_c = 1.25$

stable
unstable

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$
t/T$_{rot}$
\[
\left| \omega \right| t/T_{rot} = 2.03 \\
\left| \omega \right| t/T_{rot} = 4.06 \\
\left| \omega \right| t/T_{rot} = 6.1
\]
(a) $\omega_\theta$

(b) $u'_{rad}$
Figure 14
(a) $t/T_{rot} = 1.01$
(b) $t/T_{rot} = 2.03$
(c) $t/T_{rot} = 3.05$
(d) $t/T_{rot} = 4.06$
(e) $t/T_{rot} = 5.08$
(f) $t/T_{rot} = 6.1$
(a) $t/T_{rot} = 1.02$

(b) $t/T_{rot} = 2.03$

(c) $t/T_{rot} = 3.05$

(d) $t/T_{rot} = 4.06$

(e) $t/T_{rot} = 5.08$

(f) $t/T_{rot} = 6.1$
(a) $t^* = 2.23$

(b) $t^* = 3.61$

(c) $t^* = 4.99$

(d) $t^* = 6.37$
(a) Crow\textsubscript{1a}: $n_z = 40$

(b) Crow\textsubscript{1b}: $n_z = 79$

(c) Crow\textsubscript{1c}: $n_z = 157$

(d) Crow\textsubscript{1d}: $n_z = 313$
(a) $t^* = 6.48$

(b) $t^* = 6.59$