

# Boundary conditions for acoustic eigenmode computations in gas turbine combustion chambers

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Understanding and predicting acoustic instabilities in gas turbine combustion chambers requires the knowledge of the acoustic behaviour of all the elements feeding the combustion chamber (characterized by their impedance). Inlets and outlets of chambers are often represented as one-dimensional ducts and existing methods to evaluate impedances of choked and unchoked nozzles are described: (1) analytical formulae,<sup>1-3</sup> (2) numerical methods using the linearized Euler equations and a finite-difference solver in Fourier space and (3) full space-time solver where the response of the nozzle is studied by forcing its inlet (or outlet) and measuring its response in the time domain. These three methods are compared in reference cases (a straight duct and a subsonic distributor). Practical implications for gas turbines are then discussed: in such cases, if the diffusers and distributors connected to the chamber are not choked, it is shown that solving the acoustic problem becomes very difficult as compressors and turbines impedances are usually not known. This paper finally shows how these impedances control the acoustic oscillations of a combustion chamber by presenting an example of acoustic eigenmode calculations in a realistic gas turbine, using a Helmholtz solver and various impedances for the inlet and outlet of the combustor.

## Nomenclature

$\dot{m}$	Mass flow rate ( $kg/m^{-3}$ )
$\mathcal{A}$	Acoustic wave amplitude ( $Pa$ )
T	Transmission coefficient

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$\vec{n}$	Unit normal vector
$\vec{x}$	Position vector
$A$	Section ( $m^2$ )
$c$	Sound speed ( $m/s$ )
$c_p$	Mass specific heat capacity for constant pressure ( $J/kg/K$ )
$c_v$	Mass specific heat capacity for constant volume ( $J/kg/K$ )
$f$	Frequency ( $Hz$ )
$g_1$	Real-valued function
$g_2$	Real-valued function
$h$	Space step ( $m$ )
$K$	Real constant
$k$	Wave number ( $rad/m$ )
$L$	Nozzle length ( $m$ )
$N$	Number of discretisation nodes
$p$	Pressure ( $Pa$ )
$p_t$	Total pressure ( $Pa$ )
$R$	Reflection coefficient
$r$	Molar perfect gas constant ( $J/mol/K$ )
$s$	Entropy ( $J/kg/K$ )
$T$	Temperature ( $K$ )
$t$	Time ( $s$ )
$W$	Molar mass ( $kg/mol$ )
$W_{ac}$	Total acoustic energy flux
$Y$	Reduced acoustic admittance
$Z$	Reduced acoustic impedance

#### *Subscripts*

1	Upstream value
2	Downstream value
$x$	x-axis value
$y$	y-axis value

#### *Special notations*

$(\cdot)'$	Fluctuating values
$[A], [C]$	$N \times N$ real matrices
$[B]$	$N \times N$ complex matrix
$[D]$	$2 \times 2$ invertible matrix

$\bar{(\cdot)}$	Mean values
$\delta^-$	First order upwind differences
$\Delta_0$	Second order central differences
$\{p'\}$	Acoustic pressure amplitude $N$ vector

#### *Superscripts*

$(j)$	Discretisation index
$+$	Downstream travelling wave
$-$	Upstream travelling wave
$\epsilon$	Parameter
$I$	Incident wave
$R$	Reflected wave

#### *Greek letters*

$\delta$	Adimensionalized volume mass fluctuation
$\gamma$	Polytropic gas constant
$\lambda$	Wave length ( $m$ )
$\nu$	Adimensionalized velocity fluctuation
$\Omega$	Computational domain
$\omega$	Angular frequency ( $rad/s$ )
$\Omega'_T$	Heat release amplitude( $rad/s$ )
$\phi$	Real-valued function
$\rho$	Volume mass ( $kg/m^3$ )
$\varphi$	Adimensionalized pressure fluctuation

## 1 Introduction

Combustion instabilities (CI) are an important research theme for the gas turbine industry.<sup>4</sup> The central role played by acoustics in many combustion instability scenarios has been the subject of long controversies but is now more and more widely recognized. Two classes of methods can be used today to study oscillations in combustors.<sup>5-9</sup>

Large Eddy Simulation (LES) is becoming a standard tool to study the stability of many modern combustion devices such as aero or industrial gas turbines, rocket engines or industrial furnaces. When LES is performed using compressible Navier-Stokes equation solvers, it directly provides both the time-resolved flow and the acoustic field. Multiple recent papers have demonstrated that LES accurately predicts mean flow fields in turbulent combustors.<sup>10-13</sup> Other studies<sup>14-16</sup> show that LES of compressible flows also correctly predicts

the acoustic fluctuations in turbulent combustors and therefore can be used to study CI. However, a major drawback of LES remains its cost and especially when describing compressible flows (small time steps). Moreover, extracting an acoustic mode from a turbulent flow is a hard task, which increases the difficulty to point out the phenomena which control the appearance of CI.

Since LES remains expensive, developing alternative approaches is an interesting path: studying the linear stability of combustors is the most obvious method. This is usually achieved by Helmholtz codes, which solve the linearized wave equation to predict the frequencies and the growth rates of the acoustic modes of the combustor.<sup>5,7,8,17,18</sup> These codes run much faster than LES and work in the frequency domain while LES works in the time domain. They rely on crude modeling assumptions for the base flow (which is often supposed to be the mean flow) and for the effect of the flame on the acoustic field (using a flame transfer function). They are well suited to parametric studies, for example to optimize damping devices or combustion chamber shapes which minimize oscillations.

Both compressible LES and Helmholtz solvers naturally include acoustics and will give acoustic fields correctly provided crucial and often forgotten information is available: the acoustic impedances of all inlets, outlets and walls<sup>1</sup>. In real engines, the combustion chamber is fed by a diffuser and flows into a distributor (Fig.1). A central question to evaluate impedances is to know which elements of the engine really determine the impedance of the diffuser and of the distributor. The latter are usually choked so that most authors consider that the throat is a proper place to specify acoustic boundary condition. However, in certain cases, distributors are not choked (during start-up for example where instabilities are often observed) and in most cases, diffusers are not choked. For those situations, the evaluation of impedances can become a critical issue. The objectives of this paper are:

- to review analytic methods to evaluate impedances of one-dimensional ducts and extend the approach of Marble and Candel,<sup>2</sup>
- to present a tool able to predict the impedance of combustor inlets and outlets by approximating them as nozzle flows,
- to give simple academic examples of impedance calculations,
- to discuss the conditions that allow an evaluation of the impedances of a real gas turbine, taking into account only the geometry of the diffuser and distributor as well

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<sup>1</sup>Rigid walls obviously correspond to simple zero velocity conditions which are simple to implement but gas turbine chambers often use multiperforated walls to cool the combustion chamber and these walls have complex impedances.<sup>19</sup> Multiperforated walls are not considered here.

as the mean flow.

- to present an example of the importance of inlet and outlet impedances in the computation of modes in a real gas turbine chamber.

Reduced acoustic impedances measure the ratio between the fluctuations of acoustic pressure and velocity at a given frequency :

$$Z(\omega) = \frac{p'}{\bar{\rho} c \vec{u}' \cdot \vec{n}}, \quad (1)$$

Impedance is a complex-valued function of  $\omega$ . In fluids at rest, its real part is closely linked to the acoustic energy flux  $p' \vec{u}'$  (a zero real part means the acoustic flux is zero since pressure and velocity have opposite phases).<sup>9</sup> It is well known in some particular cases :  $Z = 1$  for an infinitely long duct,  $Z = 0$  for an outlet at constant pressure and  $Z \rightarrow \infty$  on a rigid wall. In the last relation, impedance is often replaced by its inverse called admittance and defined as :  $Y = 1/Z$ .

There are three main methods to evaluate impedances (figure 2). First, in simple cases and at low frequencies, analytical formulations may be available<sup>2,3</sup> and are described in Section 2. Secondly, for cases where the geometry of the considered component (diffuser or distributor) becomes complex and the frequency is not low, the one-dimensional Euler equations can be linearized around the mean state. Impedances at one end of the duct can be obtained as a function of the geometry only if the flow is choked, or as a function of the geometry and the impedance at the other side of the component if the flow remains subsonic. Such methods (called here FD method because a finite difference solver is required to solve the linearized equations in Fourier space) are presented in Section 3.3. Thirdly, a space-time solver (Euler equations for compressible flows) can also be used to force the component with harmonic waves and measure its response in terms of impedances in the time domain. This method is expensive, it is described rapidly in Section 5 and used here as a reference technique to evaluate the precision of the FD method. Finally, section 7 illustrates the role of impedances (or admittances): the principles of Helmholtz solvers are briefly described and the influence of the boundary conditions is highlighted in the case of a gas turbine chamber including all parts (casing, swiler, combustion chamber, dilution holes, choked outlet nozzle).

## 2 Analytical formulations for impedances of compact nozzles

For low frequencies, several analytical formulations for reflexion coefficients or impedances have been developed for simplified cases. Despite strong hypothesis, they remain accurate, even when nonlinear effects occur.<sup>20</sup> The acoustic waves are here considered as plane waves since their wavelengths are large in comparison with the characteristic lengths of the ducts. The general form of acoustic pressure and velocity in constant section ducts can be written:<sup>2,9</sup>

$$\frac{p'}{\gamma \bar{p}} = \mathcal{A}^+ \exp(ik^+x - i\omega t) + \mathcal{A}^- \exp(-ik^-x - i\omega t) \quad (2)$$

$$\frac{u'}{\bar{c}} = \mathcal{A}^+ \exp(ik^+x - i\omega t) - \mathcal{A}^- \exp(-ik^-x - i\omega t) \quad (3)$$

where  $k^+ = \omega/(\bar{c} + \bar{u})$  and  $k^- = \omega/(\bar{c} - \bar{u})$ . Determining the acoustic boundary conditions requires the determination of an impedance as defined by Eq.(1) or a reflection coefficient  $R$  given by:

$$R = \frac{\mathcal{A}^R}{\mathcal{A}^I} \quad (4)$$

The relation between  $R$  and  $Z$  depends on the orientation of  $\vec{n}$ . For  $\vec{n}$  pointing towards positive abscissae, it is written:

$$R = \frac{Z - 1}{Z + 1} \quad (5)$$

The simplest approximation has been described by Rienstra,<sup>21</sup> who showed that the reflection coefficient of an open outlet is  $-1$  (or zero impedance) for zero Strouhal number flows. This simplification can be used for academic experimental setups, but in the context of gas turbines, its application remains limited as combustion chambers rarely flow into open atmosphere but into high pressure stators followed by turbines.

A first approach to determine reflection coefficients or impedances of these passages is to consider them as compact nozzles (figure 3), as proposed by Marble and Candel.<sup>2,3</sup> When the characteristic scale of the fluctuations is large compared to the duct length (i.e  $\lambda \gg L$ ), the duct can be seen as a discontinuity in the wave propagation. Thus, it is not necessary to solve the flow in the duct and its influence can be replaced by matching conditions (built in the same way as jump conditions through a shock wave).<sup>2</sup> For example, in the distributor following the combustion chamber of aircrafts, a typical length scale is 10 cm (the length

of the distributors vanes). For burnt gases at 1200 K with a sound speed of 700 m/s, a reasonable compactness condition ( $\lambda \approx 10L$ ) leads to a critical frequency  $f_c$  of the order of 700 Hz : the distributor nozzle can then be considered as compact for all frequencies up to 700 Hz. Marble and Candel<sup>2</sup> have used conservation laws in subsonic and choked compact nozzles to derive reflection coefficients. The conservation of mass flow ( $\dot{m} = \rho u A$ ) between inlet (side 1) and outlet (side 2) can be written:

$$\dot{m}_1 = \dot{m}_2. \quad (6)$$

If there is neither heat losses nor normal shock, the total temperature, defined by  $T_t = T \left(1 + \frac{\gamma-1}{2} M^2\right)$ , with  $M = u/c$ , is also conserved between side 1 and side 2, leads to:

$$T_{t,1} = T_{t,2}. \quad (7)$$

The two conservation equations Eq.(6) and (7) imply that:

$$\frac{\dot{m}'_1}{\dot{m}_1} = \frac{\dot{m}'_2}{\dot{m}_2} \quad \text{and} \quad \frac{T'_{t,1}}{T_{t,1}} = \frac{T'_{t,2}}{T_{t,2}}, \quad (8)$$

with:

$$\frac{\dot{m}'}{\dot{m}} = \frac{1}{M} \frac{u'}{c} + \frac{\rho'}{\rho}, \quad (9)$$

and :

$$\frac{T'_t}{T_t} = \frac{1}{1 + \frac{\gamma-1}{2} M^2} \left[ \gamma \left( \frac{p'}{\gamma \bar{p}} \right) - \frac{\rho'}{\bar{\rho}} + (\gamma - 1) \bar{M} \frac{u'}{\bar{c}} \right]. \quad (10)$$

## 2.1 Choked distributor

If the distributor is choked, and if the frequency of perturbations is low, the flow rate can be assumed to match its choked value at all times. The mass flow rate is then directly linked to the thermodynamic state upstream of the nozzle:

$$\dot{m} = \frac{p_{t,1}}{\sqrt{\gamma r T_{t,1}}} A_* \gamma \left( \frac{\gamma + 1}{2} \right)^{-\frac{\gamma+1}{2(\gamma-1)}} \quad (11)$$

with  $p_{t,1} = p_1 \left(1 + \frac{\gamma-1}{2} M_1^2\right)^{\frac{\gamma}{\gamma-1}}$  is the total pressure upstream of the nozzle (Fig. 3). The logarithmic derivative of  $\dot{m}_1$  can be calculated from Eq.(11), using the fractional variation

of total pressure at side 1 :

$$\frac{p'_{t,1}}{\bar{p}_{t,1}} = \frac{1}{1 + \frac{\gamma-1}{2}\bar{M}_1^2} \left[ \gamma \left( 1 - \frac{1}{2}\bar{M}_1^2 \right) \left( \frac{p'_1}{\gamma\bar{p}_1} \right) + \gamma\bar{M}_1 \frac{u'_1}{\bar{c}_1} + \frac{1}{2}\gamma\bar{M}_1^2 \frac{\rho'_1}{\bar{\rho}_1} \right], \quad (12)$$

and Eq.(10) written at side 1. This leads to<sup>2</sup>:

$$\frac{\dot{m}'_1}{\bar{m}_1} = \frac{p'_{t,1}}{\bar{p}_{t,1}} - \frac{1}{2} \frac{T'_{t,1}}{T_{t,1}} = \frac{1}{1 + \frac{\gamma-1}{2}\bar{M}_1^2} \left[ \frac{\gamma}{2} (1 - \bar{M}_1^2) \left( \frac{p'_1}{\gamma\bar{p}_1} \right) + \frac{\gamma+1}{2} \bar{M}_1 \frac{u'_1}{\bar{c}_1} + \frac{1}{2} (1 + \gamma\bar{M}_1^2) \frac{\rho'_1}{\bar{\rho}_1} \right]. \quad (13)$$

Then, combining Eq.(9) written at side 1 and (13) entails:

$$\frac{u'_1}{\bar{u}_1} = \frac{1}{2} \frac{T'_1}{T_1} \quad \text{or} \quad \frac{u'_1}{\bar{c}_1} - \frac{\gamma}{2} \bar{M}_1 \left( \frac{p'_1}{\gamma\bar{p}_1} \right) + \frac{1}{2} \bar{M}_1 \frac{\rho'_1}{\bar{\rho}_1} = 0, \quad (14)$$

which is equivalent to write that the fluctuating Mach number is zero :

$$M'_1 = \bar{M}_1 \left( \frac{u'_1}{\bar{u}_1} - \frac{1}{2} \frac{T'_1}{T_1} \right) = 0.$$

Supposing that there is no incoming entropy wave  $s'_1 = 0$ , so that :  $\frac{\rho'_1}{\rho_1} = \frac{p'_1}{\gamma\bar{p}_1}$ , Eq.(14) leads to an expression for the acoustic inlet impedance :

$$Z_1 = \frac{2}{\bar{M}_1(\gamma-1)} \quad \text{hence} \quad R_1 = \frac{\mathcal{A}_1^-}{\mathcal{A}_1^+} = \frac{1 - \frac{1}{2}(\gamma-1)\bar{M}_1}{1 + \frac{1}{2}(\gamma-1)\bar{M}_1} \quad (15)$$

$\mathcal{A}_1^-$  does not depend on the downstream flow as no wave can come from downstream of the throat. Eq.(15) also shows that a choked nozzle behaves nearly like a rigid wall ( $R = 1$ ) for low upstream Mach number ( $M_1 \rightarrow 0$ ), which is a commonly used boundary condition for choked nozzles.

## 2.2 Choked diffuser

The case of a choked diffuser is particular. Diffusers generally disembody in a chamber casing or a plenum, where the flow is subsonic: if the diffuser is choked, there is also a shock between its throat and its outlet. Stow et al<sup>22</sup> present an analysis of the acoustic response of a choked diffuser with a normal shock in the expansion part of the duct. Assuming there is no disturbance coming from upstream ( $\mathcal{A}_1^+ = 0$ ), they showed that:

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<sup>2</sup>This relation can also be written at side 2 but this has no interest here.



$$R_2 = \frac{1 - \gamma \bar{M}_2 + (\gamma - 1) \bar{M}_2^2}{1 + \gamma \bar{M}_2 + (\gamma - 1) \bar{M}_2^2}. \quad (16)$$

A generalization for transmission coefficients has also been derived by Moase et al.,<sup>23</sup> assuming there can be perturbation upstream of the shock and  $\mathcal{A}_1^+$  is known.

## 2.3 Unchoked distributor / diffuser

In the case of a subsonic flow in the nozzle, information can propagate in both directions. Thus, information on the other side of the duct is needed to determine reflection coefficients. Conservation of mass flow rate and total temperature (Eq.(8) with Eq.(9) and (10)) are used to write a system of two equations with 6 unknowns :  $u'_1, u'_2, \rho'_1, \rho'_2, p'_1$  and  $p'_2$ . Supposing again that there is no entropy wave ( $s'_1 = s'_2 = 0$ ) to eliminate  $\rho'_1$  and  $\rho'_2$  gives the following system of two equations with 4 unknowns:

$$\frac{1}{\bar{M}_1} \frac{u'_1}{\bar{c}_1} + \frac{p'_1}{\gamma \bar{p}_1} = \frac{1}{\bar{M}_2} \frac{u'_2}{\bar{c}_2} + \frac{p'_2}{\gamma \bar{p}_2} \quad (17)$$

$$\frac{1}{1 + \frac{\gamma-1}{2} \bar{M}_1^2} \left[ \frac{p'_1}{\gamma \bar{p}_1} + \bar{M}_1 \frac{u'_1}{\bar{c}_1} \right] = \frac{1}{1 + \frac{\gamma-1}{2} \bar{M}_2^2} \left[ \frac{p'_2}{\gamma \bar{p}_2} + \bar{M}_2 \frac{u'_2}{\bar{c}_2} \right] \quad (18)$$

Writing pressure and velocity perturbations as waves (Eq.(2) and (3)) in Eq.(17) and (18) gives :

$$\frac{1}{\bar{M}_1} (\mathcal{A}_1^+ - \mathcal{A}_1^-) + \mathcal{A}_1^+ + \mathcal{A}_1^- = \frac{1}{\bar{M}_2} (\mathcal{A}_2^+ - \mathcal{A}_2^-) + \mathcal{A}_2^+ + \mathcal{A}_2^- \quad (19)$$

$$\frac{1}{1 + \frac{\gamma-1}{2} \bar{M}_1^2} [\mathcal{A}_1^+ + \mathcal{A}_1^- + \bar{M}_1 (\mathcal{A}_1^+ - \mathcal{A}_1^-)] = \frac{1}{1 + \frac{\gamma-1}{2} \bar{M}_2^2} [\mathcal{A}_2^+ + \mathcal{A}_2^- + \bar{M}_2 (\mathcal{A}_2^+ - \mathcal{A}_2^-)] \quad (20)$$

If the outlet reflection coefficient  $R_2 = \frac{\mathcal{A}_2^-}{\mathcal{A}_2^+}$  of the distributor is known, system (19)-(20) gives the reflection coefficient at the inlet :

$$R_1 = \frac{\mathcal{A}_1^-}{\mathcal{A}_1^+} = \frac{\bar{M}_1 + 1}{\bar{M}_1 - 1} \frac{(\bar{M}_1 - \bar{M}_2)(\bar{M}_2 + 1)(1 - \frac{\gamma-1}{2} \bar{M}_1 \bar{M}_2) + (\bar{M}_1 + \bar{M}_2)(\bar{M}_2 - 1)(1 + \frac{\gamma-1}{2} \bar{M}_1 \bar{M}_2) R_2}{(\bar{M}_1 + \bar{M}_2)(\bar{M}_2 + 1)(1 + \frac{\gamma-1}{2} \bar{M}_1 \bar{M}_2) + (\bar{M}_1 - \bar{M}_2)(\bar{M}_2 - 1)(1 - \frac{\gamma-1}{2} \bar{M}_1 \bar{M}_2) R_2}, \quad (21)$$

while the transmission coefficient is:<sup>2</sup>

$$T_1 = \frac{\mathcal{A}_2^+}{\mathcal{A}_1^+} = \frac{2 \bar{M}_2}{1 + \bar{M}_2} \frac{(\bar{M}_1 + 1)(1 + \frac{\gamma-1}{2} \bar{M}_2^2)}{(\bar{M}_1 + \bar{M}_2)(1 + \frac{\gamma-1}{2} \bar{M}_1 \bar{M}_2)} \quad (22)$$

The same derivation for a diffuser gives  $R_2$  as a function of  $R_1 = \frac{\mathcal{A}_1^+}{\mathcal{A}_1^-}$ :

$$R_2 = \frac{\mathcal{A}_2^+}{\mathcal{A}_2^-} = \frac{1-\bar{M}_2}{\bar{M}_2+1} \frac{(\bar{M}_2-\bar{M}_1)(\bar{M}_1-1)(1-\frac{\gamma-1}{2}\bar{M}_1\bar{M}_2)+(\bar{M}_1+\bar{M}_2)(1+\bar{M}_1)(1+\frac{\gamma-1}{2}\bar{M}_1\bar{M}_2)R_1}{(\bar{M}_1+\bar{M}_2)(1-\bar{M}_1)(1+\frac{\gamma-1}{2}\bar{M}_1\bar{M}_2)+(\bar{M}_1-\bar{M}_2)(1+\bar{M}_1)(1-\frac{\gamma-1}{2}\bar{M}_1\bar{M}_2)R_1} \quad (23)$$

and the transmission coefficient is:<sup>2</sup>

$$\mathcal{T}_2 = \frac{\mathcal{A}_1^-}{\mathcal{A}_2^-} = \frac{2\bar{M}_1}{1-\bar{M}_1} \frac{(1-\bar{M}_2)(1+\frac{\gamma-1}{2}\bar{M}_1^2)}{(\bar{M}_1+\bar{M}_2)(1+\frac{\gamma-1}{2}\bar{M}_1\bar{M}_2)} \quad (24)$$

Those results, summarized in Table 1, generalize the formulae given by Marble and Candel:<sup>2</sup> for  $R_2 = 0$ , Eq.(21) is identical to their results, while Eq.(23) matches their results when  $R_1 = 0$ . A similar work, introducing entropy disturbances, has recently been done by Moase et al.<sup>23</sup>

### 3 Numerical evaluation of impedances using the linearized Euler equations

The hypothesis of compacity is only applicable when the acoustic wave length is much larger than the nozzle length. This is too strict an approximation as soon as perturbation frequencies are no more small. The problem has then to be solved in more details, e.g mean flow and then fluctuating pressures and velocities have to be calculated in the duct <sup>3</sup>. The first consequence of taking finite length ducts into account is that impedances now depend on frequency, which is a major improvement in comparison with the previous approximations.

The inlet impedance (or outlet impedance) of a distributor (diffuser) can be obtained by solving the linearized Euler equations under some assumption: the mean flow in these ducts is quasi one-dimensional and isentropic, all geometrical complexities are included in the one-dimensional cross section evolution of the inlet or outlet ducts, which are described like simple nozzles, and, finally, the flow is a perfect gas with constant composition (no chemistry) and constant heat capacities. With these assumptions, the Euler equations are written:

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<sup>3</sup>Marble and Candel present an analytical resolution<sup>2</sup> for a finite length choked nozzle in which the mean axial velocity varies linearly along the duct. This last approximation provides the impedance everywhere in the nozzle. Nevertheless, in general, the mean velocity evolution is not linear and the acoustic problem must be solved numerically.

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho u A}{\partial x} = 0 \quad (25)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (26)$$

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0 \quad (27)$$

The system is completed by the equation of state for perfect gas :  $p = \rho \frac{R}{W} T$ .

### 3.1 Acoustics in a nozzle

For the linearization of system (25) to (27),<sup>1-3,24</sup> the variables (velocity, pressure or density) are first decomposed in two parts: a mean component (overbarred values), which only varies with space coordinates and a small fluctuating one (primed values), which depends on both space and time. The quasi one-dimensional Euler equations (25)-(27) are then linearized:

$$\frac{\partial}{\partial t} \left( \frac{\rho'}{\bar{\rho}} \right) + \bar{u} \frac{\partial}{\partial x} \left( \frac{\rho'}{\bar{\rho}} + \frac{u'}{\bar{u}} \right) = 0 \quad (28)$$

$$\frac{\partial}{\partial t} \left( \frac{u'}{\bar{u}} \right) + \bar{u} \frac{\partial}{\partial x} \left( \frac{u'}{\bar{u}} \right) + \left( \frac{\rho'}{\bar{\rho}} + 2 \frac{u'}{\bar{u}} \right) \frac{d\bar{u}}{dx} = \frac{p'}{\bar{p}} \frac{d\bar{u}}{dx} - \frac{\bar{p}}{\bar{\rho}\bar{u}} \frac{\partial}{\partial x} \left( \frac{p'}{\bar{p}} \right) \quad (29)$$

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left( \frac{s'}{c_v} \right) = 0 \quad (30)$$

Introducing the mean velocity  $\bar{u}$  and density  $\bar{\rho}$  allows to completely hide the section changes  $A(x)$  in Eq.(28) to (30). For the present approach, entropy waves have not been taken into account even though the interaction of entropy waves with nozzles can be a source of instabilities.<sup>2,22,23</sup> Since combustion is a strong source of entropy perturbations, it is logical to assume that these waves participate to some combustion instabilities. Some recent studies, however, suggest that the entropy waves have a limited influence on the resonant modes in swirled combustors.<sup>25</sup> Even though this simplification may not be justified for all cases, it was used in the present study and may have to be relaxed for future work. We thus consider that entropy fluctuations are zero:

$$\frac{p'}{\gamma \bar{p}} - \frac{\rho'}{\bar{\rho}} = \frac{s'}{c_p} = 0 \quad \text{so that} \quad \frac{\rho'}{\bar{\rho}} = \frac{p'}{\gamma \bar{p}} \quad (31)$$

Under the assumption of monochromatic fluctuations, one can write:

$$\frac{p'}{\bar{p}} = \varphi(x)e^{-i\omega t}; \quad \frac{u'}{\bar{u}} = \nu(x)e^{-i\omega t}; \quad \frac{\rho'}{\bar{\rho}} = \delta(x)e^{-i\omega t} = \frac{1}{\gamma}\varphi(x)e^{-i\omega t} \quad (32)$$

Using Eq.(28), (29) and (31) and introducing the local Mach number, system (28)-(29) for the perturbed pressure  $\varphi$  and velocity  $\nu$  becomes:

$$-i\omega\varphi + \bar{u}\frac{d\varphi}{dx} + \gamma\bar{u}\frac{d\nu}{dx} = 0 \quad (33)$$

$$\frac{\bar{u}}{M^2}\frac{d\varphi}{dx} + \gamma\bar{u}\frac{d\nu}{dx} - (\gamma - 1)\frac{d\bar{u}}{dx}\varphi + \gamma\left[2\frac{d\bar{u}}{dx} - i\omega\right]\nu = 0 \quad (34)$$

Introducing Eq.(33) in Eq.(34) also gives for the momentum equation:

$$\left(\frac{1}{M^2} - 1\right)\bar{u}\frac{d\varphi}{dx} - \left[(\gamma - 1)\frac{d\bar{u}}{dx} - i\omega\right]\varphi + \gamma\left[2\frac{d\bar{u}}{dx} - i\omega\right]\nu = 0 \quad (35)$$

Using reduced values, the acoustic impedance can be written:

$$Z = \frac{1}{\gamma M} \frac{\varphi}{\nu} \quad (36)$$

Eq.(33) and (35) can be solved if appropriate boundary conditions are specified as described below. It should be noted here that only longitudinal acoustic modes are considered in this resolution, whereas higher order acoustic modes can occur in a duct.<sup>9</sup> Nevertheless, as the main context is the study of the thermo-acoustic stability of a combustion chamber (especially gas turbines), only the lower frequency acoustic eigenmodes (hence longitudinal ones) are of interest.

### 3.2 Impedance of a choked nozzle

Distributors are most often choked in gas turbines (except for idle operating conditions). In this particular case, knowing the mean flow is enough to determine the inlet acoustic impedance of a choked nozzle. In the time domain, system (33)-(35) is strictly hyperbolic (terms in  $-i\omega$  are replaced by time derivatives) and its two characteristic velocities (eigenvalues) are:

$$\left(\frac{dx}{dt}\right)_{\mathcal{A}^\pm} = \bar{u} \pm \bar{c} \quad (37)$$

At the throat of a choked nozzle,  $\bar{u}_* = \bar{c}_*$  which implies that the  $\mathcal{A}^-$  wave does not move at the throat: no information can propagate upstream of the throat. Mathematically, it means

that  $x = x_*$  is a singular location for system (33)-(35). Eq.(35) at the throat with  $M^* = 1$  allows to write the throat impedance<sup>4</sup>:

$$Z_* = \frac{1}{\gamma \bar{M}_*} \frac{\varphi_*}{\nu_*} = \frac{2 \frac{d\bar{u}}{dx}|_* - i\omega}{(\gamma - 1) \frac{d\bar{u}}{dx}|_* - i\omega} \quad (38)$$

### 3.3 Numerical resolution

To determine inlet or outlet acoustic impedance of ducts, the mean flow needs to be characterized first. Its computation within a nozzle of cross section  $A(x)$  is a simple textbook problem: as  $\bar{M} = \bar{u}/\bar{c}$  is the local Mach number, one can write, using Hugoniot relations derived from the equations (25)-(27) for stationary flow :

$$\frac{A(x)}{A_*} = \frac{1}{\bar{M}} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} \bar{M}^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (39)$$

The Mach number is supposed to be 1 at  $x = x_*$ . Knowing  $\bar{M}$  at a boundary section enables to calculate the critical section  $A_*$ . Then, the Mach number can be determined everywhere else using an iterative Newton-Raphson method for equation (39). All other variables can be calculated with isentropic flow relations using total values:

$$\frac{\bar{\rho}(x)}{\bar{\rho}_t} = \left( 1 + \frac{\gamma - 1}{2} \bar{M}^2 \right)^{\frac{-1}{\gamma-1}} \quad \text{and} \quad \frac{\bar{p}(x)}{\bar{p}_t} = \left( 1 + \frac{\gamma - 1}{2} \bar{M}^2 \right)^{\frac{-\gamma}{\gamma-1}} \quad (40)$$

Knowing the mean flow, the impedance can be computed as follows: for a given time frequency  $\omega$ , the computational domain is discretized in space  $x = \{x^{(j)}\}_{1 \leq j \leq N}$  and the system (33)-(35) becomes:

$$-i\omega \varphi^{(j)} + \frac{\bar{u}^{(j)}}{h} \delta_\epsilon^- \varphi^{(j)} + \gamma \frac{\bar{u}^{(j)}}{h} \delta_\epsilon^- \nu^{(j)} = 0 \quad (41)$$

$$\left( \frac{1}{(\bar{M}^{(j)})^2} - 1 \right) \frac{\bar{u}^{(j)}}{h} \delta_\epsilon^- \varphi^{(j)} - \left[ \frac{(\gamma - 1)}{h} \Delta_0^\epsilon \bar{u}^{(j)} - i\omega \right] \varphi^{(j)} + \gamma \left[ \frac{2}{h} \Delta_0^\epsilon \bar{u}^{(j)} - i\omega \right] \nu^{(j)} = 0 \quad (42)$$

where the difference operators  $\Delta_0^\epsilon$  and  $\delta_\epsilon^-$  are defined by:

$$\delta_\epsilon^- \varphi^{(j)} = \epsilon(\varphi^{(j)} - \varphi^{(j-1)}) \quad \text{and} \quad \Delta_0^\epsilon \bar{u}^{(j)} = \frac{\epsilon}{2}(\bar{u}^{(j+1)} - \bar{u}^{(j-1)}) \quad (43)$$

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<sup>4</sup>For low pulsations and constant section, Eq.(33) and (35) reduce to Marble and Candel's result (Eq.(15)): in this case, Eq.(33) and (35) lead to  $\frac{d\varphi}{dx} = \gamma \frac{d\nu}{dx}$ , so that  $\varphi = \nu_*$  and  $\nu = \nu_*$ , and the impedance at the inlet is  $Z_1 = \frac{1}{\gamma \bar{M}_1} \frac{\varphi_1}{\nu_1} = \frac{1}{\gamma \bar{M}_1} \frac{\varphi_*}{\nu_*}$ . At the throat for  $\omega \rightarrow 0$ , Eq.(38) gives  $Z_* = \frac{2}{\gamma-1}$ ,  $\frac{\varphi_*}{\nu_*} = \frac{2\gamma}{\gamma-1}$  and  $Z_1 = \frac{2}{(\gamma-1)\bar{M}_1}$  which is Eq.(38).

The parameter  $\epsilon$  depends on the type of flow: for a diffuser,  $\epsilon = 1$ , while for a distributor  $\epsilon = -1$ . The system (41)-(42) is integrated from where the impedance is given ( $x = x^{(1)}$ ) to where it is searched ( $x = x^{(N)}$ ) as shown in figure 4. The system can thus be written in compact form :

$$[D]^{(j)} \begin{pmatrix} \varphi^{(j)} \\ \nu^{(j)} \end{pmatrix} = \begin{pmatrix} g_1(\varphi^{(j-1)}) \\ g_2(\nu^{(j-1)}) \end{pmatrix} \quad (44)$$

$[D]^{(j)}$  is a  $2 \times 2$  matrix and the acoustic impedance can be expressed at  $x = x^{(j)}$ :

$$Z^{(j)} = \frac{1}{\gamma M^{(j)}} \frac{\varphi^{(j)}}{\nu^{(j)}} \quad (45)$$

At  $x = x^{(1)}$ , only  $Z = Z^{(1)}$  is known: typically, section 1 must be the throat for choked flows or a section where  $Z$  is known<sup>5</sup>. The system (44) with Eq.(45) is solved as follows : in the section where the impedance is known,  $\nu^{(1)}$  is set equal to a constant  $K \neq 0$  so that  $(\varphi^{(1)}, \nu^{(1)})^T = (\gamma M Z^{(1)} K, K)^T$ , which is a set of boundary conditions for system (44). From node to node,  $\forall j \in \{2, \dots, N\}$ ,  $[D]^{(j)}$  can be inverted to give  $(\varphi^{(N)}, \nu^{(N)})^T$ .  $Z^{(N)}$  is then recovered using again Eq.(45). Of course, due to the linearity of Eq.(33) and (35), the choice of  $K$  has no effect on  $Z^{(N)}$  as pointed out by Mani.<sup>26</sup> Calculation is repeated for several frequencies to obtain the impedance curve versus frequency  $Z^{(N)}(\omega)$ .

## 4 Numerical evaluation of impedance using the Euler equations

Another way to determine acoustic impedances of section varying ducts is to use a space-time solver and the Euler equation without linearization: extracting acoustic waves from the solution of the unsteady Euler equations is expensive but used here as a verification of the FD method of section 3. The solver is here based on the cell-vertex finite volume formulation of the Lax-Wendroff scheme<sup>27,28</sup> and boundary conditions are computed following the Euler Characteristic Boundary Conditions approach.<sup>29,30</sup> For example, the time simulation is solved for a distributor (figure (5)) as follows. The mean flow is first imposed in the duct. When the mean flow is established, monochromatic acoustic waves  $\mathcal{A}^+$  of angular frequency  $\omega$  are injected at the inlet (*resp.* outlet) of the computational domain. The outlet impedance is set to  $Z_2$ . The reflected waves  $\mathcal{A}^-$  are then extracted from the pressure and velocity temporal signals at the inlet, using a characteristic decomposition. Finally, a Fourier analysis

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<sup>5</sup>In the case of a choked distributor, the singularity is avoided by using relation (38) as the boundary condition for system (44), writing :  $x^{(1)} = x_*$  and  $Z^{(1)} = Z_*$ .

of ingoing (reference) and outgoing acoustic waves gives the real and imaginary parts of the inlet acoustic impedance at angular frequency  $\omega$ .

This operation is repeated for several angular frequencies to obtain a map of  $Z(\omega)$ . Care has to be taken with the use of the inlet boundary condition, since it must impose both the mean flow and an acoustic wave.<sup>31</sup> Moreover, it must totally evacuate the reflected waves  $\mathcal{A}^-$  (non reflecting inlet boundary condition), otherwise the calculation of the inlet impedance is biased. In the following comparisons, only the case of the distributor is shown even though validations have also been made for diffusers. The procedure is similar for diffusers, except that the inlet impedance  $Z_1$  is imposed and the outlet impedance  $Z_2$  is calculated.

## 5 Code verification

The FD method described in Section 3.3 is validated here in two cases: a constant section duct where an analytical solution is available (Section 5.1) and a distributor (unchoked) with a fixed outlet pressure where a space-time solver (Section 4) is used to evaluate the impedance by forcing the inlet in the time domain (Section 5.2).

### 5.1 Comparison of FD method and analytical formulae in a constant section duct

For simple test cases as the constant section duct, a trivial analytical solution can be derived. The acoustic pressure and velocity of non-zero Mach number flow in a constant section duct of length  $L$  both verify:

$$\frac{d^2\phi}{dt^2} - \bar{c}^2 \frac{\partial^2\phi}{\partial x^2} = 0 \quad (46)$$

where  $\phi = \phi(x, t)$  is either  $p'$  or  $u'$  and  $d/dt = \partial/\partial t + \bar{u}\partial/\partial x$ . The solutions of Eq.(46) are given by Eq.(2) and (3), section 2. Given the impedance at the outlet of the duct, one can write it at the inlet for  $\bar{M} < 1$ . It has been calculated for flow conditions summarized in Table 2 and outlet impedances  $Z_2 = 0$  and  $Z_2 = \infty$ .

The analytical formulations of inlet impedance in those two cases are the following :

$$Z_1|_{Z_2=0} = -i \tan\left(\frac{kL}{1 - \bar{M}^2}\right) \quad (47)$$

$$Z_1|_{Z_2=\infty} = i \cot\left(\frac{kL}{1 - \bar{M}^2}\right) \quad (48)$$

Comparisons between imaginary parts of inlet impedances (real parts are zero) given by the FD method and Eq.(47)-(48) are shown on figures 6 and 7 for a frequency range from 1 to 1000 Hz. The agreement between the two methods is very good even for higher frequencies.

## 5.2 Comparison of analytical formulae, FD technique and space-time solver for an unchoked distributor

The comparison between the different methods to calculate impedances is performed here for an unchoked distributor. The geometry (figure 8) is discretized using a non-regular Cartesian grid for the Euler equations solver and the characteristics of the flow are gathered in Table 3. The outlet impedance at  $x = L_x$  is set to  $Z_2 = 0$ .

Figure 9 shows a good agreement between the results of the space-time solver Euler equations solver and Eq.(39) for the mean flow in the distributor, characterized here by  $y$ -integrated Mach number  $\bar{M}(x) = \int_{L_y} \bar{M}(x, y) dy$ . Real and imaginary parts of inlet acoustic impedance are plotted on figures 10 and 11. Here again, the agreement between the FD method and the space-time solver is good. Moreover, the asymptotic behaviour at low frequencies for both approaches coincides with the impedance analytically obtained from Eq.(21). Nevertheless, as in the case of a constant section duct, the real and imaginary parts of the impedance quickly evolve with frequency. It clearly shows *a posteriori* the importance of solving the system (44) and the limits of analytical formulae (such as those obtained in Section 2). The peaks observed on the real part (and change of sign of the imaginary part) correspond to the acoustic modes of the nozzle. The first one is the quarter wave mode  $\lambda = 4L_x$  and  $f=435$  Hz. Figures 12 and 13 show that the low frequency approximation (Eq.(21)) is valid up to frequencies  $f$  of approximately 200 Hz and a wavelength  $\lambda \approx 1.7$  m  $\approx 10L_x$ . Note that the real part of the inlet impedance is negative for some range of frequencies. This is equivalent to a reflection coefficient modulus greater than 1. It is possible because of the presence of a mean flow, as pointed out by Munjal<sup>32</sup> and Mechel et al.<sup>33</sup> In such a case, the average flux through a surface and over a period of a fluctuation is written :

$$W_{ac} = \frac{|\mathcal{A}^I|^2}{2\rho c} \left[ (1 + M)^2 - |R|^2 (1 - M)^2 \right] \quad (49)$$

Eq.(49) shows that  $W_{ac}$  can be positive (acoustic loss) even if  $|R| > 1$ , as soon as there is a mean flow ( $M \neq 0$ )<sup>6</sup>.

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<sup>6</sup>As indicated by Munjal, this paradox ( $|R| > 1$ ) is only apparent due to the choice of the variables  $u'$  and  $p'$  to define the reflection coefficient in Eq.(5). If proper aeroacoustic variables adapted to non-zero mean flow are used to define  $R$  (the perturbed mass flow rate  $\dot{m}'$  and the perturbed total enthalpy  $h'_t$ ), the



## 6 Impedances of combustion chambers in gas turbines

As recalled in Section 2, it is common to set  $Z = 0$  for a flow that discharges in the atmosphere. Besides,  $Y = 0$  is also often used for the inlet of a duct where the mass flow rate is imposed constant in time. These boundary conditions can be used in the case of academic configurations but not in gas turbines, where distributors and diffusers must be taken into account. The previous sections have shown that methods exist to evaluate the inlet impedance of distributors or the outlet impedances of diffusers when the impedance of the other side of the nozzle is known. The most favorable case is when the ducts connected to the casing or the chamber are choked. Indeed, the impedance at location  $x = x^{(1)}$  is known. As shown in section 3, the impedance at the throat of a choked distributor can be expressed through the relation (38). For a choked diffuser, a normal shock (or a shock network) exists between the throat and the casing inlet. Then, relations derived by Stow et al<sup>22</sup> and Moase et al<sup>23</sup> can be used to determine the reflection coefficient of the shock.

Unfortunately, diffusers and distributors are not always choked and the flow can be fully subsonic, especially at idle regimes. In such cases, location  $x = x^{(1)}$ , where the calculation using FD method is started, is often the inlet or the outlet of a rotating machines. A first approximation is to use  $Y^{(1)} = 0$  at the inlet of the diffuser (to model a constant flow rate) or  $Z^{(1)} = 0$  at the outlet of the distributor (to model a constant pressure), but those are crude approximations. As the accuracy of the resolution of (44) strongly relies on those values, one would also rather try to model the compressor and the turbine with complex impedances by studying their acoustic response. This could be done using the work of Kaji and Okazaki<sup>34</sup> (and the concept of semi-actuator disk), if little information on the turbomachinery blade rows is provided. A finer way to determine complex impedances of turbomachines lying upstream and downstream the chamber is to experimentally measure or compute the scattering of acoustic waves through them.<sup>35,36</sup>

The next section shows the effects of impedances changes for diffusers and distributors in the case of a realistic gas turbine. It confirms their importance and the necessity of modeling them accurately.

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reflection coefficient is always less than one.

## 7 Application of impedance methods in a Helmholtz solver for a real chamber

This section shows how Helmholtz solvers are built and provides one example of real gas turbine application where the result of the solver is directly controlled by the acoustic impedances used at inlets and outlets. Multidimensional Helmholtz codes<sup>15,37–39</sup> solve the linearized acoustic wave equation in the Fourier domain: they give the complex frequency (stability analysis) and spatial structure of all acoustic eigenmodes.

The Helmholtz equation is obtained by linearizing the reactive Navier-Stokes equations under the following assumptions:<sup>6,39</sup> low Mach number, no volume forces, linear acoustics, large scale fluctuations (long wavelength), homogeneous mean pressure, constant polytropic coefficient  $\gamma$ . The fluctuating pressure field is then given by the wave equation<sup>7</sup>:

$$\vec{\nabla} \cdot (c^2 \vec{\nabla} p') + \omega^2 p' = 0 \quad (50)$$

with the following boundary conditions:

$$Z(\omega) = \frac{p'}{\bar{\rho} c \vec{\nabla} p' \cdot \vec{n}}, \quad (51)$$

where  $Z(\omega)$  is the local impedance on the domain boundaries. The system (50)-(51) is discretized using finite element formulation<sup>39</sup> as follows: to be able to handle complex geometries such as gas turbine combustors, the computational domain is decomposed in linear P1 elements (triangles in 2D, tetrahedra in 3D) and equations (50) and (51) are solved using a lumped mass matrix Galerkin finite element method (see Nicoud<sup>18</sup>). The discretized equation can be put in a matrix form:

$$[A]\{p'\} + \omega[B(\omega)]\{p'\} + \omega^2[C]\{p'\} = 0 \quad (52)$$

where  $[B(\omega)]$  contains boundary conditions. The matrix problem (52) is a non-linear eigenvalue problem and is reformulated to be linear. It is then solved using a parallelized implementation of the Arnoldi method.<sup>40</sup> Several examples of its application have already been referenced<sup>15,16,18</sup> and shown the accuracy of the solver.

Compared to Navier-Stokes solvers in the space-time domain, Helmholtz codes are faster and directly supply the frequency and structure of all acoustic modes. Moreover these codes can easily handle complex-valued boundary conditions that vary with frequency, whereas

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<sup>7</sup>For this example, the effects of the active flame on the pressure field are neglected<sup>9,39</sup>

it remains a challenge in the time domain, especially in complex geometries, despite some developments<sup>41–43</sup> making use of discrete convolutions.

As an example to show the influence of the boundary conditions, several calculations have been realized on a real annular combustor (see Fig. 14 and 15). In this configuration, both the chamber and its casing are considered and the swirler is also meshed. The only differences between the computations presented below are the acoustic boundary conditions at the chamber inlet and outlet (which are respectively linked with the diffuser and the distributor). The boundary conditions are either obtained using the analytical formulae of Table 1 or computed using the FD method of Section 3.

For the diffuser, the boundary condition upstream of the throat is supposed to be either  $Z_{1,in} = 0$  or  $Y_{1,in} = 0$  (Fig. 15). For the distributor, the acoustic condition downstream of the nozzle correspond either to  $Z_{2,out} = \infty$  or to a  $Z_{2,out} = Z_*$  choked nozzle. The results of the Helmholtz code are limited here to the first mode (lowest frequency). Table 4 provides the frequency of this first mode for each case. The structures of the modes (adimensionalized pressure modulus in the central plane of the chamber) are displayed in figures 18 and 19. The results of the calculation illustrates the effects of impedances on the results. A comparison between Case A and B highlights the impact of the inlet boundary condition, on the diffuser side. Both the structure of the mode and its frequency (that has shifted from 245 to 360 Hz) are modified when the inlet impedance varies from 0 (Case A) to  $\infty$  (Case B). The approximation of  $u' = 0$  (infinite impedance) for the outlet to model a choked nozzle (case B) gives very similar results to the right acoustic impedance calculated using the FD solver (case C), at least for low frequencies. Analytical formulae or impedances given by the FD method (case D) should be preferred since the the frequency and structure of the first mode depends strongly on their values.

## 8 Conclusion

To determine the acoustic eigenmodes of gas turbine combustors, Helmholtz solvers are a very interesting and commonly used method.<sup>5,7,8,17,18</sup> This paper shows that their accuracy directly depends on the boundary conditions used for inlets and outlets. Some analytical expressions have been derived in the past,<sup>2</sup> extended here and shown to be efficient at low frequencies. In practical cases, they are not often sufficient. A methodology to calculate those acoustic boundary conditions, by approximating the ducts connected to the chamber as one dimensional nozzles and solving linearized Euler equations, has been presented and verified with different academic cases including a comparison with a full Euler equations

solver. It enables the quick determination of inlet or outlet impedances of section varying ducts (choked or not) in the frequency domain. The application of this method in a realistic gas turbine chamber with a subsonic diffuser and a choked distributor has shown that even when the whole chamber is computed (including casing and swirlers), results still depend on the boundary conditions used for the inlets and outlets. This means that a proper prediction of acoustic modes cannot be performed without these impedances. Results show that while impedances can be calculated on the distributor side where a choked nozzle is usually found, the problem is more complex for the inlet where diffusers are often unchoked and where a location to identify a proper acoustic boundary condition is difficult to find. The study of the scattering of acoustic waves has then to be considered to derive accurate impedances.

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<sup>41</sup>Fung, K., Ju, H., and Tallapragada, B., “Impedance and Its Time-Domain Extensions,” *AIAA J.*, Vol. 38, No. 1, 2000, pp. 30–38.

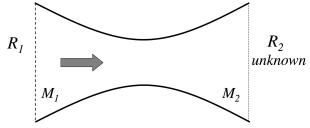
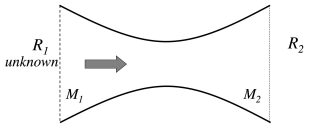
<sup>42</sup>Fung, K. and Ju, H., “Broadband Time-Domain Impedance Models,” *AIAA J.*, Vol. 39, No. 8, 2001, pp. 1449–1454.

<sup>43</sup>Ju, H. and Fung, K., “Time-Domain Impedance Boundary Conditions with Mean Flow Effects,” *AIAA J.*, Vol. 39, No. 9, 2001, pp. 1683–1690.

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	Diffuser	Distributor
		
Choked	$R_2 = \frac{1-\gamma\bar{M}_2+(\gamma-1)\bar{M}_2^2}{1+\gamma\bar{M}_2+(\gamma-1)\bar{M}_2^2}$	$R_1 = \frac{1-\frac{1}{2}(\gamma-1)\bar{M}_1}{1+\frac{1}{2}(\gamma-1)\bar{M}_1}$ or $Z_1 = \frac{2}{M_1(\gamma-1)}$
Unchoked	$R_2$ : see Eq.(23) $T_2$ : see Eq.(24)	$R_1$ : see Eq.(21) $T_1$ : see Eq.(22)

**Table 1:** Reflection coefficients depending on the acoustic boundary conditions and the flow regime in the low frequency limit.

Parameters	Value
$L$	0.2
$\rho$	1.1723
$r = R/W_{air}$	288.19
$T$	300
$M$	0.2

**Table 2:** Main parameters of the flow in the 1D tube.

Geometry	Value	Boundary conditions	Value
$L_x$	0.2	$u_{in}$	30
$L_y(inlet)$	0.015	$p_{out}$	1.013
$N_x$	101	$\rho_{in}$	1.1723
$N_y$	11	$T_{in}$	300
$\gamma$	1.399	$r = R/W_{air}$	288.19

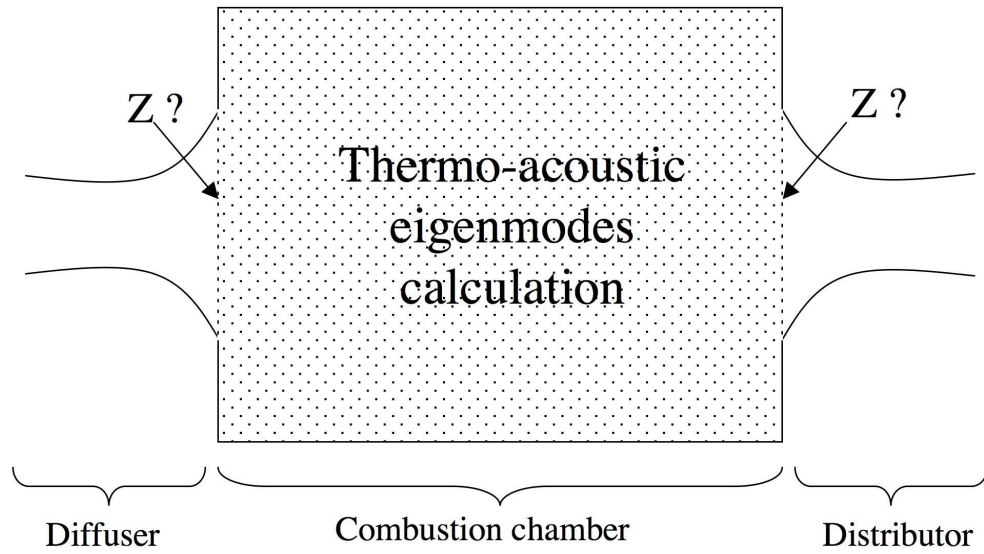
**Table 3:** Characteristics of the variable section duct and boundary conditions.

Cases	Inlet impedance $Z_{2,in}$	Outlet impedance $Z_{1,out}$	1 <sup>st</sup> mode frequency (Hz)
<b>A</b>	0 (Eq.(23) with $M_1 \approx M_2$ and $R_1 = -1$ )	$\infty$ (Eq.(15) with $M_1 \rightarrow 0$ )	245
<b>B</b>	$\infty$ (Eq.(23) with $M_1 \approx M_2$ and $R_1 = 1$ )	$\infty$ (Eq.(15) with $M_1 \rightarrow 0$ )	360
<b>C</b>	$\infty$ (Eq.(23) with $M_1 \approx M_2$ and $R_1 = 1$ )	$Z_{1,out}$ (FD method)	357
<b>D</b>	$Z_{2,in}$ (FD method)	$Z_{1,out}$ (FD method)	330

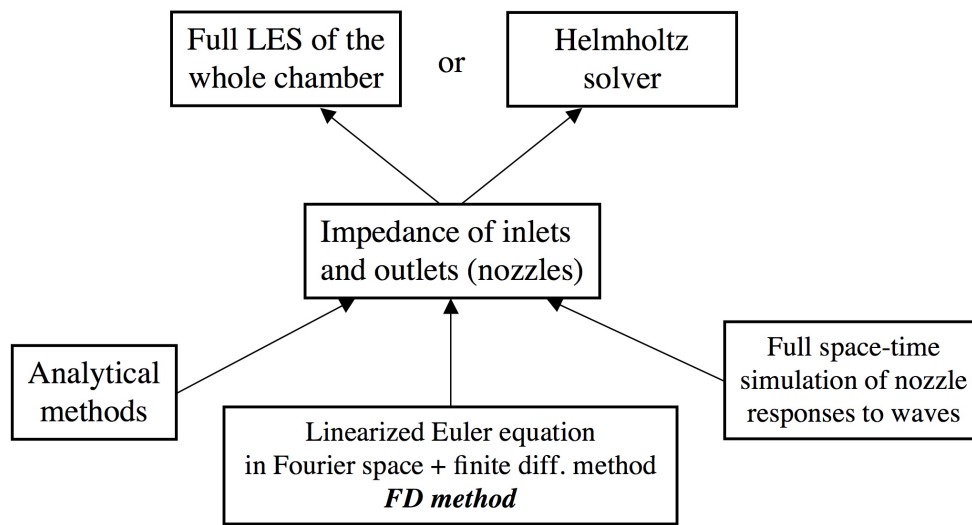
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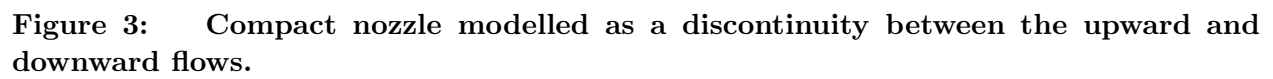
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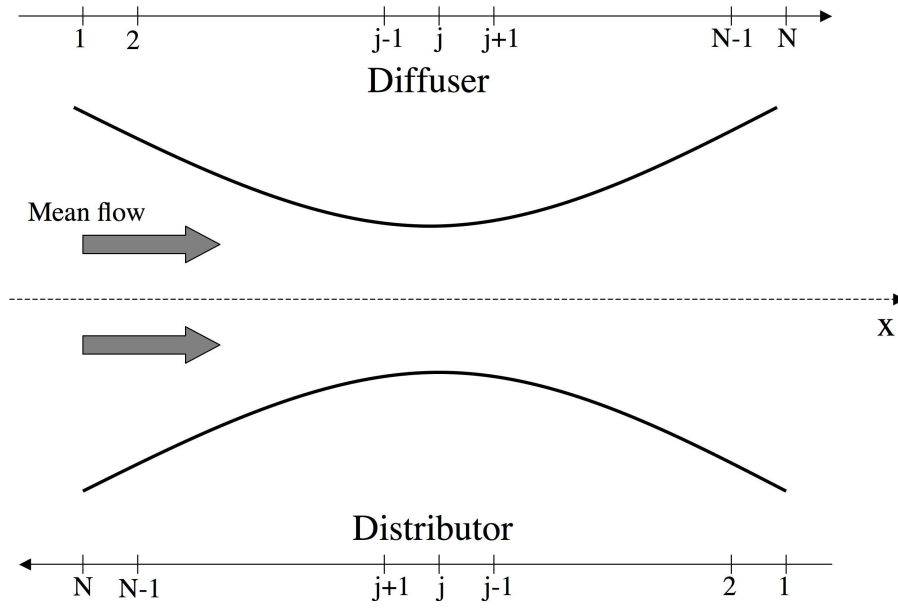
**Figure 1:** Combustion chamber with diffuser and distributor. Ducts feeding the chamber are replaced by acoustic impedances for the eigenmode calculation.



**Figure 2:** Connections between the different calculations needed to study the thermoacoustic stability of combustion chambers.







**Figure 4:** System (41)-(42) integration for diffusers and distributors. Boundary conditions are given at  $x = x^{(N)}$ .

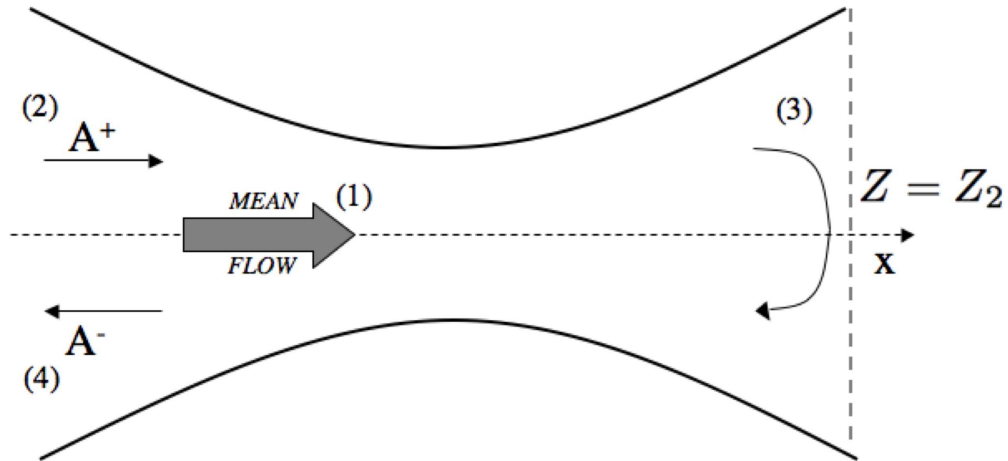
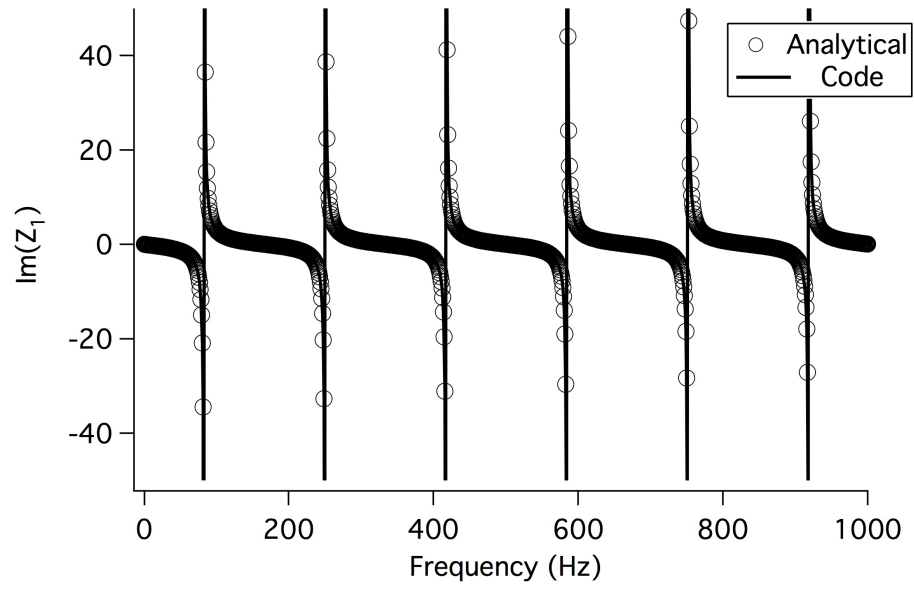
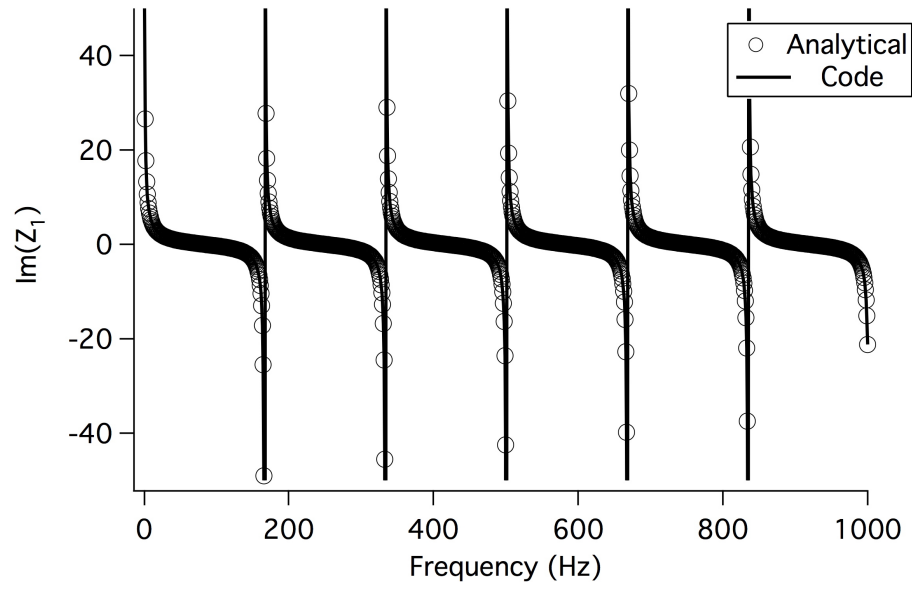


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**Figure 6:** Comparison between inlet impedances given by the analytical expression (47) and the FD method. The outlet impedance is  $Z_2 = 0$ .



**Figure 7:** Comparison between inlet impedances given by the analytical expression (48) and the FD method. The outlet impedance is  $Z_2 = \infty$ .

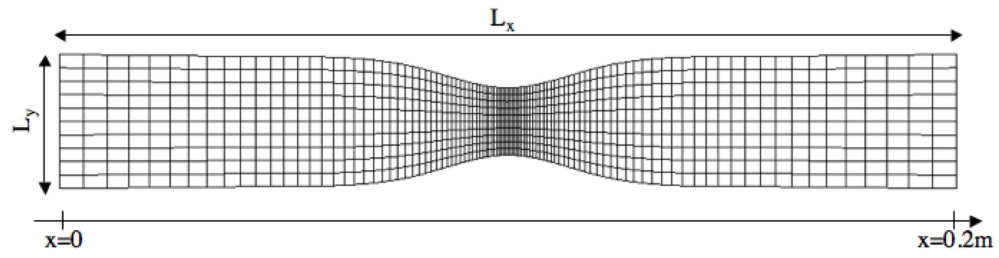
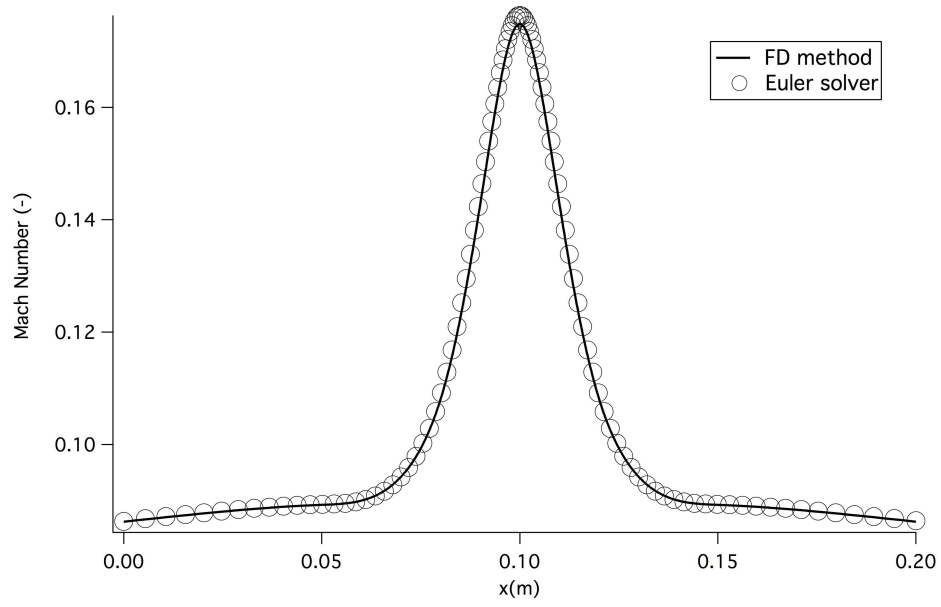
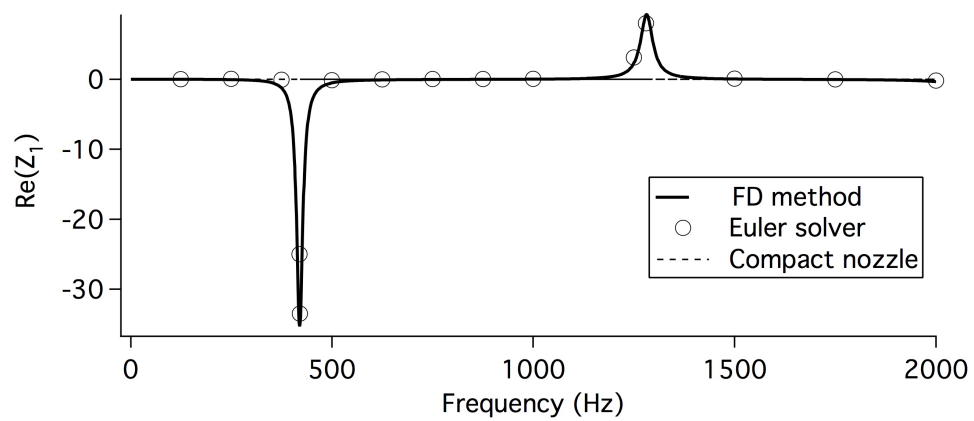


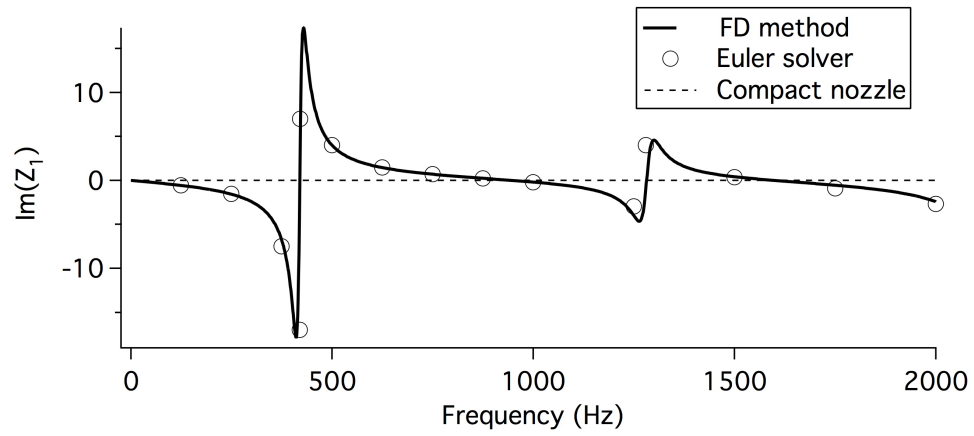
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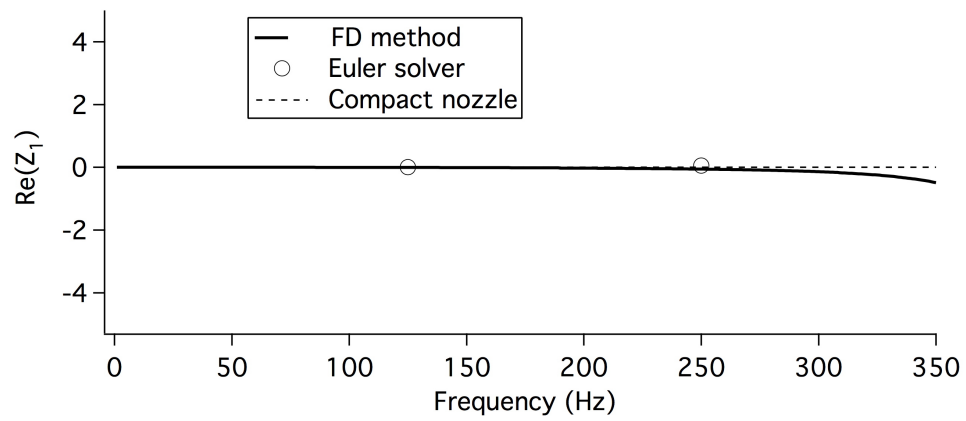


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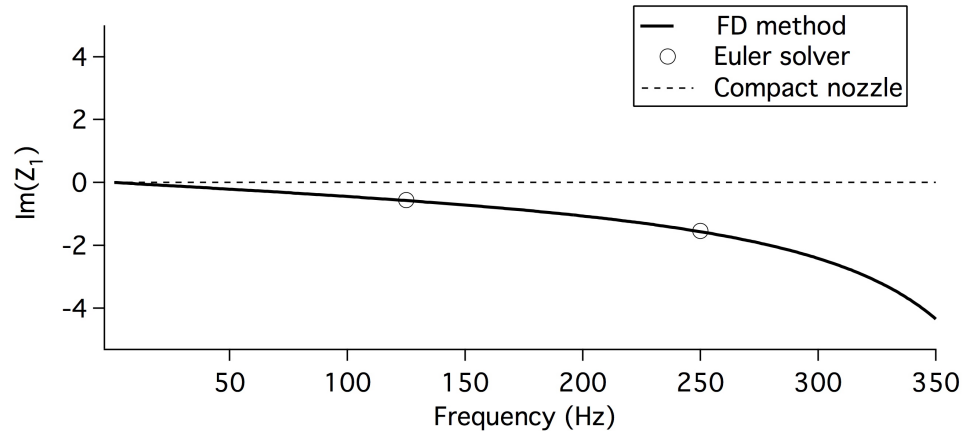


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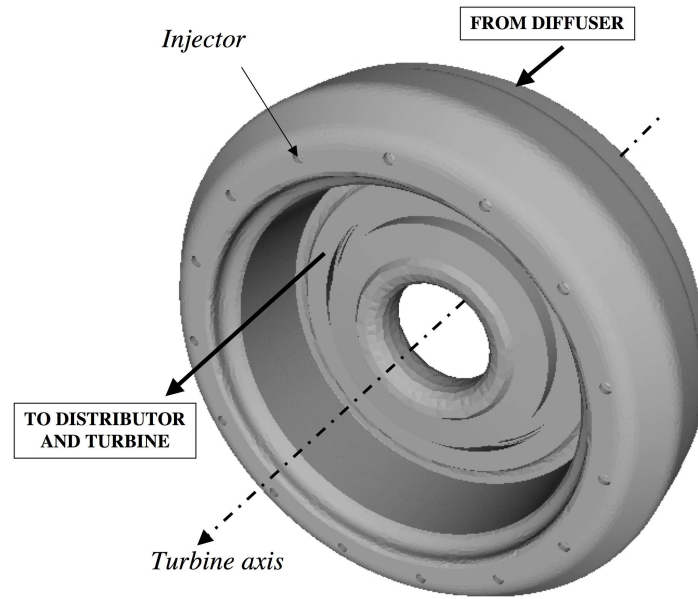


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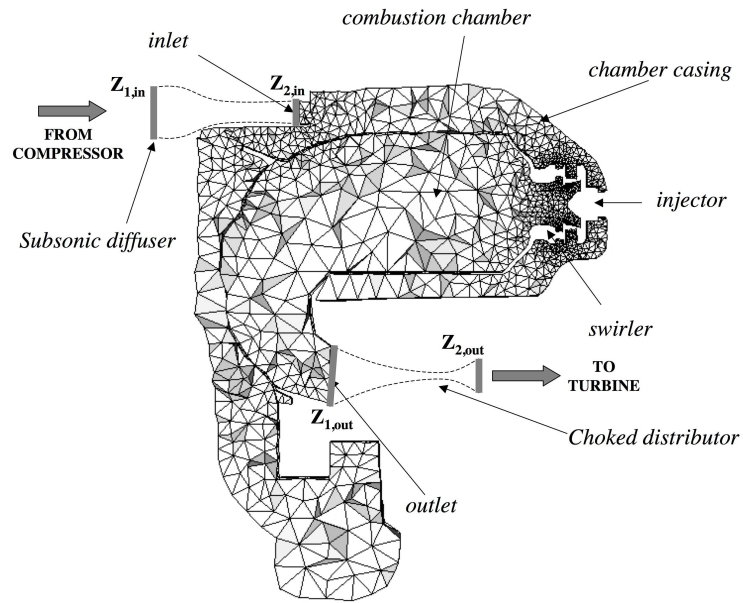
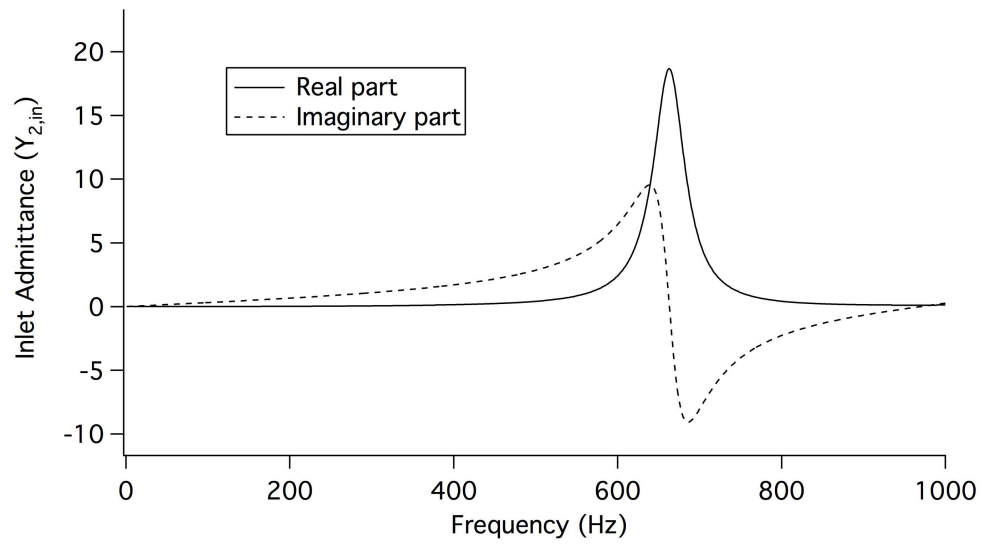
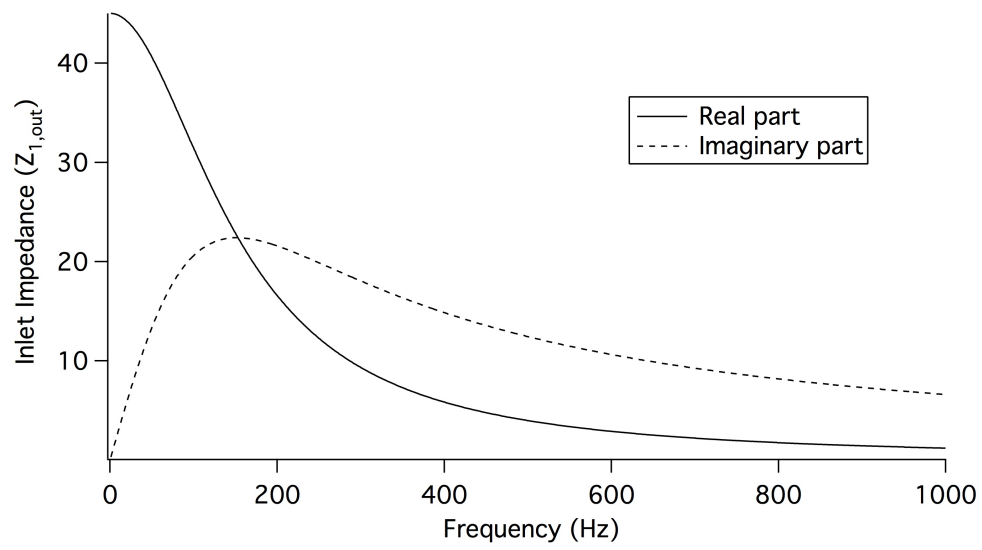


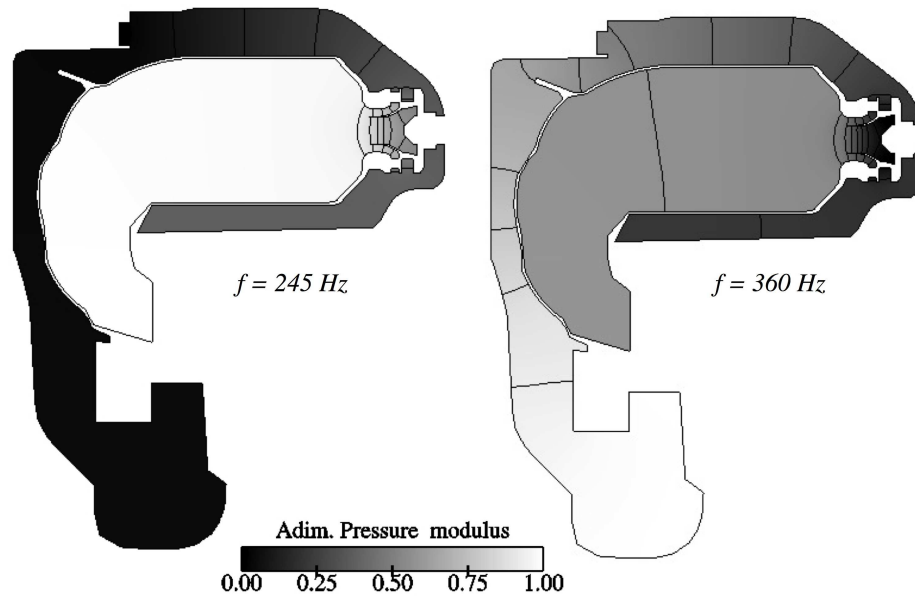
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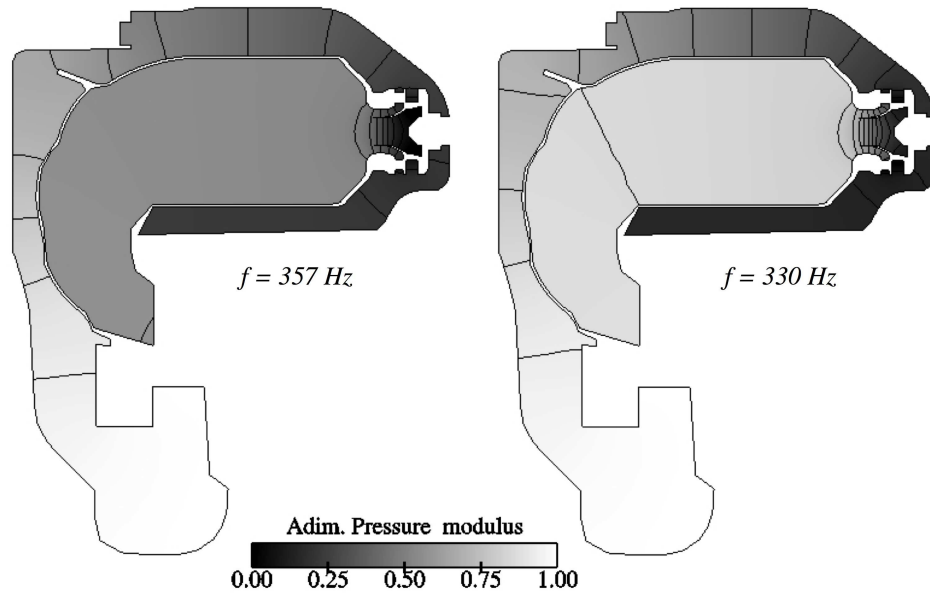
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