Direct and large-eddy simulations of merging in co-rotating vortex system

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Direct and large-eddy simulations were used to study the process of merging of co-rotating vortex system, an important topic of aircraft wake dynamics. A simplified configuration was chosen, consisting of a couple of co-rotating, symmetrical vortices whose initial distribution of vorticity obeys an analytical law. The Lamb-Oseen model and the two-scale VM2 model by Fabre and Jacquin (Phys. Fluids, 16, 2004) were used to initialize the simulations in the two-dimensional (stable) and three-dimensional (unstable) cases, for Reynolds numbers based on circulation ranging from $Re_T = 750$ to $240000$, and for various initial ratios $a_0/b_0$ between vortex core size and vortex spacing. In all cases, the onset of merging is associated to an exchange of vorticity between vortices; in the unstable cases this process is faster as it is triggered by a short-wave elliptical instability whose wavelength depends on the actual ratio $a/b$ at the beginning of the linear regime - which may be relevant for example for the development of wake control strategies. The structure of the final merged vortex is characterized by a two-scale azimuthal velocity profile, with a steeper intermediate power-law region in the case of unstable merging compared to stable merging.

I. Introduction

The merging of two- and three-dimensional vortices occurs in a number of fundamental processes of fluid mechanics, such as two-dimensional$^2$ and three-dimensional$^3$ turbulence, as well as in meteorology and geophysical flows.$^4$ Merging plays a major role in the formation of aircraft trailing vortices. In the initial stages of its formation, the wake of an aircraft is a complex vortex system composed of multiple interacting counter- and co-rotating vortices that are generated by the roll-up of the vorticity sheet shed by the wing and its components like flaps, spoilers and engine nacelles. All these vortices merge into a pair of coherent
counter-rotating vortices and represent an hazard for following aircrafts, in particular during take-off and landing phases. This motivated the extensive research that has been conducted both in US and Europe to characterize the structure of such vortices and to develop concepts to accelerate their decay (see the classical reviews by Spalart, Rossow and the recent special issue by Crouch and Jacquin and references therein). The present study focuses on one aspect of the problem, namely the process of merging of two co-rotating vortices in the extended near-field of the wake, i.e. one to ten wingspans behind the aircraft. This occurs for example in high-lift wing configurations where the two vortices are generated at the wing-tip and flap positions, respectively.

The experimental work by Cerretelli and Williamson first showed that at low Reynolds number vortex merging can be represented in four stages: a first diffusion stage, a convective stage, a second diffusion stage and an axisymmetrization stage (the main features of each phase will be discussed in detail in the analysis of the numerical simulations in Sec. III.C). They also found that the onset of merging is related to the generation of an antisymmetric vorticity field that brings the two vortices together, and observed that the final merging time depends on the Reynolds number as well as the initial ratio \( a_0/b_0 \) between the vortex core size \( a \) and spacing \( b \). Meunier et al. found that a pair of two-dimensional co-rotating vortices start to merge when this characteristic ration exceeds a critical threshold \( (a/b)_c \approx 0.22 \).

These results were later on confirmed by Leweke et al. both experimentally and numerically. In real aircraft wakes the initial ratio \( a_0/b_0 \) of the primary co-rotating vortices is often smaller than \( (a/b)_c \); however, the size of vortex core increases during the roll-up of the vortex sheet, while at the same time the interaction with remnants of the vortex sheet and the multiple vortices present in the wake reduces the vortex spacing. This increases the ratio \( a/b \) and ultimately leads to the merging of the vortices. The influence of the Reynolds number on the two-dimensional merging was studied in detail by Ferreira and Pereira by means of direct numerical simulation. Although limited to low Reynolds numbers and two-dimensional flows, their analysis focused on the third stage of merging which consists of a rotational movement coupled with a diffusion process. In particular, they observed a merging in two stages for Reynolds number less than 500. Orlandi recently investigated the dependence of three-dimensional merging on Reynolds number and the characteristic ratio \( a/b \) using direct simulations. He showed that merging is associated to a large increase of pressure extremum and that the complex feature of three-dimensional merging (compared to two-dimensional case) can be related to the axial disturbances imposed to the two vortices. It is known that, depending on the initial spacing and core size, two co-rotating vortices can be unstable due to the strain field that each vortex exerts on the other. (See the classical work by Moore and Saffmann and the review by Kerswell and references therein for various applications in science and engineering). The resulting “elliptical” instability is

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characterized by a three-dimensional sinusoidal deformation of the core, and a wavelength of the order of the vortex core size. In the context of aircraft wakes, this is usually referred to as short-wave instability to distinguish from long-wave or Crow instability whose wavelength is of the order of the vortex spacing. The elliptical instability was observed numerically and experimentally both in counter-rotating and co-rotating vortices. Le Dizès and Laporte derived an analytical relation that predicts the growth rate of the elliptical instability during the linear regime of its evolution as a function of $a_0/b_0$ and Reynolds number. This theoretical analysis was based on Gaussian distribution of the initial vorticity field and was supported by direct and large-eddy simulations.

Meunier et al. recently reviewed the physics of vortex merging and identified some key issues that deserve further investigation, both from a modeling and a computational point of view: (i) the effects of initial non-Gaussian vortex profiles in determining the characteristics of merging process; (ii) the effects of non-symmetric vortex profiles; and (iii) the effects of axial velocity deficit. In this study we aim at elucidating the first of these issues using direct and large-eddy simulations of trailing vortices. In particular, we are interested in evaluating the impact of vortex modeling on the development and wave selection of the elliptical instability (which is an important issue in aircraft wakes where short- and long-wavelength instabilities may occur together), as well as on the structure of the final merged vortex. The first model we analyze is the classical, one-scale (gaussian) Lamb-Oseen vortex model, which is used for validation and comparison with previous studies; the second model is the VM2 two-scale model proposed by Fabre and Jacquin that is supposed to model the roll-up of the vortex sheet and was found to best fit the vortex profile in the extended near-field of a model-scale wake.

The paper is organized as follows. The governing equations, the LES model and the numerical method are described in Sec. II. The results of the simulations are presented in Sec. III: the vortex models used to initialize the simulations are detailed in Sec. III.A; the computational results of stable merging are presented in Sec. III.C and are compared to analytical predictions and experimental results available in the literature; the analysis of three-dimensional merging, and its relation with the development of elliptical instability are presented in Sec. III.D. Conclusions are given in Sec. IV.

II. Governing equations and numerical model

II.A. Governing equations

In conservative form, the transport equations of a compressible gas read:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho u) = 0$$

(1)
\[
\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \rho \mathbf{I}) - \nabla \cdot \mathbf{T} = 0
\]  
(2)

\[
\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho \mathbf{H} \mathbf{u}) - \nabla \cdot (\mathbf{T} \mathbf{u}) + \nabla \cdot \mathbf{q} = 0
\]  
(3)

where \( \rho \) is the gas density and \( \mathbf{u} \) is the velocity vector. The total energy \( E \) is the sum of internal energy \( e = C_v \cdot T \) and kinetic energy \( K \equiv 1/2 \| \mathbf{u} \|^2 \): \( E = C_v \cdot T + K \) where \( T \) is the temperature, \( R \) is the gas constant for air, \( C_v = R/(\gamma - 1) \) and \( C_p = C_v + R \) are, respectively, the specific heats at constant volume and constant pressure, and \( \gamma = C_p/C_v = 1.4 \) is the ratio of specific heats. Temperature and pressure are related through the equation of state \( p = \rho RT \), and total enthalpy is defined by \( H = E + p/\rho = C_p \cdot T + K \). The stress tensor is given by \( \mathbf{T} = 2\mu (\mathbf{S} - 1/3 \nabla \cdot \mathbf{u} \mathbf{I}) \) where the strain rate tensor \( \mathbf{S} = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \) is the symmetric part of the velocity gradient and \( \mu \) is the dynamic viscosity given by Sutherland’s law, \( \mu/\mu_{\text{ref}} = (T/T_{\text{ref}})^{0.76} \). The heat fluxes is given by Fouriers law, \( \mathbf{q} = -\lambda \nabla T \) where the thermal conductivity \( \lambda = \mu C_p / \text{Pr} \) is constructed by setting the molecular Prandtl number to \( \text{Pr} = 0.75 \), which is a typical value in the atmosphere.  

\[ II.B. \ LES \text{ approach and subgrid scale model description} \]

In the large-eddy simulation (LES) approach Eqs. 1-3 are filtered spatially, so that any variable \( \phi(x) \) is decomposed into a resolved (or large scale) part \( \overline{\phi}(x) \) and a nonresolved (or subgrid scale) part \( \phi''(x) \), with \( \phi(x) = \overline{\phi}(x) + \phi''(x) \). This procedure may be obtained by a convolution integral of the variable with any filter function depending on a filter width \( \Delta \) (in practice, the size of the computational cell \( \Delta_x \)). In compressible flows, the large-scale filtered variables are usually recast in terms of Favre-filtered quantities defined by \( \overline{\phi} = \rho \overline{\phi}/\rho \) with \( \phi(x) = \overline{\phi}(x) + \phi'(x) \). Applying Favre-filter to Eqs. 1-3 yields

\[
\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \overline{\mathbf{u}}) = 0
\]  
(4)

\[
\frac{\partial (\overline{\rho} \overline{\mathbf{u}})}{\partial t} + \nabla \cdot (\overline{\rho} \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} + \overline{\rho} \mathbf{I}) - \nabla \cdot \overline{\mathbf{T}} = -\nabla \cdot \mathbf{T}_{\text{sgs}}
\]  
(5)

\[
\frac{\partial (\overline{\rho} E)}{\partial t} + \nabla \cdot (\overline{\rho} \overline{\mathbf{H}} \overline{\mathbf{u}}) - \nabla \cdot (\overline{\mathbf{T}} \overline{\mathbf{u}}) + \nabla \cdot \overline{\mathbf{q}} = \nabla \cdot \mathbf{q}_{\text{sgs}} - \nabla \cdot \mathbf{k}_{\text{sgs}}
\]  
(6)

where the classical assumptions by Erlebacher et al.\textsuperscript{24} have been made: (i) the Favre-filtered shear stresses and the heat and scalar fluxes are identified with their filtered counterparts and (ii) the SGS correlations involving fluctuations of molecular viscosity are neglected, i.e. \( \overline{\mathbf{T}} \approx \tilde{\mathbf{T}} \approx 2\pi \left( \overline{\mathbf{S}} - 1/3 \nabla \cdot \overline{\mathbf{u}} \mathbf{I} \right) \); \( \mathbf{q} \approx \tilde{\mathbf{q}} \approx -\overline{\mathbf{p}} C_p / \text{Pr} \nabla \tilde{T} \). The Favre-filtered equation of state becomes \( \overline{p} = \overline{\rho} R \tilde{T} \) so that the left-hand side of Eqs. 4-6 only contains resolved quantities. On the other hand, the unresolved SGS inviscid fluxes in the right-hand side, \( \mathbf{T}_{\text{sgs}} \equiv \rho \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \overline{\rho} \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} \); \( \mathbf{k}_{\text{sgs}} \equiv \rho \overline{K} \overline{\mathbf{u}} - \overline{\rho} \overline{K} \overline{\mathbf{u}} \), and \( \mathbf{q}_{\text{sgs}} \equiv \rho \overline{C_p T} \overline{\mathbf{u}} - \overline{\rho} C_p \overline{T} \overline{\mathbf{u}} \) modeled through
eddy viscosity concepts:

\[ T_{\text{sgs}} - \frac{\text{tr}(T_{\text{sgs}})}{3} I = -2\mu_{\text{sgs}} \left( \tilde{S} - \frac{\nabla \cdot \tilde{u}}{3} I \right), \quad (7) \]
\[ k_{\text{sgs}} = T_{\text{sgs}} \tilde{u}, \quad (8) \]
\[ q_{\text{sgs}} = -\mu_{\text{sgs}} C_p \nabla \Theta \quad (9) \]

where \( \mu_{\text{sgs}} \) is the SGS dynamic viscosity and \( \Theta = \tilde{T} - 1/2\rho C_v \text{tr}(T_{\text{sgs}}) \) is the modified temperature introduced by Lesieur and Comte\(^{25}\) with \( \text{tr}(T_{\text{sgs}}) \) the trace of the SGS stress tensor. The latter is also an unresolved term and should be modeled as done for example by Yoshizawa.\(^{26}\)

However, as shown by Ng and Erlebacher\(^{27}\) and Lesieur and Comte,\(^{25}\) \( \text{tr}(T_{\text{sgs}}) \sim M_{\text{sgs}}^2 \), with \( M_{\text{sgs}} \) the SGS Mach number that is generally negligible in weakly compressible flows as those studied in this work. Thus, we assume \( \Theta = \tilde{T} \). The turbulent Prandtl and Schmidt numbers, \( \text{Pr}_{\text{sgs}} \) and \( \text{Sc}_{\text{sgs}} \), are set equal to 0.419 as suggested by Gerz and Holzäpfel\(^{28}\) and in accordance to the results by Moin \textit{et al.}\(^{29}\) and Pitsch \textit{et al.}\(^{30}\) for turbulent shear flows. The SGS viscosity model is based on the Structure Function Model developed by Métaias and Lesieur\(^{31}\) in spectral space and then transposed into physical space by Ducros \textit{et al.}\(^{32}\)

The Structure Function is defined by

\[ \overline{F}_2(x, \Delta, t) = \langle ||\tilde{u}(x + r, t) - \tilde{u}(x, t)||^2 \rangle_{||r||=\Delta} \quad (10) \]

where \( \Delta = (\Delta_x \Delta_y \Delta_z)^{1/3} \) is the cutoff length and \( \langle \rangle \) denotes spatial averaging over a sphere of radius \( \Delta \). As the information brought by the model in Eq. 10 is local in space, it leads to a poor estimation of the kinetic energy at the cutoff wavelength, which can be improved by a suitable filtering to remove the influence of large scales on SGS viscosity. The procedure defined by Ducros \textit{et al.}\(^{32}\) consists in applying a discrete Laplacian high-pass filter to the velocity field before calculating the Structure Function. The resulting Filtered Structure Function model yields for the turbulent viscosity

\[ \mu_{\text{sgs}} = \overline{\mu}_{\text{sgs}} = \overline{\rho} \alpha^{(n)} \Delta \sqrt{\overline{F}_2^{(n)}(x, \Delta, t)} \quad (11) \]

where the superscript \( n \) indicates that the filter has been applied \( n \) times. The optimum values obtained by Ducros \textit{et al.}\(^{32}\) are \( n = 3 \) and \( \alpha^{(3)} = 0.00084 \). The formulation defined by Eqs. 7-11 insures that the SGS viscosity vanishes when there is no energy at the cutoff wavelength, which is crucial for the simulation of transitional flows, as shown in comparative analysis of SGS models for LES of trailing vortices.\(^{33}\) For example, large-eddy simulations of the elliptical stability of a vortex pair by Le Dizès and Laporte\(^{19}\) showed that the model correctly predicts the evolution of the vortex core radius and the transition to turbulence at
II.C. Non-dimensionalization

The transport equations 1-3 and 4-6 supplemented by the model 7-11 are solved in non-dimensional form using reference variables that univocally identify a reference or "infinite" state: length $l_{\text{ref}}$, density $\rho_{\text{ref}} \equiv \rho_\infty$, temperature $T_{\text{ref}} \equiv T_\infty / (\gamma - 1)$, speed of sound $a_{\text{ref}} \equiv a_\infty = \sqrt{\gamma R T_\infty}$, pressure $p_{\text{ref}} \equiv p_\infty / \gamma$, dynamic viscosity $\mu_{\text{ref}} \equiv \mu_\infty$ and specific heat $C_{p,\text{ref}} \equiv C_{p,\infty}$. Inserting Eqs. 7-9 into Eqs. 4-6 and defining the Reynolds number based on reference variables $Re = \rho_{\text{ref}} a_{\text{ref}} l_{\text{ref}} / \mu_{\text{ref}}$ finally yields (asterisk indicates non-dimensional variables):

$$\frac{\partial \bar{p}^*}{\partial t^*} + \nabla^* \cdot (\bar{p}^* \bar{u}^*) = 0,$$  \hspace{1cm} (12)

$$\frac{\partial (\bar{p}^* \bar{u}^*)}{\partial t^*} + \nabla^* \cdot (\bar{p}^* \bar{u}^* \otimes \bar{u}^* + \bar{p}^* I) = \frac{1}{Re} \nabla^* \cdot \left[ 2 \left( \bar{\mu}^* + \mu_{\text{sgs}}^* \right) \left( \bar{S}^* - \frac{\nabla^* \cdot \bar{u}^*}{3} I \right) \right],$$  \hspace{1cm} (13)

$$\frac{\partial (\bar{p}^* \bar{E}^*)}{\partial t^*} + \nabla^* \cdot (\bar{p}^* \bar{H}^* \bar{u}^*) = \frac{1}{Re} \nabla^* \cdot \left[ 2 \left( \bar{\mu}^* + \mu_{\text{sgs}}^* \right) \left( \bar{S}^* - \frac{\nabla^* \cdot \bar{u}^*}{3} I \right) \bar{u}^* \right]$$

$$+ \frac{1}{Re} \nabla^* \cdot \left[ \left( \bar{\mu}^* \mu_{\text{Pr}} + \frac{\mu_{\text{sgs}}^{* \mu_{\text{sgs}}}}{Pr_{\text{sgs}}} \right) C_{p,\text{ref}} \nabla^* T^* \right].$$  \hspace{1cm} (14)

Note that the resulting set of equations 12-14 are equivalent to the unfiltered compressible Navier-Stokes equations with the formal substitutions $\mu \rightarrow \bar{\mu} + \mu_{\text{sgs}}$ and $\lambda/C_{p} \equiv \mu/Pr \rightarrow \bar{\mu}/Pr + \mu_{\text{sgs}}/Pr_{\text{sgs}}$.

II.D. Numerical method

The non-dimensional transport equations 12-14 are discretized on collocated meshes with non-uniform grid spacing using a three-dimensional, finite differences Navier-Stokes solver.\textsuperscript{34–36} The spatial derivatives are computed using the sixth order compact scheme by Lele\textsuperscript{37} with modified coefficients to take into account the exact metrics of the mesh.\textsuperscript{35} As shown by Ga-met\textit{ et al.},\textsuperscript{35} the precision of the scheme reduces to fourth (third) order for the approximation of the first (second) derivative on stretched grids. In this work the mesh is enclosed in such a way that the vortex system is contained in a core region with uniform grid-spacing, and is stretched away using a geometric law (see Fig. 1). This insures that the scheme is still formally sixth-order accurate in the uniform region of interest where one needs highest precision to resolve the vortex instability and turbulence. On the other hand, mesh stretching insures that the boundaries of the domain are sufficiently far to reduce the influence of boundary conditions. We use symmetry boundary conditions in the transversal plane containing the vortices (this avoids inflow of spurious waves into the computational domain),\textsuperscript{38} while pe-
III. Results

III.A. Vortex models and Initial conditions

The initial velocity field in Eqs. (12)-(14) consists of a pair of two-dimensional axisymmetric vortices, co-rotating in the $x - y$ plane. Their properties can be conveniently described by considering an isolated vortex in a cylindrical coordinate system originating at the vortex center $(x_c, y_c)$. In the absence of axial flow the azimuthal velocity $v_\theta$ only depends on the radial coordinate $r$, so that the axial vorticity is given by $\omega = \frac{1}{r} \frac{\partial r v_\theta}{\partial r}$ (for notational ease we skip the asterisk symbol from non-dimensional variables). The total circulation $\Gamma$ and the second moment of vorticity $\Gamma a^2$ are given, respectively, by

$$
\Gamma = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \omega(x, y) \, dx \, dy = 2\pi \int_{0}^{+\infty} r \, \omega(r) \, dr, \quad (15)
$$

$$
\Gamma a^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} r^2 \omega(x, y) \, dx \, dy = 2\pi \int_{0}^{+\infty} r^3 \omega(r) \, dr \quad (16)
$$

where $a$ is the dispersion radius. In incompressible two-dimensional flows these equations identify the basic integral properties of the vortex: $\Gamma$ is a time invariant and represents the strength of the vortex, while $a$ represents an “objective” measure of the viscous core size of the vortex as it evolves solely by effect of viscosity. In this study we considered two analytical profiles of $v_\theta$ that have been used in the literature to model wake vortices: the Lamb-Oseen (LO) model and the VM2 model by Fabre and Jacquin. The Lamb-Oseen vortex is a classical model with Gaussian distribution of vorticity and azimuthal velocity given by

$$
v_{LO}(r) = \frac{\Gamma}{2\pi r} \left(1 - e^{-r^2/a^2}\right), \quad (17)
$$

$$
\omega_{LO}(r) = \frac{\Gamma}{\pi a^2} e^{-r^2/a^2}. \quad (18)
$$

The relevant length scale is the vortex core radius $r_c$ that identifies the location of maximum velocity $v_c \equiv v_{LO}(r_c) = \max(v_{LO})$ and separates the internal region of the vortex with solid body rotation ($v \sim r$) from the external potential-flow region ($v \sim 1/r$). From Eq. (17) it can be easily shown that the core radius is related to the dispersion radius via the relation $1 + 2(r_c^2/a^2) = \exp(r_c^2/a^2)$ or $r_c^2/a^2 \equiv \beta_{LO} \simeq 1.256$. Finally, integrating 15 and expressing...
the circulation in terms of \( r_c \) yields

\[
\Gamma = 2\pi r_c v_c \alpha_{LO}
\]

with \( \alpha_{LO} \equiv 1/[1 - \exp(-\beta_{LO})] \simeq 1.398 \).

The VM2 model\(^1\) represents the structure of vortex core by means of two length scales, an internal core scale \( a_1 \) and an external core scale \( a_2 \). As in the LO model, the flow is characterized by solid body rotation in the internal core \((r < a_1)\) and behaves as potential-flow outside the external core \((r > a_2)\). In the intermediate zone \( a_1 < r < a_2 \) velocity follows a power law \( v(r) \sim r^{-\alpha} \). Matching the three laws yields for the azimuthal velocity \(^1\)

\[
v_{VM2}(r) = \frac{\Omega_0 r}{1 + (r/a_1)^4} \left[ 2 - \left( \frac{1 + \alpha a_1}{a_1^4 + r^4} + \frac{1 - \alpha a_2}{a_2^4 + r^4} \right) r^3 \right]
\]

where \( \Omega_0 \) is the rotation rate of the internal core. Using Eqs. (15) and (21) gives the total circulation \( \Gamma = 2\pi\Omega_0 a_1^2 \left( \frac{a_2}{a_1} \right)^{1-\alpha} \). In this study we retained the characteristics parameters \( \alpha = 0.5 \) and \( a_2/a_1 = 10 \), as they were found to be representative of aircraft wakes.\(^2\) Using these values and defining the internal, \( v_1 = \Omega_0 a_1 \), and external, \( v_2 = \Gamma/2\pi a_2 \), velocity scales finally yields for the circulation

\[
\Gamma = 2\pi v_1 a_1 \alpha_{VM2}^{0.5,10}
\]

with \( \alpha_{VM2}^{0.5,10} \equiv (a_2/a_1)^{1-\alpha} \simeq 3.16 \). Inserting Eq. (21) into Eq. (16) and integrating numerically gives the dispersion radius through the relation \( a_2^2/a^2 \equiv \beta_{VM2}^{0.5,10} = 0.0294 \).

For the sake of comparison between the two vortex models we chose to conserve the same integral properties \( \Gamma \) and \( a \). Thus, for given LO parameters \( r_c \) and \( v_c \), we let \( a_1 \) and \( v_1 \) being free VM2 parameters and solve for the system:

\[
\begin{cases}
\frac{a_1^2}{\beta_{VM2}^{0.5,10}} = \frac{r_c^2}{\beta_{LO}} \\
2\pi a_1 v_1 \alpha_{VM2}^{0.5,10} = 2\pi r_c v_c \alpha_{LO}
\end{cases} \implies \begin{align*}
a_1 & = \sqrt{\frac{\beta_{VM2}^{0.5,10}}{\beta_{LO}}} \cdot r_c \simeq 0.153 r_c \\
v_1 & = \sqrt{\frac{\beta_{LO}}{\beta_{VM2}^{0.5,10}}} \cdot \frac{\alpha_{LO}}{\alpha_{VM2}^{0.5,10}} v_c \simeq 2.888 v_c
\end{align*}
\]

The initial vortex system is characterized by the ratio \( a_0/b_0 \) between the vortex core size \( a_0 \) and the vortex spacing \( b_0 \). The two values considered here, \( a_0/b_0 = 0.1 \) and 0.15 (see Tab. 1), are close to \( a/b \sim 0.13 \) which was deduced from the experimental results by Jacquin \textit{et al.},\(^2\) and can be considered as reasonable estimate in the case of merging between flap and wing-tip aircraft vortices in high-lift configuration.
III.B. Computational domain

For both two- and three-dimensional simulations the transverse domain in the $xy$ plane is defined by $L_x = L_y = 20 l_{ref}$ where the reference length $l_{ref} \equiv b_0$ (see Fig. 1), and consists of $N_x \times N_y = 401 \times 401$ gridpoints. The computational set-up for various configurations is summarized in Tab. 1 for convenience. The mesh is regular in the region of interest for the dynamics of the vortex system, $D_p = L_{xp} \times L_{yp} = 1.2 b_0 \times 1.2 b_0$, where the grid resolution is $\Delta_x = \Delta_y = 7 \times 10^{-3} b_0$. For initial ratios $a_0/b_0 = 0.1$ and 0.15 this gives, respectively, 14 and 21 points in the dispersion radius. The mesh is then stretched away to minimize the effects of the borders using a geometric law (see Sec. II.D). The maximum stretching coefficient is $\Delta x_{\text{max}}/\Delta x_{\text{reg}} \approx 1.23$. In the three-dimensional simulations the transverse domain is simply copied in the axial direction, the choice of its extention is discussed in detail in Sec. III.D.

III.C. Two-dimensional stable merging

The goal of this section is to characterize the basic two-dimensional process of stable merging using direct numerical simulations, and validate our model against theoretical results and available experimental data and correlations; the second goal is to get a reference computational set-up for the three-dimensional simulations of the elliptical instability.

The ensemble of the simulation parameters is reported in Tab.1. To illustrate the vortex dynamics during merging we report in Fig. 3 the evolution of the vorticity field for run $L2$, at Reynolds number based on vortex circulation $Re_f = \Gamma/\nu = 1500$. The figure shows that merging can be decomposed into four stages as first suggested by Cerretelli and Williamson: during the “first diffusion stage” the vortices simply turn around each other with angular velocity $\Omega_c = \Gamma/\pi b_0^2$ and turnover period $t_c = 2\pi/\Omega_c = 2\pi^2 b_0^2/\Gamma$, while their core size increases by viscous diffusion (Fig. 3a). Although the vortices rapidly become elliptical by mutual stretching, much of their vorticity is confined within the internal separatrix streamline forming a “8-shaped” loop in the reference frame rotating with the vortex system. When some portion of the fluid crosses this separatrix, it is advected along the two external streamlines and the vortices start exchanging their vorticity (Fig. 3b). This sets the beginning of the second stage or “convective stage”. Filaments of vorticity are created in the two recirculating-flow regions, which ultimately causes vortices to approach each other through the velocity induced by these filaments on each vortex. In the third stage or “second diffusion stage”, the two vorticity extrema persist until they finally merge (Fig. 3c). During the last stage (“axisymmetrization stage”), the vorticity filaments reconnect, the vortex axisymmetrizes and the core size increases by viscous diffusion (Fig. 3d).

It is instructive to analyze the merging process described through the evolution of the dispersion radius $a(t)$ and the vortex spacing $b(t)$, namely the distance between the two vortex cen-
ters (identified by local maximum of vorticity) as shown in Fig. 5 for run L2. (Note that time is rescaled by the turnover period, \( t^* = t/t_c \)). As pointed out by various authors in the literature, \(^8-\text{10}\) the onset of the vortex interaction occurs at a critical ratio \((a/b)_c \equiv a(t_1^*)/b(t_1^*)\) at the end of the first diffusion stage (when vortices start to exchange vorticity and \( b(t) \) first decreases, see Fig. 4). In all cases considered in this study we obtained \((a/b)_c \equiv a(t_1^*)/b_0 \sim 0.22\), which is independent on Reynolds number and in agreement with the results by Leweke et al.\(^10\) To evaluate \( a(t) \) before merging we used the definition given in Eq. (16) (that is valid for an isolated vortex) and limited the domain of integration to semi-infinite plane -rotating with the vortex system- that contains one of the vortices and is bounded by the orthogonal line to the vortex separatrix (see Fig. 3a). Note that this technique implicitly assumes that vortices are separated, which is not the case during the third stage of merging (see 3a), and explains the noisy behavior of \( a(t) \) in Fig. 4 for \( t_2^* < t^* < t_3^* \). Figure 5 shows the evolution of \( b(t) \) and \( a(t) \), for various Reynolds numbers and LO initial profiles. In the first stage \( 0 < t^* < t_1^* \) the computed values of \( a \) follows the theoretical law valid for an isolated vortex:\(^40\) \( a^2(t) = a_0^2 + 4\nu t \). Letting \((a/b)_c \equiv a(t_1^*)/b_0 \sim 0.22\), yields

\[
t_1^* = \frac{Re}{8\pi^2} \left[ \left( \frac{a}{b} \right)_c^2 - \frac{a_0^2}{b_0^2} \right],
\]

During the convective stage, \( t_1^* < t^* < t_2^* \), the vortex spacing decreases linearly in time (see Fig. 5) with a rate \( d(b/b_0)/dt^* = -3.25 \), again independent on Reynolds number. During the second diffusion stage, \( t_2^* < t^* < t_3^* \), \( b(t) \) continues to decrease and attains \( b \sim 0 \) at \( t = t_3^* \) (which indicates merging of the vortices). The duration of these two stages are \( t_2^* - t_1^* = \text{const} = 0.5 \) and \( t_3^* - t_2^* = \mathcal{O}(Re^{1/2}) \), respectively, and are both in agreement with the results reviewed by Meunier et al.\(^20\)

To analyze the structure of the final merged vortex at \( t^* = 1.7 \) we reconstructed the tangential velocity profile from the the radial circulation profile that is obtained by interpolating the vorticity field on a polar grid originating at the center of the vortex, \( \omega(x,y) \rightarrow \omega(\rho,\theta) \). Integrating on a disk of radius \( r \) yields:

\[
\Gamma(r) = \int_0^{2\pi} \int_0^r \omega(\rho,\theta) \rho \, d\rho \, d\theta \quad \Rightarrow \quad v_\theta(r) = \frac{\Gamma(r)}{2\pi r}.
\]

In Fig. 6 we report the evolution of the velocity profiles for run V2. The figure shows that the VM2 profile evolves towards a profile described by one length scale since the intermediate region vanishes by diffusion of the internal vortex region (the internal radius \( a_1 \) increases while the external radius \( a_2 \) remains constant). The velocity profiles become indistinguishable at \( t^* \sim 0.66 \), just before the beginning of the convective phase (note that this process is
faster than complete merging which is attained at $t^* = 1.7$, see Fig. 5). This results can be associated to the viscous relaxation process identified by Le Dizès and Verga. They demonstrated that two non-Gaussian vortices, such as in the VM2 model, evolve by slow diffusion toward a single attractive solution which corresponds to a Gaussian vortex system as in the LO model. In other words, vortices “forget” the initial two-scale structure before starting to exchange their vorticity.

At the end of the merging process both models again predict a two-scale vortex structure like in the initial VM2 model but with a steeper power-law region ($\alpha = 0.6$ instead of 0.5). This is clearly shown in the profiles of Fig. 7 that were extracted at $t^* = 1.7$ (runs L2 and V2) when the final merged vortex has one single core (one vorticity extremum). The same behavior and the same coefficient of the power-law region were obtained for all two-dimensional cases at various (low) Reynolds numbers shown in Tab. 1, and was also observed in model-scale experiments by Jacquin et al. Thus, the two-scale vortex structure seems to be intrinsic to the process of stable merging. The generalization of these results to three-dimensional and high Reynolds number flows is discussed in the next section.

### III.D. Three-dimensional unstable merging

The flow parameters and model equations used for these three-dimensional simulations are again summarized in the Tab. 1, for both LO and VM2 vortex pairs. The simulations at $Re_T = 10000$ were chosen for comparison with two-dimensional results of stable merging, while the runs at $Re_T = 240000$ allow flow conditions closer to the ones encountered in aircraft wakes. The initial condition consists of a co-rotating vortex pair located at the center of the computational domain as done in Sec. III.C. The theoretical predictions of the elliptical instability by Le Dizès and Laporte were used to determine the mode with the highest growth rate (the “most unstable” mode) as a function of the flow configuration, $Re_T$ and $a_0/b_0$. In principle, the length of the axial domain $L_z$ can then be chosen as a multiple of the (theoretical) wavelength $\lambda_{ell}^{th}$ of the elliptical instability; here we took $L_z = 3\lambda_{ell}^{th}$ that insures the confinement effects of the computational domain are limited. Each wavelength is discretized with 12 points, so that the number of gridpoints in the axial direction is $N_z = 36$. The simplest way to trigger the instability at a given wavenumber would to inject energy at the corresponding wavelength using some forcing technique. On the other hand, one motivation of this study was to understand if and how other waves other than the elliptical instability wave emerge in co-rotating vortices. Thus, instead of forcing selectively the
elliptical instability, the initial flow-field is perturbed by adding a white noise:

\[
\begin{align*}
    u_0 &= u_{LO,VM2} \times (1 + \varepsilon \text{rand}(x, y, z)) \\
    v_0 &= v_{LO,VM2} \times (1 + \varepsilon \text{rand}(x, y, z)) \\
    w_0 &= 0
\end{align*}
\]

(25)

where \text{rand} \in [-1/2;1/2] is a random number modeling white noise and \( \varepsilon = 10^{-3} \) is the magnitude of the white noise.

III.D.1. Topology of the flow

It has been demonstrated theoretically by Le Dizès and Laporte\textsuperscript{19} that a co-rotating vortex pair can be unstable when the Reynolds number based on circulation is higher than a critical value \( Re_c \) that depends on the initial ratio \( a_0/b_0 \) (see Fig. 8). Physically, this means that the diffusion of the vortex cores is sufficiently slow that three-dimensional short-wave instability develops when the vortices are still separated (before the convection stage). Figure 8 shows that for \( a_0/b_0 = 0.15 \) and \( Re_c > 3500 \) all three-dimensional simulations in Tab. 1 are potentially unstable. We use runs \( L5 \) and \( V5 \) to illustrate the three-dimensional dynamics of the vortex system. The short-wave instability is responsible for unstable merging and is shown in Fig. 9 by the sequence of the snapshots of the vorticity magnitude \( \|\omega\| \). The development of the instability is characterized by the oscillation of the vortex core in the axial direction (see Fig. 9, \( t^* = 1.95 \)). This form of this oscillation is associated to the most unstable axial mode, i.e. the mode with the highest growth rate (see discussion in Sec. III.D.2 below). In particular, the figure clearly shows the emergency of mode \( k = 4 \) for LO vortex model and mode \( k = 3 \) for VM2 vortex model. When the unstable mode saturate the flow becomes turbulent and the two vortices start to exchange their vorticity (Fig. 9, \( t^* = 2.45 \)). It can be appreciated from the vorticity iso-surfaces that the VM2 vortex simulation exhibits more small-scale turbulent structures than LO vortex at the same wake age, suggesting that the growth rate of the most unstable mode is higher in the VM2 model so that the instability saturates earlier. This qualitative feature is in good agreement with the linear stability analysis by Fabre and Jacquin\textsuperscript{1} and is explained by spectral analysis in Sec. III.D.2. The vortices merge and the resulting coherent vortex re-laminarizes at \( t^* = 3.2 \). Finally, it is interesting to point out that the development of short wave instability leads to faster merging is compared to the two-dimensional case: for the configuration \( a_0/b_0 = 0.15 \) and \( Re_T = 10000 \), the stable merging was reached at about \( t^* \sim 5.1 \) while the unstable merging is at \( t^* \sim 3.2 \).
III.D.2. Spectral analysis

A spectral analysis was performed by FFT (Fast Fourier Transform) to determine the effective mode, wavelength and growth rate of the instability. The growth rate is obtained from the Fourier decomposition of the total kinetic energy

$$\sigma_k = \frac{d\ln(E_k^{1/2})}{dt}$$

(26)

where $E_k^{1/2}$ is the Fourier coefficient of mode $k$, respectively. The corresponding wavelength is $\lambda_k = \frac{L_z}{k}$ and the wavenumber is $k_0 = \frac{2k\pi}{L_z}$.

In all cases a transition is observed before the onset of the instability (which corresponds to the emergence of the dominating unstable mode). The computed growth rates for all three-dimensional runs rare reported in Tab. 2 together with the theoretical values obtained from the analysis of linear stability of a LO vortex pair.19

As mentioned in the previous section the length of the axial domain was taken as three times the wavelength of the (theoretical) elliptical instability. However, in run $L5$ at $Re_T = 10000$ the most unstable mode obtained from Fourier analysis is $k = 4$ with a wavelength $\lambda_4/b_0 = 0.37$, while the theoretical prediction gives $k = 3$ and $\lambda_3/b_0 = 0.493$. This was qualitatively observed in Fig. 9 and is confirmed in Fig. 10: the mode $k = 3$ develops first, but rapidly saturates before growing again around $t^* \approx 1.5$ when the mode $k = 4$ emerges as the dominating mode that effectively drives the dynamics of the flow. This mismatch in wave selection is due to viscosity effects. Indeed, when the vortices start to interact at $t^* \approx 0.6$ (beginning of the linear regime) their core radius has increased by viscous diffusion to $a(0.6) \approx 1.19 a_0$ and so the effective ratio $a(0.6)/b_0 \approx 0.178$ is different from the initial $a_0/b_0 = 0.15$ that is used to get the theoretical growth rate. To clarify this point a second simulation (run $L6$) was performed with the same initial conditions but with a larger axial domain, $L_z/b_0 = 1.92$, that is three times the wavelength of the elliptical instability predicted by the theory using $a(0.6)/b_0 = 0.178$ instead of $a_0/b_0$. In this case Fig. 10 shows that mode $k = 3$ emerges as the dominating mode, although with a slightly larger growth rate than the theoretical value (see Tab. 2).

The aim of run $V5$ is to compare the instability occurring in the two vortex models at a relatively low $Re_T = 10000$ and $L_z/b_0 = 1.48$. The spectral analysis indicates that the mode $k = 3$ with $\lambda_3/b_0 = 0.493$ is the most amplified mode, followed by mode $k = 4$ (the opposite of run $L5$). This can be explained by observing that at the beginning of the linear regime the internal radius $a_1$ has increased by diffusion, $a_1(0.6)/a_1(0) \approx 1.18$, while the external radius is almost the constant, $a_2(0.6) \approx a_2(0)$, and the ratio $a(0.6)/b \approx 0.178$ as for run $L6$ (LO model with larger domain). On the other hand, Fabre and Jacquin1 showed that the
wavelength of the instability of the VM2 model is of the order of $a_1$ when the coefficient of the power-law region is $0.5 \leq \alpha < 1$, and of the order of $a_2$ when $0 < \alpha < 0.4$. At $t^* = 0.6$ \(\alpha\) decreased to some value around 0.4 (not shown), so that the results consistently give $\lambda_3 \sim a_2$ as the correct scaling of the instability. Furthermore, for $a_2/a_1 = 10$ and $\alpha = 0.5$ the predicted growth rate of the instability is 30% larger than in a LO model with the same dispersion radius. In run $L5$ we obtained $\sigma_3^* = 6.2$ while in run $L6$ it was $\sigma_3 = 5.1$ (see Tab. 2), which is consistent considering that the ratio $a_2/a_1$ slightly decreased at $t^* = 0.6$.

The last two simulations (runs $L7$ and $V6$) are useful to compare the flow instability in the two vortex models at $Re = 240000$, closer to aircraft wakes Reynolds number. The Fourier analysis shows that mode $k = 3$ is the most amplified mode in the LO model (run $L7$) as expected from linear theory (see Fig. 11). Indeed, as discussed in Sec. III.C for stable merging the dispersion core evolves very slowly at high Reynolds (see Fig. ??) so that the initial ratio $a_0/b_0 = 0.15$ is maintained at the beginning of the linear regime. In the case of VM2 vortex profile (run $V6$) the power-law region slightly reduces before the beginning of the linear regime ($t^* \approx 0.5$). The most amplified mode is $k = 7$ as also reported in Fig. 11). The corresponding wavelength $\lambda_3/b_0 = 0.211$ is closer to the external radius $a_2(0.5)/b_0 \approx 0.257$ than the internal radius $a_1(0.5)/b_0 \approx 0.09$, which again is consistent with theoretical prediction\(^1\) since the power-law coefficient $\alpha(0.5)$ is close to 0.4. In addition, the growth rate for VM2 model is higher than the Lamb-Oseen model (see Tab 2).

III.D.3. Structure of the final vortex

An important question we want to address in this study is to find our whether the different growth rates of the instability, associated to the different vortex models and flow conditions, reflect into different profiles of the final merged vortex. This is also a critical information that is needed to characterize the counter-rotating vortex pair in the extended near-field wake and to quantify the wake hazard in the following far-field wake. To extract the velocity profiles of the final merged vortex at $t^* = 3.93$ we averaged the velocity fields in the axial direction and interpolated these fields into polar grid as done in Sec. III.C (see Eq. 24). These profiles are plotted in Fig. 12 for runs $L5$, $V5$ and $L7$, $V6$. It is interesting to observe that all curves exhibit two-scale vortex profile as in VM2 model. It is interesting to observe that the coefficient of the power law region is roughly the same for all cases analyzed, $\alpha = 0.35 - 0.4$ (although is different from the value $\alpha = 0.5$ used to initialize VM2 runs). According to the above findings, we can reasonably conclude that the structure of the merged vortex seems to be independent on Reynolds numbers and the initial conditions (vortex profile and ratio $a/b$), but it depends on the type (stable or unstable) of merging. Indeed, in the case of stable merging we obtained $\alpha \simeq 0.6$ in the power-law region in all cases considered here.
IV. Conclusion

In this work we used direct and large-eddy simulation to analyze the process of stable and unstable merging of co-rotating vortices, and its dependence on the initial vortex profile, vortex spacing and Reynolds number. In the two-dimensional stable case, DNS at various low Reynolds numbers showed that merging is initiated by the exchange of vorticity between the vortices, which is favored by the viscous diffusion of the cores and is associated to a critical ratio of vortex dispersion radius over vortex spacing \((a/b)_c \sim 0.22\), in accordance to literature work. The results showed that Reynolds number does not affect the dynamics of the flow and the structure of the final merged vortex, but increases the final time of merging. The three-dimensional unstable merging was simulated using direct and large-eddy simulations for Reynolds numbers ranging from \(Re_T = 10000\) to \(240000\). The results showed that the elliptical instability develops in the vortex core due to the strain field which is induced by one vortex on the other. The computed growth rates agree with the theoretical predictions both for one-scale vortex pair (Lamb-Oseen model) and two-scale vortex pair (VM2 model), provided the correct development of the vortex core size is taken into account in the wave selection of the elliptical instability. It was found that the growth rate of instability is higher in a VM2 model than a LO model, although the profile of tangential velocity is the same. This is seems to be a common feature of stable and unstable merging: the structure of the final vortex is independent on Reynolds number and the initial ratio \(a_0/b_0\) and is characterized by a two-scale vortex velocity profile. However, the coefficient \(\alpha\) of the power law region, \(r^{-\alpha}\), does depend on the type of merging: for a stable merging \(\alpha \sim 0.6\) and in the other case \(\alpha \sim 0.35\).

V. Acknowledgments

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References


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Table 1. Summary of numerical configurations.
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Table 2. Spectral analysis results: Fourier mode number $k$ and corresponding instability wavelength $\lambda/b_0$ and its growth rate $\sigma^* = \sigma \times t_c = \sigma \times (2\pi^2 b_0^2/\Gamma)$. The theoretical prediction for Gaussian vortices is also noted, for the viscous configuration $\sigma^*_{thv}$ and the inviscid one $\sigma^*_{thinv}$. 
Figure 1. Computational domain and initial location of the vortex pair. Axes are normalized by the initial spacing $b_0$ between the vortices.
Figure 2. Initial profiles of the azimuthal velocity of VM2 (solid line) and Lamb-Oseen (dashed line) vortex for the simulations with \( a_0/b_0 = 0.15 \) (top, linear plot; bottom, logarithmic plot). Profiles are rescaled by circulation \( \Gamma_{\text{tot}} = \Gamma_1 + \Gamma_2 \) and initial dispersion radius \( a_0 \); the distance \( r \) from the vortex center is normalized by the initial spacing \( b_0 \).
Figure 3. Evolution of the normalized vorticity isocontours ($\omega/\Gamma/a_o^2$) during the process of two-dimensional merging (run $V2$). The streamlines are represented by solid black lines and are calculated in the reference frame rotating at the angular velocity of the vortex system, $\Omega = \Gamma / \pi b_o^2$. 

a) First diffusion stage $t^* = 0.19$

b) Convective stage $t^* = 1.1$

c) Second diffusion stage $t^* = 1.52$

d) Axisymmetrization stage $t^* = 2.02$
Figure 4. Evolution of the normalized vortex spacing $b/b_0$ (solid line) and dispersion radius $a/b_0$ (dashed line) during stable merging (run L2).
Figure 5. Evolution of the normalized spacing $b/b_0$ (top) and dispersion radius $a/a_0$ (bottom) for various $Re$ (symbols). The theoretical diffusion law $a(t) = \sqrt{4\pi t + a_0^2}$ is also represented (solid line).
Figure 6. Evolution of the velocity profiles of a VM2 vortex (run V2): solid line: \( t^* = 0 \); dashed line: \( t^* = 0.11 \); dash-dotted line: \( t^* = 0.5 \) (top: linear plot; bottom, logarithmic plot). Velocity is rescaled by the total circulation \( \Gamma_{\text{tot}} = \Gamma_1 + \Gamma_2 \) and the initial dispersion radius \( a_0 \).
Figure 7. Velocity profiles of the final merged vortex at $t^* = 1.7$ for an initial VM2 vortex (run $V_2$, symbols) and Lamb-Oseen vortex (run $L2$): top, linear plot; bottom, logarithmic plot. Profiles are rescaled by total circulation $\Gamma_{\text{tot}} = \Gamma_1 + \Gamma_2$ and the initial dispersion radius $a_0$. 

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Figure 8. Theoretical prediction by Le Dizès and Laporte\textsuperscript{19} of the critical Reynolds number as a function of the ratio $a/b$ for co-rotating vortices (solid line) and counter-rotating vortices (dashed line). Symbol correspond to the values of $Re$ and $a_0/b_0$ used in the simulations of Tab. 1: $\circ$, two-dimensional merging; $+$, three-dimensional merging.
Figure 9. Evolution of two selected isosurface of vorticity magnitude during the unstable merging, \( \omega_1 = \frac{|\mathbf{\omega}|}{(\Gamma/a_0)} \sim 0.18 \) and \( \omega_2 \sim 0.11 \) and colored by the instantaneous level of vorticity magnitude. The isocontours of the normalized vorticity magnitude are also represented in the first plane (32 levels \( \in [\omega_{\text{min}}, \omega_{\text{max}}] \)). Left panel: initial Lamb-Oseen vortex system (run L5); right panel: initial VM2 vortex system (run V5). Top to bottom: \( t^* = 1.95, 2.42, 3.29 \).
Figure 10. Time history of the energy of the most unstable Fourier modes (run L5). Solid line: $L_z/b_0 = 1.48$ and $k = 3$; dashed line, $L_z/b_0 = 1.48$ and $k = 4$; dashed-dot line $L_z/b_0 = 1.92$ and $k = 3$. 
Figure 11. Time history of the energy of the most unstable Fourier modes at high Reynolds, $Re = 240000$. Dashed line, initial Lamb-Oseen vortex (run L0); solid line, initial VM2 vortex (run V6).
Figure 12. Velocity profiles of the final vortex at the end of unstable merging. Dashed line: initial Lamb-Oseen vortex (run \(L5\) et \(L6\)); solid line: initial VM2 vortex (run \(V5\) et \(V6\)). Thick line at \(t^* = 3.29\) and \(Re_T = 10000\) and thin line at \(t^* = 3.93\) for \(Re_T = 240000\).