Non-uniform time sampling for multiple-frequency harmonic balance computations

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Abstract

A time-domain harmonic balance method for the analysis of almost-periodic flows is presented. This kind of method uses the Fourier analysis to derive a more efficient time marching scheme than classical ones, on such flows. It has recently gain important interest, especially in the turbomachinery field where the flow spectrum is essentially a combination of the blade passing frequencies. Traditionally, researchers have used a uniform sampling but in the case of several frequencies non necessarily multiple of each other, harmonic balance methods can face stability issues due to a bad condition number of the Fourier operator. Two algorithms are derived to find a non uniform time sampling in order to minimize this condition number. Their behavior are studied on a wide range of frequencies and a model problem of a 1D flow with pulsating pressure enables to prove their efficiency. Finally, a turbomachinery flow is analyzed with different frequency sets. It demonstrates the stability and robustness of the present non-uniform Harmonic Balance method regardless of the frequency set.

Keywords: Harmonic balance, almost-periodic flow, time sampling, condition number, turbomachinery

1 Introduction

Up to now, the industrial design of multistage turbomachines was based on steady analysis, for which the most advanced tools were three-dimensionnal Reynolds-Averaged Navier-Stokes steady computations. With the need to ever increase performance, aggressive design choices foster unsteady phenomenon, such as: blade interactions in compact turbo-engines, separated flows at, or close to stable operability limits or aeroelastic phenomenon, to name but a few. In such a context, engineers now need tools to account for these effects as early as possible in the design cycle. With the increase of computational power, unsteady computations are entering industrial practice, but the associated restitution time remains an obstacle for daily basis applications. For this reason, efficient and/or accurate unsteady approaches are receiving a lot of attention. Different ways can be pursued to achieve an appropriate trade-off between efficiency and accuracy.

A first approach is to deal with the model equations: the Unsteady Reynolds-Averaged Navier-Stokes (U-RANS) equations can be simplified using some level of linearization (see Refs. [18, 7, 20]), to obtain a fast solution but with some limitations in nonlinear regimes (see Ref. [8] for an example of accuracy issues, and Ref. [20] for some cure of stability problems). Conversely, the Large-Eddy Simulation (LES) approach can be used to increase accuracy [15, 34], but at prohibitive cost.

A second approach, usually based on the U-RANS equations but not necessarily, is to work on the time-integration algorithm to reduce the computational cost as compared to standard techniques. To achieve this, Fourier-based methods for periodic flows have undergone major developments in the last decade (see He [21] for a recent review, or the special issue of the International J. of CFD [23]). The basic idea is to decompose time-dependent flow variables into Fourier series, which are then injected into the equations of the problem. The time-domain problem is thus made equivalent to a frequency-domain problem, where the complex Fourier coefficients are the new unknowns. At this point, two strategies exist to obtain the solution. The first one is to solve directly for the Fourier coefficients, using a dedicated frequency-domain solver, as proposed by He and Ning [22, 32]. The second strategy is to cast the problem back
to the time domain using inverse Fourier transform, as proposed by Hall [17, 13] with the Harmonic Balance (HB) method. The unsteady time-marching problem is thus transformed into a set of steady equations coupled by a source term that is a high-order spectral evaluation of the time-derivative of the initial equations. The main advantage of solving in the time domain is that it can be implemented in an existing classical RANS solver, taking advantage of all classical convergence-acceleration techniques for steady state problems. The HB approach has demonstrated significant reduction of computational time, typically of a factor 2 to 10.

In turbomachines, the relative motion of fixed and rotating blades gives rise to deterministic unsteady interactions at frequencies termed BPFs (for Blade Passing Frequencies). In a multi-stage turbomachine, a row sandwiched between two others is submitted to (at least) two BPFs (see Tyler and Sofrin [40] for instance), hence the need for multiple frequency methods. Initially developed for single frequency problems, harmonic methods have been extended to account for multiple frequencies [14, 9, 10]. All the variations of the HB technique proposed in the literature rely on a uniform time sampling of the longest period of interest (though the number of samples can differ). Ekici and Hall [9] mentioned the use of non-uniform sampling but did not develop it. However, when the fundamental frequencies involved are significantly different, uniform sampling leads to an unnecessary high number of time samples: given that the shortest period has to be discretized by at least three instants (Shannon [35] requires at least two instants per period to capture a frequency but stability issues imply an odd number of samples [41]), uniform sampling of the longest period requires a total number of samples that grows with the ratio of the largest to shortest periods. This can compromise the efficiency of the method, as too many time samples are computed. Besides, as demonstrated in the present contribution, uniform time sampling can also raise stability issues. To overcome these computational limitations, a new approach using non-uniform time sampling is proposed in the present contribution.

This paper is organized as follows: First, in section 2, mono- and multi-frequency HB methods are presented, and the impact of time sampling on numerical stability is discussed. Then, two algorithms for an automatic choice of the time samples are presented and compared in section 3. The proposed non-uniform sampling is assessed for a model problem in section 4. Finally, section 5 is dedicated to the application to a turbomachinery configuration, with emphasis on the choice of frequencies.

2 Time-domain harmonic balance technique

The Unsteady Reynolds-Averaged Navier-Stokes (U-RANS) equations in integral form are given by

$$
\int_{\Omega} \frac{\partial W}{\partial t} dV + \oint_{\partial \Omega} \vec{F} \cdot \vec{N} ds = 0,
$$

where $\vec{F}$ is the flux across $\partial \Omega$ and $W$ is the vector of the conservative unknowns (conservative variables plus complementary turbulent variables). Assuming $\Omega$ is a control volume, the semi-discrete finite-volume form of the Navier-Stokes equations is obtained from Eq. (1):

$$
\frac{d}{dt} (V \bar{W}) + R(\bar{W}) = 0,
$$

with $V$ the volume of the cell $\Omega$, $R$ the residual resulting from the discretization of the fluxes and the source terms (including the turbulent equations) and $\bar{W}$ the mean of the unknowns.
over the control volume. In the following, the over line symbol is dropped out for clarity. Moreover, the mesh is considered not deformable, which allows to remove the volume $V$ of the time derivative in Eq. (2), and simplifies explanations. However, the treatment remains valid if the mesh is deformable (see Ref. [8] for instance).

2.1 Periodic flows

If the mean flow variables $W$ are periodic in time of period $T = 2\pi/\omega$, so are the residuals $R(W)$ and the Fourier series of Eq. (2) is

\[ \sum_{k=-\infty}^{\infty} \left( ik\omega V\hat{W}_k + \hat{R}_k \right) e^{ik\omega t} = 0, \]  

(3)

where $\hat{W}_k$ and $\hat{R}_k$ are the Fourier coefficients of $W$ and $R$ corresponding to mode $k$:

\[ W(t) = \sum_{k=-\infty}^{\infty} \hat{W}_k e^{ik\omega t}, \quad R(t) = \sum_{k=-\infty}^{\infty} \hat{R}_k e^{ik\omega t}. \]  

(4)

The complex exponential family forming an orthogonal basis, the only way for Eq. (3) to be true is that the weight of every mode $k$ is zero, which leads to an infinite number of steady equations in the frequency domain:

\[ ik\omega V\hat{W}_k + \hat{R}_k = 0, \quad \forall k \in \mathbb{Z}. \]  

(5)

McMullen et al. [30, 31, 29] solve a subset of these equations up to mode $N$, $-N \leq k \leq N$, yielding the Non-Linear Frequency Domain (NLFD) method.

The principle of the time-domain Harmonic Balance approach, sometimes referred to as Time Spectral Method (TSM) [13, 36], is to use an Inverse Discrete Fourier Transform (IDFT) to cast the equations back into the time domain. The IDFT then induces linear relations between Fourier coefficients $\hat{W}_k$ and a uniform sampling of $W$ at $2N+1$ instants in the period:

\[ W_n = \sum_{k=-N}^{N} \hat{W}_k \exp(i\omega n\Delta t), \quad 0 \leq n < 2N + 1, \]  

(6)

with $W_n \equiv W(n\Delta t)$ and $\Delta t = T/(2N+1)$. This leads to a new system of $2N+1$ mathematically steady equations coupled by a source term:

\[ R(W_n) + VD_t(W_n) = 0, \quad 0 \leq n < 2N + 1. \]  

(7)

The source term $VD_t(W_n)$ appears as a high-order formulation of the initial time derivative in Eq. (2). This new time operator connects all the time levels and can be expressed analytically by

\[ D_t(W_n) = \sum_{m=-N}^{N} d_m W_{n+m}, \]  

(8)
with
\[
d_m = \begin{cases} 
\frac{\pi}{T} (-1)^{m+1} \csc \left( \frac{\pi m}{2N+1} \right), & m \neq 0, \\
0, & m = 0. 
\end{cases}
\] (9)
A similar derivation can be made for an even number of instants, but it is proved in Ref. [41] that it can lead to a numerically unstable odd-even decoupling.

A pseudo-time \( \tau_n \) derivative is added to Eqs. (7) to time march the equations to the steady-state solutions of all the instants:
\[
V \frac{\partial W_n}{\partial \tau_n} + R(W_n) + V D_t(W_n) = 0, \quad 0 \leq n < 2N + 1.
\] (10)
This time step is local to a given cell and different for all HB instants. For stability reasons, its computation step is modified [41] to take into account this additional source term,
\[
\Delta \tau_n = \text{CFL} \frac{V}{\|\xi_n\| + \omega NV}.
\] (11)
The extra term \( \omega NV \) is added to the spectral radius \( \|\xi_n\| \) to restrict the time step. Equation (11) implies that a high frequency and/or a high number \( N \) of harmonics can considerably constrain the time step, especially for explicit Runge Kutta time integration scheme, as mentioned in [17]. Several implicit schemes, which are more stable and thus allow larger CFL number, have been derived for the HB method: Krylov-space based methods are used in [39, 43], and Antheaume et al. [1] propose a point Jacobi algorithm. The present paper uses the block-Jacobi algorithm derived in Ref. [36] to improve robustness and efficiency.

This time-domain harmonic balance method was implemented in the elsA solver developed by Onera [5] and Cerfacs. This code solves the RANS equations using a cell-centered approach on multi-blocks structured meshes. Significant savings in CPU costs have been observed in various applications such as dynamic derivatives computation [19], aeroelasticity [8] and rotor/stator interactions [37]. However, this approach is limited to periodic flows (i.e. a single fundamental frequency) and is unfit when the main frequencies of the system are not integer multiple of each others (e.g. a sandwiched row in a multi-stage turbomachine, which is submitted to the upstream and downstream blade passing frequencies). The single-frequency HB method is therefore extended to the case where the flow is not periodic in time but is almost periodic.

2.2 Almost-periodic flows

2.2.1 Mapping on a set of arbitrary frequencies
If the flow variables are composed of non-harmonically related frequencies (i.e. the flow spectrum has high-energy discrete-frequency modes), the flow regime can be termed almost-periodic [2]. Instead of a regular Fourier series, the U-RANS equations are mapped on a set of complex exponentials with arbitrary angular frequencies \( \omega_k \). The conservative variables and the residuals are then approximated by
\[
W(t) \approx \sum_{k=-N}^{N} \hat{W}_k e^{i\omega_k t}, \quad R(t) \approx \sum_{k=-N}^{N} \hat{R}_k e^{i\omega_k t},
\] (12)
where \( \hat{W}_k \) and \( \hat{R}_k \) are the coefficients of the almost-periodic Fourier series for the frequency \( f_k = \omega_k/2\pi \). Injecting this decomposition in Eq. (2) yields
\[
\sum_{k=-N}^{N} \left( i\omega_k V \hat{W}_k + \hat{R}_k \right) e^{i\omega_k t} = 0. \tag{13}
\]
Sampling in time onto a set of \( 2N + 1 \) time levels to solve Eq. (13), the following matrix formulation is obtained:
\[
A^{-1} \cdot \left( iVP\hat{W}^* + \hat{R}^* \right) = 0, \tag{14}
\]
where the almost-periodic inverse discrete Fourier transform (IDFT) matrix reads:
\[
A^{-1} = \begin{bmatrix}
\exp(i\omega_{-N}t_0) & \cdots & \exp(i\omega_0t_0) & \cdots & \exp(i\omega_Nt_0) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\exp(i\omega_{-N}t_k) & \cdots & \exp(i\omega_0t_k) & \cdots & \exp(i\omega_Nt_k) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\exp(i\omega_{-N}t_{2N}) & \cdots & \exp(i\omega_0t_{2N}) & \cdots & \exp(i\omega_Nt_{2N})
\end{bmatrix}, \tag{15}
\]
with \( \omega_0 = 0, \; t_0 = 0 \) and
\[
P = \text{diag}\left(-\omega_N, \ldots, \omega_0, \ldots, \omega_N\right),
\hat{W}^* = \left[\hat{W}_{-N}, \ldots, \hat{W}_0, \ldots, \hat{W}_N\right]^\top,
\hat{R}^* = \left[\hat{R}_{-N}, \ldots, \hat{R}_0, \ldots, \hat{R}_N\right]^\top. \tag{16}
\]
As opposed to the case of periodic flow, the arbitrary complex exponentials family does not form, \textit{a priori}, an orthogonal basis.

Knowing a time sampling that allow \( A^{-1} \) to be invertible, the almost-periodic Fourier coefficients can be approximated thanks to
\[
\begin{cases}
\hat{W}^* = AW^*, & \text{with } W^* = [W(t_0), \ldots, W(t_i), \ldots, W(t_{2N})]^\top, \\
\hat{R}^* = AR^*, & \text{with } R^* = [R(t_0), \ldots, R(t_i), \ldots, R(t_{2N})]^\top.
\end{cases} \tag{17}
\]
Eq. (14) thus becomes
\[
iVA^{-1}PA + R^* = VD_t[W^*] + R^* = 0, \tag{18}
\]
where the multiple-frequency HB time-derivative operator \( D_t[] = iA^{-1}PA \), the HB source term, can not be easily derived analytically, and has to be numerically computed.

At this step of the derivation of the method, the time sampling \([t_0, \ldots, t_{2N}]\) remains to be specified.

2.2.2 Condition number and convergence

Kundert \textit{et al.} \cite{26} show that the condition number of \( A \), and thus \( A^{-1} \), has a salient role in the convergence of a Harmonic Balance computation. The condition number of the almost-periodic DFT matrix \( A \) is defined as
\[
\kappa(A) = \kappa(A^{-1}) = \|A\| \cdot \|A^{-1}\|, \quad \kappa(A) \geq 1, \tag{19}
\]

where \( \| \cdot \| \) denotes a matrix norm. Considering the resolution of \( Ax = b \), if \( A \) is invertible and if \( \delta A \), \( \delta x \) and \( \delta b \) are the numerical errors associated to the computation of \( A \), \( x \) and \( b \) respectively, then

\[
(A + \delta A)(x + \delta x) = b + \delta b.
\]

Therefore, the condition number sets an upper bound of the error made on \( x \):

\[
\frac{\| \delta x \|}{\| x \|} \leq \kappa(A) \left[ \frac{\| \delta A \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} \right].
\]

(21)

The error on the iterative resolution of the U-RANS equations can therefore be amplified by the HB source term. This amplification is led by the condition number of the almost-periodic DFT matrix. This also means that if the errors are high but the condition number is small and vice-versa, the computation can diverge too. However, the errors can not be a priori controlled, thus the will to minimize the condition number.

In the case of periodic-flows, the DFT matrix is well-conditioned: the uniform sampling for harmonically related frequencies leads to a condition number equal to 1, which is the theoretical lower bound for the condition number. This is linked to the orthogonality of the complex exponential family. Unlike the periodic case, it is usually impossible to choose a uniform set of time instants over which the almost-periodic DFT matrix \( A \) is well conditioned when the frequencies are arbitrary. In fact, it is common for uniformly-sampled sinusoids at two or more frequencies to be nearly linearly dependent, which causes them not to be orthogonal, leading to the ill-conditioning encountered in practice. As the frequency set is chosen by the user, the only degrees of freedom left to get a well-conditioned matrix are the time levels. The following section describes two algorithms to find a non-uniform time sampling that minimizes the almost-periodic DFT matrix condition number.

3 Non-Uniform time sampling algorithms

Two algorithms that automatically choose the time levels in order to minimize the condition number are presented: first, the Almost Periodic Fourier Transform (APFT) algorithm proposed in the electronics literature is described, then a gradient-based optimization algorithm over the condition number (OPT) is presented.

3.1 The APFT algorithm

Based on the work of Kundert et al. [26] in electronics, the APFT algorithm has been implemented. The aim of the APFT algorithm is to maximize the orthogonality of the almost-periodic DFT matrix in order to minimize its condition number. It is based on the Gram-Schmidt orthogonalization procedure. First, the greatest period \( 1/\min_k(f_k) \) is oversampled with \( M \) equally-spaced time levels, \( M \gg 2N + 1 \) being specified by the user and \( N \) the number of frequencies. Considering these time levels, a rectangular almost-periodic DFT matrix is built. Noting that every row of this matrix is a vector, a set of \( M \) vectors is obtained, numbered from 0 to \( M - 1 \), and of length \( 2N + 1 \). The first vector \( V_0 \) (corresponding to \( t = 0 \)) is arbitrarily chosen as the first time level and any component in the direction of \( V_0 \) is removed from the following vectors using the Gram-Schmidt formula:

\[
V_s = V_s - \frac{V_0^\top \cdot V_s}{V_0^\top \cdot V_0} V_0, \quad s = 2, \cdots, M.
\]

(22)
The remaining vectors are now orthogonal to $V_0$. Since the vectors had initially the same Euclidean norm, the vector having the largest norm is the most orthogonal to $V_0$. It is assigned to $V_1$. The previous operations are then performed on $M - 2$ remaining vectors using $V_1$ as $V_0$. This process is repeated until the required $2N + 1$ vectors are defined. As a time instant corresponds to a vector, it yields $2N + 1$ time levels, which enables the construction of the almost-periodic DFT matrix. This algorithm is summarized in Algo. 1.

**Algorithm 1** The Almost Periodic Fourier Transform Algorithm.

\[
\omega_{\text{min}} \leftarrow \min (|\omega_k|, \ 1 \leq k \leq N)
\]

for \( m \leftarrow 0, \cdots, M - 1 \) do

\[ t_m \leftarrow \frac{2\pi}{\omega_{\text{min}}M} \]

end for

for \( n \leftarrow 1, \cdots, 2N \) do

for \( m \leftarrow n + 1, \cdots, M \) do

\[ V_m \leftarrow V_m - \frac{V_n^\top V_m}{V_n^\top V_n} V_n \]

end for

argmax() returns the index of the largest member of a set

\[ k = \text{argmax} (\|V_s^n\|, \ n + 1 \leq s \leq M) \]

swap($V_{n+1}, V_k$)

swap($t_{n+1}, t_k$)

end for

\[ T_{\text{optimized}} \leftarrow [t_0, \cdots, t_{2N}] \]

3.2 Gradient-based optimization algorithm (OPT)

A more direct approach is to seek directly a set of time levels that minimize the condition number of the associated almost-periodic DFT matrix, instead of using orthogonality properties. This minimization problem can be solved numerically by an optimization algorithm.

The limited memory optimization method of Broyden-Fletcher-Goldfarb-Shannon (L-BFGS-B, [4]) is used to look for a minimum of the condition number of the almost-periodic IDFT matrix $\kappa (A[T])$ as function of the time levels vector $T$. This quasi-Newton algorithm approximates the inverse Hessian matrix $H(\kappa (A[T]))^{-1}$ with the BFGS formula in order to decrease the objective $\kappa (A[T])$ in the direction $-H(\kappa (A[T]))^{-1}\nabla \kappa (A[T])$. This descent direction is associated to the search for a zero of the gradient, which is a necessary condition for an extremum, in a second order Taylor series. Finally, a line search on $\alpha$ is performed to minimize $\kappa (A[T - \alpha H(\kappa (A[T]))^{-1}\nabla \kappa (A[T])])$. In the present case, the derivative $\nabla \kappa (A[T])$ of the objective with respect to the time levels is approximated by first-order finite differences. An open-source implementation of this reference and broadly used algorithm is employed [44].

Gradient descent methods being local, L-BFGS-B converges to a local minimum of the condition number. This minimum is unsatisfying if the starting point $T$ is not well chosen, therefore a strategy to find an appropriate one is required. As shown in the following comparison, APFT or uniform-sampling time levels do not always guarantee acceptable condition numbers, and so cannot be used to provide a starting point for L-BFGS-B. For this reason, an initial sampling of evenly-spaced time levels taken in fractions of the greatest period is used. The almost-periodic
IDFT matrix is built for each of these time levels and the corresponding condition numbers are computed. A large number, typically thousands, of fractions of the greatest period gives a large set of potential time levels vectors. This is acceptable given the very low cost of the computation of the condition number on such small matrices of size \((2N+1) \times (2N+1)\). From this set, the time levels vector associated to the almost-periodic IDFT matrix having the smallest condition number is taken as starting point. The optimization algorithm actually achieves a local adjustment of the time levels.

In this way, the exploitation capability of the gradient-based optimizer is well combined with the exploration capacity of the sampling. This finally gives solutions that are always close to the ideal value of 1, as shown in Tab. 1. The OPT method is summarized in Algo. 2.

**Algorithm 2** The gradient-based optimization algorithm (OPT).

\[
\omega_{\text{min}} \leftarrow \min (|\omega_k|, \quad 1 \leq k \leq N)
\]

\[
\text{for } m \leftarrow 0, \cdots, M - 1 \text{ do}
\]

\[
\omega_m \leftarrow \frac{m+1}{M} \cdot \omega_{\text{min}}
\]

\[
\text{for } i \leftarrow 0, \cdots, 2N \text{ do}
\]

\[
t_i \leftarrow \frac{i \cdot 2\pi}{\omega_m \cdot (2N+1)}
\]

\[
\text{end for}
\]

\[
T_m \leftarrow [t_0, \cdots, t_i, \cdots, t_{2N}]
\]

\[
C_m \leftarrow \kappa (A[T_m])
\]

\[
\text{end for}
\]

**argmin()** returns the index of the smallest member of a set

\[
k \leftarrow \text{argmin} (C_m, \quad 0 \leq m \leq M - 1)
\]

**min_l-bfgs-b** \((\kappa (A[T]), T_{ini})\) returns the optimal time levels vector \(T\) with the condition number \(\kappa (A[T])\) as objective function using the L-BFGS-B algorithm and \(T_{ini}\) as starting point.

\[
T_{optimized} \leftarrow \text{min}_l\text{-bfgs-b} (\kappa (A[T]), T_{ini} = T_k)
\]

### 3.3 Assessment of the algorithms

To assess the presented algorithms, a plot of a non dimensional frequency \(\delta^*_f\) against the condition number of the almost-periodic DFT matrix \(\kappa(A)\) is drawn Fig. 1. \(\delta^*_f\) is defined as:

\[
\delta^*_f : \begin{cases} 
[0 : f_1] & \rightarrow [0 : 2] \\
\ f & \rightarrow 2 \cdot \frac{f_1 - f}{f_1 + f}
\end{cases}
\]

(23)

By taking \(f_1\) constant, and having \(\delta^*_f\) sampled between 0 and 2, the whole range of \(f \leq f_1\) is explored. Moreover, as \(\delta^*_f\) is anti-symmetric \((\delta^*_f(-f) = -\delta^*_f(f))\) and the almost-periodic IDFT matrix is symmetric \(A[-f] = A[f]\) since it is only a permutation of the vectors, then:

\[
\kappa (A[\delta^*_f(-f)]) = \kappa (A[-\delta^*_f(f)]) = \kappa (A[\delta^*_f(f)]),
\]

(24)

meaning that the case \(f \geq f_1\) can be deduced in a straightforward way.

For each value of \(\delta^*_f\), the condition number of the almost-periodic IDFT matrix \(\kappa(A)\) is computed, highlighting the ability of the different algorithms to choose the time levels that
minimize the condition number, for any input frequencies. This assessment is only valid for two frequencies, but the tendency is kept when increasing the number of frequencies. Two frequencies are involved thus five time levels are required. The results of three algorithms are depicted Fig. 1: (i) APFT: the Almost Periodic Fourier Transform algorithm, (ii) OPT: the gradient-based optimization algorithm and (iii) EQUI: evenly spaced time levels oversampling the largest period as done in Gopinath et al. [14] using $2N + 1$ time levels and in Ekici and Hall [9, 10] using $3N + 1$ time levels.

![Figure 1: Comparison of the presented algorithms.](image)

The EQUI algorithms give fair results ($\kappa(A) \leq 2$) only at discrete points, corresponding to the particular cases where $f$ is a multiple of $f_1$, which are similar to the single-frequency case. Oversampling improves the results but the almost-periodic DFT matrix becomes rectangular and the memory cost of such a computation increases drastically preventing the use of such an approach. The APFT algorithm improves the results, as it gives results with $\kappa(A)$ close to unity for $0.3 < \delta_f^* < 2$. However, when $\delta_f^*$ tends to the boundaries (0 and 2), the condition number seems to go to infinity. These correspond to special values of $f$:

$$\delta_f^* = 0 \iff f = f_1,$$

$$\delta_f^* = 2 \iff f = 0. \tag{25}$$

This means that the APFT algorithm fails to work when the frequencies are too close to one another, and when they are significantly different. This limits the method for a range of frequencies where the HB method could give a salient gain in CPU time. Finally, the OPT algorithm gives a condition number close to unity for any value of $\delta_f^*$. The OPT algorithm thus ensures that the convergence of the HB method is not sensitive to the specified set of frequencies. Table 1 summarizes the results obtained with each algorithms.

Thus the proposed non-uniform time sampling combined with the OPT algorithm allows to tackle problems with large frequency separation. In such cases, the gain of the HB approach as compared to classical time-marching methods is expected to be significant: with a time-marching scheme, the time-step has to be small enough to discretize the shortest period, while the number of time steps of the simulation has to be long enough to reach the (almost-) periodic state (i.e. a simulation time equal to several times the longest period). Conversely, the cost of the HB method only depends on the number of frequencies to capture, regardless of their relative values.
3.4 Distribution of the time levels

For harmonically-related frequencies, the optimal time levels correspond to a uniform set sampling the fundamental frequency period as it gives the theoretical lower bound \( \kappa(A) = 1 \). Since the frequencies are harmonically related, the distribution of the time levels on the other frequencies is also uniform. Considering the frequency vector \( F = [f_1, \cdots, f_k = kf_1, \cdots, Nf_1] \) and the time levels vector \( T \):

\[
T = \left[ 0, \frac{1}{f_1 \cdot (2N + 1)}, \cdots, \frac{2N}{f_1 \cdot (2N + 1)} \right],
\]

then the product of the \( i^{th} \) term of \( T \) to its associated frequency is

\[
f_1 \cdot \frac{i}{f_1 \cdot (2N + 1)} = k f_1 \cdot \frac{i}{k f_1 \cdot (2N + 1)} = f_k \cdot \frac{i}{f_k \cdot (2N + 1)}.
\]

Eq. (27) means that evenly spaced time levels for the fundamental frequency are still seen as evenly spaced by the \( k^{th} \) harmonic. This is an explanation why the condition number of the almost-periodic IDFT matrix \( A^{-1} \) will be unity as each frequency is sampled by evenly spaced time levels [3].

Now, considering non harmonically related frequencies, there is mathematically no reason that evenly spaced time levels over the smallest frequency are seen as evenly spaced by the other frequencies in the general case. Hence the use of non-evenly spaced time levels and algorithms to automatically choose them.

Figure 2 shows the distribution of the time levels, relative to each frequency period, obtained by the presented algorithms for frequencies \( f_1 = 3 \) Hz and \( f_2 = 17 \) Hz (i.e. \( \delta_f = 1.4 \)). To do so, the chosen time levels are redistributed on the considered frequency period by applying a modulo of it. Then, they are divided by the latter, so that the results are dimensionless. In light gray line is depicted the \( y = x \) function representing the evenly spaced solution on the considered period. Keeping in mind that if each frequency sees evenly spaced time levels, then the condition number is the smallest, the optimal solution would be to have relative time levels on \( y = x \) for each period. Running the EQUI, APFT and OPT algorithms leads to a condition number of 33.1, 3.8 and 1.1, respectively. The EQUI algorithm is perfect for the period \( 1/f_1 \) but is really far from the evenly spaced time levels for period \( 1/f_2 \). The APFT algorithm is far from the evenly spaced solution for each period considered but closer than EQUI regarding period \( 1/f_2 \). Finally, the OPT algorithm is the only one to be close to the evenly spaced solution for each period considered. Hence the possibility to use the proposed HB method for any set of frequencies.

The source code of the proposed algorithms and the scripts to generate Figures 1 and 2 are available over the internet\(^1\).

\(^1\)http://cerfacs.fr/~gomar/PyLeap.html
Figure 2: Distribution of the time levels on each frequency periods.

The impact of the time sampling on HB computations is now investigated for the simple case of a channel flow with fluctuating pressure outlet.

4 Channel flow

4.1 Test case description

A channel configuration is set up to study the properties of the proposed HB method and the above algorithms for non-uniform time sampling. It is a 2D channel of length $L_x = 100$ m in the axial direction and $L_z = 1$ m in the transverse one. The boundary conditions are: (i) an injection condition for the inlet, (ii) symmetric conditions for the upper and lower bounds as the flow is assumed to be symmetric in the transverse direction, and (iii) a fluctuating pressure imposed at the outlet:

$$P_{\text{outlet}}(t) = P_m \cdot [1 + A_1 \cdot \sin(2\pi f_1 t) + A_2 \cdot \sin(2\pi f_2 t)], \quad (28)$$

where $P_m$ is the temporal average static pressure, $A_n$ the amplitude of the $n^{\text{th}}$ mode and $f_n$ its frequency. The mean outlet pressure $P_m$ is set to 60% of the inlet total pressure $P_{i0} = 101,325$ Pa.

Pressure waves travel within the flow with the velocity $u + c$ and $u - c$, where $u$ denotes the local flow velocity and $c$ the sound velocity. Since the pressure waves are generated at the outlet, only the $u - c$ waves are visible, resulting in pressure waves propagating upstream of the channel, which are damped by the effect of viscosity. Figure 3 shows a schematic diagram of the channel case, illustrating the propagation and attenuation of the pressure waves.

The mesh consists of 997 points along the axial direction and 9 in the transverse one, which amounts to almost equal spacings in both directions.

This configuration is turbulent as the Reynolds number based on the inlet flow velocity and the axial length of the channel is about $Re \approx 2.0 \times 10^5$. Turbulence is modeled using the
one-equation model of Spalart and Allmaras [38], and the third-order upwind Roe scheme [33] is used to compute the convective fluxes.

4.2 Convergence sensibility analysis

As mentioned previously, the condition number is of great importance for the convergence of the proposed HB method. To highlight this feature, the presented channel case is computed with a single frequency at the outlet: \( f_1 = 3 \) Hz with an amplitude \( A_1 = 0.05 \) for the first case and \( A_1 = 0.01 \) for the second one, the second frequency having a zero amplitude: \( A_2 = 0 \). Two frequencies are specified for the HB computation: \( f_1 \) and its first harmonic \( 2f_1 \). The time levels are chosen to reach varying condition numbers \( 1 \leq \kappa(A) \leq 3.43 \). Since the input frequencies of the HB computation are harmonically related, the minimal conditioning \( \kappa(A) = 1 \) is obtained with evenly spaced time levels. The OPT algorithm is modified by subtracting the targeted conditioning to the objective function, so that the different condition numbers can be reached. The distribution of the time levels for each condition number is shown in Fig. 4. The time levels deviate from the evenly spaced solution as the condition number grows. The results in Fig. 5 show that for a condition number \( \kappa(A) \geq 3.43 \) and wave input amplitude \( A_1 = 0.05 \), the computation diverges. However, the computations with the same condition numbers but a smaller input amplitude \( A_1 = 0.01 \) converge. In fact, the condition number amplifies the
errors made during the iterative process. When the input waves have a smaller amplitude, the iterative errors are slighter, hence the convergence as explained § 2.2.2.

\[ 10^{-4} \quad 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \quad 10^2 \]

\[
\begin{align*}
0 & \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \\
\text{Density residual} & \\
\text{Iteration (×10^3)} & \\
\kappa(A) = 1.69 & \\
\kappa(A) = 2.24 & \\
\kappa(A) = 2.82 & \\
\kappa(A) = 3.43 & \\
\end{align*}
\]

(a) \( A_1 = 0.01 \)

\[
\begin{align*}
0 & \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \\
\text{Density residual} & \\
\text{Iteration (×10^3)} & \\
\kappa(A) = 1.69 & \\
\kappa(A) = 2.24 & \\
\kappa(A) = 2.82 & \\
\kappa(A) = 3.43 & \\
\end{align*}
\]

(b) \( A_1 = 0.05 \)

Figure 5: Relation between the condition number \( \kappa(A) \) and the convergence of the solution.

### 4.3 Validation of the multi-frequency HB method

To validate the proposed HB method, two non-harmonically related frequencies are chosen as input for the outlet boundary condition: \( f_1 = 3 \) Hz and \( f_2 = 17 \) Hz.

A classical time-marching scheme is taken for comparison, namely the Dual Time Stepping scheme (DTS [24]). The DTS method is a 2\textsuperscript{nd}-order implicit time-marching scheme. Convergence in time discretization is obtained after 20 periods using 160 instants per almost-period. Since the frequencies are integers and coprime, the almost-period is \( T = 1 \) s. Iterative convergence for the inner loop is considered achieved when the normalized residuals drop by \( 10^{-2} \) within a maximum of 50 sub-iterations.

The results obtained with the DTS scheme are compared to the HB results for pressure waves amplitudes of \( A = A_1 = A_2 = 0.001 \). The transient of the DTS computation is shown Fig. 6, illustrating the wave propagation with a slight attenuation of the high-frequency waves.

The results are analyzed for frequencies \( 1 < f < 40 \) Hz and the dominant frequencies (the one that have the highest amplitudes) are set for the HB computation. To do so, pressure signals are probed upstream, in the middle and downstream of the channel at \( x = [25 \text{ m}, 50 \text{ m}, 75 \text{ m}] \) and \( z = 0.5 \text{ m} \) respectively. The spectrum of the aforementioned unsteady pressure signals, obtained with a Fourier Transform, are plotted Fig. 7. The labeled frequencies are the dominant ones, as for each probe, these have a high amplitude. They are thus selected for the HB computation. For such frequencies, the OPT algorithm gives a set of time levels leading to a condition number of 1.4.

A Discrete Fourier Transform is computed at several axis position, resulting in the spatial evolution of the different harmonics, which is used for the comparison of the HB and DTS approaches, in the middle of the canal (\( z = 0.5 \text{ m} \)). In Fig. 8, the results are plotted for the frequencies that have been set for the HB computation. The overall agreement is fair. Some local discrepancies can be observed upstream for frequencies \( f_2 + 3f_1, f_2 - f_1 \) and \( f_2 - 2f_1 \), but these are minimal regarding the temporal evolution, as shown Fig. 9, where the time evolution
of pressure signals is extracted at all probes. The difference between the HB and the DTS method is negligible, proving that the proposed HB method is capable of reproducing the unsteady almost-periodic phenomena.

The goal of this section was not to show significant CPU savings but rather the ability of the present HB method to capture an almost-periodic flow on a model problem. It is now applied to a more complex configuration, namely a turbomachinery, where its computational efficiency is also emphasized.

5 Turbomachinery Application

Under the assumption that all unsteady phenomena in a blade row during stable operation are periodic and can be correlated with the rotation rate $\Omega$ of the shaft, the dominant frequencies are those created by the passage of the neighboring blades. In a multi-row turbomachine, a blade row sandwiched between the upstream and downstream rows is subjected to wake and potential effects. In practical turbomachines, the blade count of neighboring rows are generally different and coprime. Consequently, a sandwiched blade row resolves various combinations of the frequencies, which are additions and/or subtractions of multiples of the blade passing frequencies: according to Tyler and Sofrin [40], the $k^{th}$ frequency resolved in the blade row $j$ is given by

$$\omega_{k}^{rowj} = \sum_{i=1}^{nRows} n_{k,i}B_{i}(\Omega_{i} - \Omega_{j}).$$

(29)
Here, $B_i$ and $\Omega_i$ are respectively the blade count and the rotation rate of the $i^{th}$ blade row, the $n_{k,i}$ are $k$ sets of integers driving the frequency combinations. It must be noted that only the blade rows that are mobile relative to the considered $j$ one contribute to its temporal frequencies and that every blade row solves its own set of frequencies and thus its own set of time levels. To set up a HB computation for a multistage configuration, it is of course impossible to use each and every $n_{k,i}$ possible, and the user has to choose which frequency combinations will appear in the computation of each row.

In the literature, Gopinath et al. [14] and Ekici & Hall [9] assessed their implementation of the Harmonic Balance on a 2D multi-stage compressor (namely configuration D). It is composed of a rotor sandwiched by two stators having respectively 32, 40 and 50 blades. Various combinations of the stators BPFs are considered, but with evenly-spaced time levels sampling the largest period. While Gopinath et al. use $2N + 1$ samples, Ekici & Hall over-sample this period with $3N + 1$ time levels. This leads to a rectangular $(2N + 1) \times (3N + 1)$ almost-periodic Fourier Matrix and requires the computation of its Moore-Penrose pseudo-inverse. The chosen frequencies and the a posteriori associated condition numbers of the above references are given Tab. 2. For $N = 4$, the $3N + 1$ instants oversampling approach of Ekici & Hall efficiently reduces the condition number. But for this case, the use of evenly-spaced time levels is sufficient as the condition number seems to be small enough for the considered magnitude of unsteadiness. However, such an approach fails when dealing with more widely-separated frequencies as illustrated in the present contribution § 3. Moreover, using an oversampling increases the CPU cost and memory required as the number of steady computations to solve simultaneously is higher. These two reasons highlight the need for a non-uniform HB method proposed in the current paper.
Figure 8: Spatial evolution of the amplitude of the dominant frequencies in the channel, for $f_1 = 3$ Hz and $f_2 = 17$ Hz.
Figure 9: Unsteady pressure signals at different axial positions.

5.1 Boundary conditions for sector reduction

Section 3 showed how to contain the problem size by reducing the time span over which the solution is sought. In the following sections, it is explained how to cut down the mesh size by using a grid that spans only one blade passage per row.

5.1.1 Phase-lagged azimuthal boundary conditions

In a single blade passage computation of a multi-row configuration, the phase-lag condition \([11]\) needs to be used to take the space-time periodicity into account. It states that the flow in one blade passage \(\theta\) is the same as next blade passage \(\theta + \Delta \theta\) but at another time \(t + \delta t\):

\[
W (\theta + \Delta \theta, t) = W (\theta, t + \delta t),
\]

where \(\Delta \theta\) is the pitch of the considered row. Assuming that every temporal lag is associated to a rotating wave of rotational speed \(\omega_k\), the constant time lag can be expressed as

\[
\delta t = \frac{\beta_k}{\omega_k}, \quad \forall k,
\]
Table 2: Frequency combinations and associated condition number of computations made in the literature.

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where

$$\beta_k = 2\pi\text{sign}(\omega_k) \left( 1 - \frac{1}{B} \sum_{i \neq k} n_{k,i} B_i \right),$$

(32)

the $n_{k,i}$ being the integers specified for the computation of the frequencies from Eq. (29), and $B_i$ the number of blades in row $i$.

The phase-lag condition was adapted to time-domain HB by Gopinath et al. [13]. The derivation starts with the almost-periodic Fourier transform of Eq. (30):

$$\sum_{k=-N}^{N} \tilde{W}_k(\theta + \Delta \theta, t) e^{i\omega_k t} = \sum_{k=-N}^{N} \tilde{W}_k(\theta, t) e^{i\omega_k \delta t} e^{i\omega_k t}. $$

(33)

Thus, the flow spectrum from one blade passage is equal to that of the next blade passage modulated by the inter-blade phase angle $\beta_k$:

$$\tilde{W}_k(\theta + \Delta \theta, t) = \tilde{W}_k(\theta, t) e^{i\omega_k \delta t} = \tilde{W}_k(\theta, t) e^{i\beta_k}.$$ 

(34)

Using the same notation as previously, the following matrix formulation is obtained:

$$W^* = A^{-1}MAW^*(\theta),$$

(35)

where

$$M = \text{diag}(-\beta_N, \ldots, \beta_0, \ldots, \beta_N),$$

(36)

and $A^{-1}$ is given by Eq. (15).
5.1.2 Stage coupling

Each blade row has its own frequency set and therefore its own time sampling. Therefore, the $n^{th}$ time level in the $j^{th}$ and $(j + 1)^{th}$ rows do not necessarily match the same physical time. Consequently, at the interface between adjacent blade rows, the flow field on the donor side needs to be generated for all the time levels of the receiver side using a spectral interpolation. A non-abutting join interface is used to perform the spatial communication between the two rows [28]. In order to account for the pitch difference and relative motion, a duplication of the flow is performed in the azimuthal direction using the phase-lag periodicity. Moreover, as described in Ref. [37], the time levels at the interface are oversampled and filtered to prevent aliasing.

5.2 Application to a subsonic compressor

The studied subsonic compressor case is a mid-span slice of the inlet guide vanes (IGV) and first stage of the axial compressor CREATE [16], located in Lyon (France) at the Laboratoire de Mécanique des Fluides et Acoustique (LMFA). This configuration is composed of 32 IGV blades, 64 rotor (RM1) blades and 96 stator (RD1) blades.

5.2.1 Mesh and numerical parameters

As shown Fig. 10, the blade passages are meshed with a block-structured topology. It is composed of five grid points in the radial direction, 33 in the azimuthal direction and 100 in the axial direction for both rows. This leads to a total number of approximately 50,000 mesh cells.

![Figure 10: Geometry of the studied compressor slice.](image)

The IGV blade is not actually meshed but taken into account through a non-uniform injection boundary condition that represents the wake of the IGV entering the RM1 domain. This injection follows the self-similarity law of Lakshminarayana and Davino [27] which states that the spatial evolution of a wake can be described by a Gaussian function. As $B_{RD1} = 3 \cdot B_{IGV}$, the frequency content remains mono-frequential in the rotor (i.e., the BPF of the downstream rotor is just an harmonic of the IGV’s BPF). Therefore the number of blades composing the
IGV has been changed from 32 to 80 so that the configuration still presents a $2\pi/16$ periodicity but the frequency content is now multi-frequential in the rotor.

The outlet duct is modeled through a valve condition coupled with a simplified radial equilibrium. The reference outlet static pressure $P_s$ for the radial equilibrium integration is imposed according to the formula $P_s = P_{ref} + \lambda(Q/Q_{ref})^2$, as illustrated Fig. 11. $P_{ref}$ is a reference static pressure chosen such that when $\lambda = 0$ Pa the compressor is choked, and $Q_{ref}$ is the corresponding mass flow. $Q$ is the current mass flow and $\lambda \geq 0$ is a user-defined pressure. Its different values allow to move along the compressor map: when $\lambda$ increases, the outlet static pressure rises and the mass flow rate decreases and vice-versa (Fig. 11). At the blades walls, wall laws [12] are imposed. The lower and upper radial conditions are slip walls.

The convective fluxes are discretized using the second-order Jameson scheme [25] with added artificial viscosity or a second-order Roe scheme [33, 42]. For this study, the turbulent viscosity is computed with the one-equation model proposed by Spalart and Allmaras [38].

The DTS scheme is used to get a numerical reference solution. The periodicity of the different blade passages is such that a $2\pi/16$ periodicity is enough to perform the unsteady computations. To reach an established periodic state, 67 passages (using 400 instants per azimuthal period) of the periodic sector are necessary. Figure 12 plots the time evolution of the fluid density $\rho$ and its associated spectrum downstream of the rotor. The spectrum is not only composed of the blade passing frequencies and their harmonics but also combinations of their frequencies as estimated by Tyler and Sofrin [40]. The amplitude of a frequency combination may also be higher than an harmonic of a blade passing frequency. For example, $\text{BPF}_{IGV} - \text{BPF}_{RD1}$ is higher than the third and fourth harmonics of $\text{BPF}_{RD1}$. This highlights the necessity of being able to take into account these frequency combinations in a HB computation.

![Figure 11: Valve condition at the outlet.](image-url)
The convergence of the Harmonic Balance computations is done in two steps: first 15,000 iterations with a second order Roe scheme, then 10,000 iterations with the Jameson scheme (with the artificial dissipation coefficients $k_2 = 1.0$ and $k_4 = 0.032$). To understand how the frequency set influences the convergence of the computations, several of them were chosen accordingly to the spectral analysis of the time signal from Fig. 12. They range from three to six frequencies. The frequency combinations used are summarized Tab. 3. This table shows the coefficients $n_{k,i}$ from Eq. (29) chosen for each blade row. They are given by the immediately adjacent rows. For example, for $N = 4$ v1, the frequency set is $[BPF_{RM1}, 2BPF_{RM1}, 3BPF_{RM1}, 4BPF_{RM1}]$ in the IGV (i.e. $j = 1$ in Eq. (29)) and in the RD1 (i.e. $j = 3$) (which means that for these rows the frequency content is mono-frequential) whereas, it is $[BPF_{IGV} - BPF_{RD1}, BPF_{IGV}, BPF_{RD1}, BPF_{IGV} + BPF_{RD1}]$ in the RM1 (i.e. $j = 2$ in Eq. (29)). It is clear that the different blade rows have different frequency sets. The upstream injection block and RD1 only solve for the BPF of the rotor and its harmonics, thus the classic Fourier analysis ensures that the best conditioning of the matrix $A^{-1}$ is given by evenly distributed time levels over the period $T = 1/BPF_{RM1}$. At this point, it should be noted that 80 and 96 are multiples of 16 (i.e. blade number of RD1 minus blade number of IGV). Thus all the frequency combinations of Tab. 3 for the rotor are multiples of the base frequency $BPF_{IGV} - BPF_{RD1}$. However, unlike the mono-frequential case, not all the intermediate harmonics need to be taken into account. For example, in a six-frequency set, the highest frequency is $2BPF_{RD1}$ which is also $12 \times (BPF_{IGV} - BPF_{RD1})$. To
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Table 3: Frequency combination coefficients.
perform a mono-frequential harmonic computation taking into account \(2 \text{BPF}_{RD1}\), one would thus need \(\text{BPF}_{IGV} - \text{BPF}_{RD1}\) as the fundamental and the eleven following harmonics, which implies a computation with twenty five time samples. Such an approach would be inefficient, as the intermediate harmonics are not relevant here (see Fig. 12). The present multi-frequential HB method allows to perform the computation only on a set of chosen frequencies.

For such frequency ratios, the APFT algorithm provides good enough condition numbers of the matrix \(A^{-1}\) (as shown Tab. 3) and the use of the OPT algorithm was not mandatory. As it would be too long and tedious to present the time levels distribution for all the frequency sets, Fig. 13 focuses on the three sets of four frequencies. It allows to observe, for the same number of frequencies, the impact of the frequency set on the APFT algorithm. The first remark that can be drawn from Fig. 13 is that the APFT algorithm is not always needed: for \(N = 4\ v_1\), the best time levels distribution for the rotor is given by a uniform sampling while the APFT algorithm gives a condition number of 2.16. However, the gain is significant for the two other configurations: with evenly-spaced time levels, the condition numbers of the matrix \(A\) are respectively of \(6.74 \cdot 10^{15}\) for \(N = 4\ v_2\) and \(2.62 \cdot 10^{15}\) for \(N = 4\ v_3\), while with the time levels issued from the APFT algorithm they go respectively down to 1.76 and 2.0.

![Figure 13: Distribution of the time levels in the rotor for four frequencies over the base frequency \(\text{BPF}_{IGV} - \text{BPF}_{RD1}\).](image)

The convergence of the HB computations depends on the choice of the frequency set as shown Fig. 14, which depicts the convergence history at the peak-efficiency operating points. For all the computations, the residuals drop at least three orders of magnitude, which is considered to be enough to ensure convergence [6]. Figure 15 plots the mass flow rate convergence for the first set of four frequencies. It also emphasizes the importance of the spatial discretization, as the instantaneous mass flow rates (and thus the mean mass flow rate) differ between the Roe and Jameson schemes. This effect is beyond the scope of this work, and the Jameson scheme is considered as the reference scheme for the rest of the study.
5.2.3 Time-averaged compressor performance

Figures 16 and 17 show the computed compressor map: the total pressure ratio \( \Pi \) and the isentropic efficiency \( \eta_{is} \) are plotted against the massflow. They are non-dimensionalized by the values at the maximum-efficiency point. Concerning the total pressure ratio, there is an overall good agreement between the DTS and the HB techniques, regardless of the number of frequencies. The maximum relative difference is 0.4\%. The isentropic efficiency is more sensitive to the number of frequencies as there is a 1\% difference for \( N = 3 \), which reduces below 0.1\% for more frequencies. However, there are not much differences for more than four frequencies. Therefore, in terms of global data, the computations are converged with respect to the frequency content. Four frequencies are consequently a minimum to compute correctly the aerodynamic performance.

5.2.4 Instantaneous results

Figure 18 shows the instantaneous entropy flow field for the maximum-efficiency operating point \( \eta_{is} = 1 \). For the HB computations, the computed passage was duplicated using phase-lag
to check that the azimuthal phase-lag boundary condition ensures the continuity of the flow field between the original blade passage and the duplicated ones. All the wakes are correctly convected downstream and very few differences can be seen in the different flow fields. Some numerical wiggles can be observed downstream the RM1/RD1 interface, but the higher the number of frequencies, the better the solution as already shown by Sicot et al. [37].

5.2.5 Unsteady results

To analyze the prediction of unsteady row interactions within the rotor, Fig. 19 shows the azimuthal evolution, in the relative frame, of the non-dimensional axial speed downstream of the rotor, as a function of time, for the DTS, HB $N = 3$, HB $N = 4$ v1 and HB $N = 6$ computations. In this diagram, the horizontal bands of low axial speed correspond to the wakes of the rotor itself, which remain steady in the relative rotating frame. The IGV wakes cropped by the rotor and convected within the passage can also be observed as "oblique strips" of low velocity. Comparing figures 19(a) and (b) clearly shows that only three frequencies are not sufficient to reproduce correctly the time and space evolutions of the wakes. The four-frequency set (Fig. 19(c)) gets the rotor and IGV wakes clearly visible, but a bit more twisted than in the reference solution. The minimum value is also under-predicted by 3.6%. The six-frequency solution (Fig. 19(d)) has the same features as the four-frequency one, except that the IGV wake is slightly better predicted and the minimum is now correct.

To facilitate comparisons between the different HB frequency set and DTS, the azimuthal evolution of the non-dimensional fluid density $\rho$ along a line of constant radius is plotted Fig. 20 for $t = 0$. This amounts to extracting a vertical line at $t = 0$ in Fig. 19, but this time density was chosen as it is a conservative variables and is more varying than others. For the sake of clarity, the results for the second four-frequency ($N = 4$ v2) and five-frequency sets are not shown here. The results for three frequencies oscillate around the values of the DTS. Underlying the comments made on the convergence history, the two sets of four frequencies give quite different azimuthal results. Both results for four frequencies follow the variations of the DTS. The results for the first set of four frequencies are quite fair. Given the quality of the results given by HB $N = 4$ v1, it is surprising that HB $N = 6$ v1 does not perform better since its frequency
Figure 18: Comparison of the entropy flow fields at $\eta_{*} = 1$ and $t = 0$. 
Figure 19: Rotor exit: Azimuth-time map of the non-dimensional axial speed at $\eta_{\text{sa}} = 1$. 
content is merely an enrichment of HB $N = 4$ v2.

To further analyze unsteady interactions within the rotor, a probe was positioned downstream of the rotor in the middle of the passage. The unsteady density signal is plotted Fig. 21. This amounts to extracting an horizontal line in Fig. 19, but for the density variable. For four frequencies, the variations of non-dimensional $\rho$ in time are almost the same and are matching the evolution of the DTS. To have an accurate approximation of the flow field, four frequencies seems to be the minimum required.

The unsteady pressure coefficient $C_p$ at mid-span of the rotor blade is now studied, and the contribution of the upstream and downstream rows are isolated. Figure 22(a) depicts the mean value on the rotor blade along the normalized curvilinear coordinates for DTS, HB $N = 3$, HB $N = 4$ v1 and HB $N = 6$, whereas (b) and (c) plot the amplitude evolution for, respectively, the upstream and the downstream blade passing frequency. The leading edge corresponds to $s = 0$ or $s = 1$, whereas the trailing edge is located at $s = 0.5$. Between 0.0 and 0.5 is the suction side and between 0.5 and 1 is the pressure side. As shown Fig. 22(a), three frequencies are enough to capture the mean $C_p$ value around the blade. All four-frequency set and the six-frequency set fit perfectly the DTS amplitudes for the passing frequency of the IGV blades, except for an oscillation at the end of the suction side. Concerning the amplitudes of the
passing frequency of RD1, HB $N = 4 \ v1$ and HB $N = 6$ correctly predict the suction side and HB $N = 4 \ v3$ under-predicts the maximum of the amplitude. All frequency sets have trouble predicting the amplitude right after the trailing edge at the pressure side. Surprisingly, it is HB $N = 4 \ v3$ which is the best match, whereas one would have rather expected HB $N = 6$ to be.

![Image](a) Mean

(b) Amplitude evolution of IGV BPF mode  
(c) Amplitude evolution of RD1 BPF mode

Figure 22: Rotor blade: Fourier analysis of $C_p$ for $\eta_{\text{is}} = 1$.

5.2.6 HB computations with only BPF of adjacent rows

The previous computations were made in the ideal case in which the flow spectrum is known a priori. This allows to choose the frequencies that are the most likely to give the best results. From this standpoint, HB $N = 4 \ v1$ is an especially good example. However, in a design process, one does not have such an information. The usual first guess consists in using only the blade passing frequencies of the adjacent rows. This leads to two new sets of frequencies (one of four and one of six frequencies), which are summarized Tab. 4.

Figures 23 and 24 show the associated time distributions found thanks to the APFT algorithm on both base frequencies (BPF$_{IGV}$ and BPF$_{RD1}$) with the associated condition numbers.
Table 4: Frequency combination coefficients to compute only harmonics of the fundamental blade-passing frequencies.

As done previously, the first step consists in checking the aerodynamic values. Figures 25 and 26 plot respectively the total pressure ratio and the isentropic efficiency for the new frequencies. Regarding both values, the relative error margin is almost the same as in the previous HB computations.

The resulting entropy flow fields for these new frequency sets are shown Fig. 27. The reference DTS field is also plotted as a reminder (a). They do not show any significant discrepancy with the previous figures. In compliance with the comments made on Fig. 18, some wiggles manifest at the interface RM1/RD1 with HB \( N = 4 \ v4 \), but disappear as the number of frequencies is increased. Figures 28 and 29 compare respectively the azimuthal and temporal evolution of the new frequency sets with the old ones of corresponding number of frequencies. It comes out from Fig. 28 that, in this case, the importance of the blade passing frequencies cannot be denied since with enough harmonics of the passing frequencies (top) the DTS curve is very well-matched by HB \( N = 6 \ v2 \). With fewer harmonics (bottom), HB \( N = 4 \ v4 \) behaves like HB \( N = 4 \ v2 \).

Figure 29 shows no noticeable improvement (or deterioration) of the local time evolution with the change of frequencies.

The time-azimuth maps are given Fig. 30. The main difference is the shape of the bubble in the wake of the rotor, which is better captured by the six-frequency set.

The previous figures show that the performance of HB \( N = 6 \) are not as good as HB \( N = 6 \ v2 \). Figure 31 shows the \( C_p \) for both six-frequency sets. The mean value evolution (a) exhibits no difference between the two frequency sets. The same remark can be made for the IGV BPF (b) except for a minor difference at 80% of the suction side. Concerning RD1’s BPF (c), HB \( N = 6 \ v2 \) gives a better overall match with the DTS than HB \( N = 6 \ v1 \) and especially in the last third of the pressure side.

### 5.2.7 Computational gain

Figure 32 shows that the HB computations allow a reduction of the CPU cost by a factor 4.5 for four frequencies, the gain being higher with fewer harmonics. However, it should be
Figure 23: Distribution of the time levels over the period of the IGV.

Figure 24: Distribution of the time levels over the period of the RD1.

Figure 25: Non-dimensional total pressure ratio map $\Pi^*$.  

Figure 26: Non-dimensional isentropic efficiency map $\eta_{is}^*$.  

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Figure 27: Comparison of the entropy flow field at $\eta_{ls}^* = 1.0$ and $t = 0$. 
kept in mind that the reference DTS simulations are done on a $2\pi/16$ periodic sector, whereas practical turbomachinery configurations usually do not have such periodicity, thus requiring simulations on the whole $360^\circ$ machine. In this case, an additional factor 16 in gain can thus be estimated, suggesting a gain of almost two orders of magnitude. Since the present mesh does not allow multigrid computation, it is also possible to expect a gain even higher as multigrid is a very efficient convergence-acceleration technique for steady computations. This leaves room for further improvements in CPU time reduction.

6 Conclusion

In this study, two algorithms have been derived to find a non-uniform sampling for state-of-the-art time-domain harmonic balance simulations: the APFT algorithm improves the Fourier matrix orthogonality in order to reduce its condition number, while the OPT algorithm directly minimizes the condition number thanks to a gradient-based optimization method. A model problem has proved the ability of the present implementation to accurately capture a flow driven by two frequencies non multiple of each other. Then, a turbomachinery configuration has been tested with several frequency sets. It has shown that nonlinear flows can be modeled to engineering accuracy with only four frequencies. In this case, the HB method is 4.5 times faster than a classical time-marching scheme. The main reasons are the reduced time span (only the largest period) and the reduced grid span (one blade passage vs. $2\pi/16$ sector, although the latter could be even higher for an industrial turbomachinery). All our conclusions for the present test-cases have been extended to 3D flows without any new assumptions. Our current focus is on multistage turbomachinery and counter-rotating open-rotors.
Figure 29: Comparison of the temporal evolution of non-dimensional $\rho$ at $\eta^* = 1$.

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References

References


Figure 30: Azimuth-time map of the axial speed at $\eta_{ls}^* = 1$.


Figure 31: Fourier analysis of $C_p$ at $\eta_{ls} = 1$.


Figure 32: CPU ratio DTS / HB.


