Sensitivity Analysis and Multi-Objective Optimization
for LES Numerical Parameters

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Running head: Sensitivity Analysis and Multi-Objective Optimization for LES Numerical Parameters

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Abstract

Accuracy and reliability of large-eddy simulation data in a really complex industrial geometry are investigated. An original methodology based on a response surface for LES data is introduced. This surrogate model for the full LES problem is built using the Kriging technique, which enables a low-cost optimal linear interpolation of a restricted set of LES solutions. Therefore, it can be used in most realistic industrial applications. Using this surrogate model, it is shown that: (i) optimal sets of simulation parameters (subgrid model constant and artificial viscosity parameter in the present case) can be found (ii) optimal values, as expected, depends on the cost functional to be minimized. Here, a realistic approach which takes into account experimental data sparseness is introduced. It is observed that minimization of the error evaluated using a too small subset of reference data may yield a global deterioration of the results.

1 Introduction

Large-eddy simulation (LES) [1, 2] is now commonly recognized as a reliable numerical approach for the prediction of complex turbulent flows. The assessment of quality and reliability of LES results is an important issue, since error estimation and control is a necessary step toward an increased use of LES for practical applications. Several works dealing with the analysis of numerical schemes within the LES framework [3, 4, 5, 6] and subgrid modelling error [7, 8] have been performed during the last decade. Most of them put the emphasis on the improvement of the computed results on simple test flows, but the issue of estimating the error on complex flows
is still an important open issue.

A priori and a posteriori error estimation for LES has recently been addressed in a few research works. While some adjoint-based error estimation techniques have been implemented for grid refinement by Hoffman and coworkers (see [9] for a global survey of this approach), simplest methods for computing an error index have been proposed [10, 11]. Another strategy is the error map approach proposed by Geurts and coworkers [7, 8, 12] which is certainly the most used one. Following this approach, the response surface of the LES computation is drawn by running a large set of simulations, varying the values of the computational setup parameters. This approach yielded very interesting results in isotropic turbulence, making it possible to derive optimal trajectories for the setup parameters which lead to the lowest possible error.

The underlying idea here is not to compute the local sensitivity of the LES solution with respect to the setup parameters (as for the adjoint-based approach), but to parameterize the space of possible solutions spanned by the numerical tool (including the numerical scheme, boundary and initial conditions, subgrid model, ...). The number of samples required can be dramatically reduced considering advanced response surface techniques. Pseudo-spectral modelling can be performed using the generalized Polynomial Chaos approach if the density probability of the possible variations of the setup parameters is prescribed. The first use of this technique to analyze LES sensitivity is very recent [13]. Another popular approach for response surface building is the Kriging method [14], which is an optimal linear approximation technique. This technique is now more and more popular in the field of aerodynamics, and was used to analyze RANS solution sensitivity and optimality [15] or for
surrogate model based shape optimization [16].

The search for the minimal errors solution requires the foreknowledge of a reference solution, and is therefore not possible in all realistic configurations. Therefore, the estimation of error is not always possible, and it must be replaced by an evaluation of the robustness of the solution. Robustness is an important feature of the solution in engineering applications, since a very sensitive solution, i.e. a solution which varies a lot if setup parameters undergo small variations, are of restricted interest for practical purposes.

The goal of the present paper is twofold: (i) to assess the possibility of building reliable Kriging-based response surface for LES of complex flows and (ii) to derive optimal LES solutions in realistic applications at affordable cost. The selected case is the LES of jet in crossflow (JICF) on unstructured grids, including wall heat transfer.

The main features of the numerical method are presented in section 2. The flow configuration and the numerical setup are displayed in section 3. The construction of the Kriging surface response (considering variations of both the subgrid and the artificial viscosities), the analysis of the solution sensitivity and the definition of optimal LES solutions are discussed in section 4. It is worth noting the different error norms (or equivalently cost functions) are introduced, which require different amount of data related to a reference solution. The optimized LES solutions are presented in section 5. Conclusions are drawn in section 6.
2 Governing Equations

The aim of the LES is to resolve the large scale of the turbulence while the small ones are modeled because of their universality. It is one of the main discrepancies with the RANS simulation where all scales are modeled. Consequently, the LES will be easier to use in industrial complex configurations in which the large scales are dominant. Moreover, the unsteady behavior of the JICF will be predicted more precisely by an unsteady LES approach.

2.1 Physical model

The removal of small scales in LES-type simulations is often modelled as the application of a filtering operator to the continuity equation, the transport equations of momentum, energy and species. In this case a Favre filtering is used:

\[
\overline{\rho f(x, t)} = \overline{\rho f(x, t)} = \int_{-\infty}^{+\infty} \rho f(x', t) \mathcal{G}(x' - x) \, dx',
\]

where \( \mathcal{G} \) is the filter function, \( \rho \) is the density and \( f \) can be either the velocity vector, the energy or the mass fraction of species. It leads to the following equations:

\[
\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{u}_i}) = 0,
\]

\[
\frac{\partial}{\partial t}(\overline{\rho \, \overline{u}_i}) + \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{u}_i \, \overline{u}_j}) = -\frac{\partial \overline{\rho}}{\partial x_j} + \frac{\partial \overline{\tau}_{ik}}{\partial x_k} - \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{T}_{ij}}),
\]

\[
\frac{\partial}{\partial t}(\overline{\rho \, \overline{E}}) + \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{u}_i \, \overline{E}}) = -\frac{\partial \overline{\rho}}{\partial x_j} + \frac{\partial}{\partial x_j}[\overline{(\tau_{ij} - \rho \, \delta_{ij}) \, u_i}] - \frac{\partial}{\partial x_j}(\overline{\rho \, \overline{Q}_{ij}}) - \frac{\partial}{\partial x_j}(\overline{\rho \, \overline{T}_{ij} \, \overline{u}_i}),
\]

\[
\frac{\partial}{\partial t}(\overline{\rho \, \overline{Y}_\alpha}) + \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{u}_i \, \overline{Y}_\alpha}) = -\frac{\partial \overline{\mathcal{F}_i^\alpha}}{\partial x_i} - \frac{\partial}{\partial x_i}(\overline{\rho \, \overline{F}_i^\alpha}).
\]

where

\[
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij}, \quad q_i = -\lambda \frac{\partial T}{\partial x_i}, \quad J_i^\alpha = -\rho \frac{\partial Y_\alpha}{\partial x_i}.
\]
The terms $\tilde{T}_{ij}$ and $\tilde{F}_i$ need to be modeled to close the LES equations. To realize that, an eddy viscosity is introduced to replace the sub-grid scales action:

$$\tilde{T}_{ij} = (\tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j) = -2 \nu_t \tilde{S}_{ij} + \frac{1}{3} \tilde{T}_{kk} \delta_{ij},$$

(4)

with,

$$\tilde{S}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_i} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial x_k} \delta_{ij}.$$ 

(5)

So it remains just to define the sub-grid scale turbulent viscosity $\nu_t$. In this study, the Standard Smagorinsky model is used [17]:

$$\nu_t = (C_S \Delta)^2 \sqrt{2 \tilde{S}_{ij} \tilde{S}_{ij}},$$

(6)

with $C_S = 0.18$ the standard value of the model constant. The cutoff length $\Delta$ is taken equal to the cubic root of the control cell volume.

### 2.2 Flow solver

The simulation is performed with the turbulent compressible Navier-Stokes code AVBP developed at CERFACS. This solver uses structured, unstructured and hybrid grids to resolve complex configurations to model unsteady flows thanks to the LES approach. In order to handle such grids, the CFD code AVBP is based on a cell-vertex finite-volume approximation. Furthermore, the LES approach employs the second order scheme of Lax-Wendroff for spatial discretization, and for time integration an explicit method is used.

As it was said previously, AVBP uses a numerical discretization which is spatially centered. These types of schemes are known to be subject to small oscillations (wiggles) in the solution variations. In order to solve this problem, a common practice is
to add an artificial viscosity to smooth the gradients. The use of artificial viscosity is triggered by a wiggle detector based on pressure. Several artificial viscosity models are used in AVBP and all are based on a combination of two operators: the second order artificial viscosity operator which smoothes gradients and introduces an artificial dissipation, and the fourth order artificial viscosity operator which allows to control high frequency wiggles. In this paper, the influence of the second operator is more investigated. It is based on this principle:

- First, a cell contribution of the 4\textsuperscript{th} order artificial viscosity is computed on each vertex of the cell \( \Omega_j \):

\[
R_{k \in \omega_j} = \frac{1}{N_v \Delta t_{\Omega_j}} \frac{V_{\Omega_j}}{\text{smu4}} [(\nabla \cdot \omega)_{\Omega_j} (x_{\Omega_j} - \bar{x}_k) - (\omega_{\Omega_j} - \omega_k)]
\]  

(7)

with: \text{smu4} the fourth order artificial viscosity constant which is tunable by the user

(8)

- Then every cell contributions are added what gives the nodal value:

\[
d\omega_k = \sum_j R_{k \in \omega_j}
\]

(9)

3 Computational Setup

The selected case presented in this document is the interaction between a hot JICF (see Margason [18] for a detailed review on JICF’s and their applications) and the wall of a turbo-fan engine nacelle. The interaction takes place on the leading edge of the nacelle where hot air is injected through the wall of the nacelle in order to avoid icing on the exterior (see Fig. 1). The hot air is collected in a scoop and ejected
on the leading edge of the nacelle to form the JICF. Since the velocities are of the order of tens of \( m.s^{-1} \), temperature can be considered as a passive scalar.

The objective of engine designers is to determine the essential characteristics of the jet and its development, and not to simulate the flow around the turbo-fan engine. The jet is thus studied in the simplified geometry of the flat plate configuration. This choice is justified by the fact that the nacelle has relatively low curvature and the jet diameter is very small in comparison with the dimension of the turbo-fan engine. The zone where the jet mixes with the cross flow can then be regarded as quasi-planar. Experimental data to be used in the validation of the numerical methods for this industry-like configuration were obtained using an apparatus located at ONERA.

![Diagram](image)

Figure 1: **Schematic illustration of the engine anti-icing system**

As it is shown on Fig. 2, the computational domain is composed of one jet injected vertically, whose injection diameter \( D \) measures 30 \( mm \) and the exit is flushed to the flat plane. The jet inlet is located \( X/D = 2.8 \) downstream the main inlet of the domain, and \( X/D = 33.3 \) upstream of the outlet. Each lateral face is located
$Y/D = 27.8$ of the jet inlet, and the domain size is $37.2D \times 46.5D \times 33.3D$. So the spreading of the jet is not restricted by the domain. On this domain a fine mesh composed of 2,740,000 cells is used (Fig. 3). It is refined along the jet directory. To keep a relatively reasonable number of cells, a wall law is used for the bottom wall. Consequently, the grid has a first cell from the bottom wall whose the height corresponds to $\Delta y^+ = 150$.

Figure 2: Computational domain considered for LES: (a) Top view and, (b) Lateral view, and (c) Front view.
Figure 3: Fine Mesh used for the LES Computations.

Thanks to experimental measurements, the flow in the main section is fixed by imposing the velocity profile with a boundary layer profile at the inlet of the domain. The initial boundary layer height is $\delta = 2D$ and the free-stream velocity is $U_0 = 47.1 \, m.s^{-1}$. At the injection pipes which compose the jet scoop, a mean turbulent pipe profile is applied, using a standard 1/7 power law approximation (Table 1). For these inlets, ”injection of turbulence” is not employed because it would be negligible in front of the turbulence generated by the shear of the two fluids. All these velocity profiles agree with the mass flux of the experiments.

Because of acoustic wave reflections due to the compressible flow solver, a boundary condition of relaxation is applied on the velocity, the temperature and the mass fraction of the species for the inlets. The same boundary condition is applied for the outlet but just on the pressure. With regards to the plate, which is divided in two
parts, a no-slip isothermal wall law condition is applied on the so-called "grid", and on the remainder part a no-slip adiabatic wall law condition is used. At the lateral surfaces, a symmetry condition is used, and a wall-slip-adiabatic condition is forced at the top surface. The use of a wall-slip adiabatic condition on the top surface is valid in this case because it is placed sufficiently far from the jet exit. So it seems reasonable to consider that no fluid leaves this wall, and there is not a reflection of energy back into the domain: the use of NSCBC (Navier–Stokes Compressible Boundary Conditions: see Poinso et al. [19], Polifke et al. [20]) on this surface is thus not necessary.

<table>
<thead>
<tr>
<th>Table 1: Inlet characteristics.</th>
</tr>
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<tbody>
<tr>
<td>Patch</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>Inlet1 &amp; Inlet2</td>
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<tr>
<td>Inflow</td>
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</table>

4 Definition of the Optimization Problem

4.1 Optimization framework

The present numerical method involves artificial dissipation. It is known that numerical dissipation and subgrid model induced dissipation are in competition, and that they must be tuned in and ad hoc, case-dependent way to recover the best possible results (e.g. [21, 22]). It is chosen in the present work to retain both the
Smagorinsky constant $C_s$ and the artificial fourth-order dissipation parameter $smu4$ as optimization parameters.

Cost function evaluations require expensive LES simulations that is why the optimization method was to provide results in as few evaluations as possible excluding genetic algorithms and most global optimization methods. Despite gradient methods converge quickly to a local minima, these methods cannot treat multiple cost functions and were not adapted to this problem. Response surface based optimization seemed the best method to solve this low-dimensional problem. Once the sample database computed and the inexpensive response surfaces built, a Pareto front of solution can easily be plotted by a simple exploration of the domain of solution. Moreover, as the problem depends on two variables the robustness of solutions can be assessed by plotting the cost functions on the whole domain.

### 4.2 Space of possible solutions and solution sensitivity

A range of ±30% around the standard values $C_s = 0.18$, $smu4 = 0.01$ was studied. As only two parameters were considered, a sufficient exploration of the research space was possible to build globally precise response surfaces using 4 points per direction for the sample database. The space filling distribution method was a 16-points uniform grid sampling.

- For Smagorinsky constant: $0.126 \leq C_s \leq 0.234$ with a step of 0.036.

- For the fourth-order smoothing constant: $0.007 \leq smu4 \leq 0.013$ with a step of 0.002.
The sensitivity of the mean flow with respect to these two parameters is illustrated in Figs. 4 and 5, which compare the standard LES solution, the experimental data, and the extreme values found exploring the response surface built for the mean flow solution at each grid point. The sensitivity of the solution is directly related to the differences between the two extrema profiles. The robustness of the solution is observed to depend on both the location and the physical variable under consideration.

The figures reveal the existence of two different cases. In the first case, the reference data lie between the two extrema and it may be possible to tune the setup parameters to match the experimental data, leading to a very accurate LES solution at the considered location. In the second case, the experimental data are outside the response surface, meaning that the exact solution does not belong to the space of solutions spanned by the LES method used here.
Figure 4: Range of velocity response in the damping parameter space.
Figure 5: Range of cooling effectiveness response in the damping parameter space.

4.3 Cost functions for multidisciplinary optimization

We now address the issue of defining and computing the optimal LES solution. To this end, some cost functions to be minimized must be defined, the optimal LES solution being the one which minimizes one or several cost functions.

4.3.1 Definition of cost functions

A total of four cost functions were computed. Mean and mean squared error was computed along two planar cut at locations $X/D = 1$ and $X/D = 8$. 
The following total cost function is used in the following:

\[
F^{(l)} = \sqrt{\sum_i \alpha_i E_i^2} = \sqrt{\frac{1}{2} (E_u^{(l)} + E_v^{(l)} + E_w^{(l)})^2 + \frac{1}{2} E_\eta^{(l)}}
\]  

(10)

where:

- in \( L_1 \) Norm:

\[
E_u^{(1)} = \frac{1}{\Delta z} \int_0^{\Delta z} \frac{|U_{num} - U_{exp}|}{\|U_{exp}\|_{\text{max}}} \, dz \\
E_v^{(1)} = \frac{1}{\Delta z} \int_0^{\Delta z} \frac{|V_{num} - V_{exp}|}{\|V_{exp}\|_{\text{max}}} \, dz \\
E_w^{(1)} = \frac{1}{\Delta z} \int_0^{\Delta z} \frac{|W_{num} - W_{exp}|}{\|W_{exp}\|_{\text{max}}} \, dz \\
E_\eta^{(1)} = \frac{1}{\Delta y} \int_0^{\Delta y} \frac{|\eta_{num} - \eta_{exp}|}{\eta_{exp_{\text{max}}}} \, dz
\]  

(11)

- Or, in \( L_2 \) Norm:

\[
E_u^{(2)} = \frac{1}{\Delta z} \int_0^{\Delta z} \left( \frac{U_{num} - U_{exp}}{\|U_{exp}\|_{\text{max}}} \right)^2 \, dz \\
E_v^{(2)} = \frac{1}{\Delta z} \int_0^{\Delta z} \left( \frac{V_{num} - V_{exp}}{\|V_{exp}\|_{\text{max}}} \right)^2 \, dz \\
E_w^{(2)} = \frac{1}{\Delta z} \int_0^{\Delta z} \left( \frac{W_{num} - W_{exp}}{\|W_{exp}\|_{\text{max}}} \right)^2 \, dz \\
E_\eta^{(2)} = \frac{1}{\Delta y} \int_0^{\Delta y} \left( \frac{\eta_{num} - \eta_{exp}}{\eta_{exp_{\text{max}}}} \right)^2 \, dz
\]  

(12)

where the thermal efficiency is defined as

\[
\eta = \frac{<T_w> - T_0}{T_i - T_0}
\]

(13)
with $< T_w >, T_j$ and $T_0$ the temporally averaged local wall temperature, the jet temperature and the temperature of the cross flow upstream of the jet exit, respectively.

The cost function (10) which allows to test the sensitivity to the damping parameters is well adapted to the configuration. Indeed, it is possible to test the sensitivity of the mixing prediction to these parameters via the effectiveness ($E_n$), and also the sensitivity of the velocity predictions ($E_U, E_V, E_W$). Moreover, thanks to non-dimensional errors and the fact that $E_V$ and $E_W \ll E_U$, the sensitivity to the viscosity parameters takes into account as much the mixing prediction as the velocity ones (particularly the $U$ field).

The Eq. 10 can be thus simplified:

$$ F(t) \leq \sqrt{\frac{1}{2} E_U^2(t) + \frac{1}{2} E_n^2(t)} $$

(14)

### 4.4 Response surface construction using the Kriging method

Originally developed for application in geostatistics [14], the Kriging method was extended to various applications including computer experiments (Design and Analysis of computer experiments) [23]. Among all techniques available to predict functions at unknown locations from data observed at sampled locations (polynomial surfaces, artificial neural networks, etc.), the Kriging is a linear unbiased statistical predictor. Linear estimators express unknown values from linear combination of sample points. As the model is unbiased, the model can not be fitted by minimizing least square error as for polynomial regression, but rather using a statistical likelihood maximization.
General Kriging model is a sum of a polynomial regression model plus a gaussian process. Most of the time as no assumption can be made a priori on the global trend of the studied function, a constant is used for the regression part (Ordinary Kriging).

Given \( n \) samples \( X^T_s = \{s^1, ..., s^i, ..., s^n\} \) in the parameter space \( S \subset \mathbb{R}^2 \) and the evaluation of the studied function \( f \) at those control points \( F^T_s = \{f(s^1), ..., f(s^i), ..., f(s^n)\} \),

the linear predictor at a location \( x \in S \) is:

\[
\tilde{f}(x) = \sum_{i=1}^{n} \gamma_i(x) f(s^i) = \Gamma^T(x) F(X_s)
\]

with \( \{\gamma_i(x)\} \) the weighting functions respecting the isotropic stationary model assumption and the unbiasedness constraints.

The model being stationary, the covariance \( C \) of the surface function \( f \) between two locations \((s^i, x)\) depends only on the distance between these two locations and the variance \( \sigma^2 \) of the samples.

\[
C[f(s^i), f(x)] = E[(f(s^i) - E(f(s^i))) (f(x) - E(f(x)))] = C(|s^i - x|)
\]

Covariances are usually represented as a matrix called the covariance matrix:

\[
C = \begin{pmatrix}
\sigma^2 & C(||s^1 - s^2||) & \cdots & C(||s^1 - s^n||) \\
C(||s^2 - s^1||) & \sigma^2 & \cdots & C(||s^2 - s^n||) \\
\vdots & \vdots & \ddots & \vdots \\
C(||s^n - s^1||) & C(||s^n - s^2||) & \cdots & \sigma^2 = C(0)
\end{pmatrix}
\]
The covariance vector $\bar{c}$ at unknown location is:

$$
\bar{c}(x) = \begin{pmatrix}
C(||x - s_1||) \\
C(||x - s_2||) \\
\vdots \\
C(||x - s_n||)
\end{pmatrix}
$$  \hspace{1cm} (18)

The estimated value $\hat{f}(x)$ will most likely differs from the actual value $f(x)$ and the difference is called the estimation error:

$$e(x) = \hat{f}(x) - f(x)$$  \hspace{1cm} (19)

To obtain an optimal function estimator, the weighting functions $\Gamma(x)$ are computed so as to minimize the variance of the error or equivalently the Mean Squared Error (MSE) of $\hat{f}(x)$:

$$\frac{\partial}{\partial \Gamma(x_p)} [MSE(\hat{f}(x_p))] = 0$$  \hspace{1cm} (20)

where $MSE$ is defined by the following formula:

$$MSE(\hat{f}(x)) = E[(\hat{f}(x) - f(x))^2]$$

$$= E[\hat{f}^2(x)] + E[f^2(x)] - 2E[f(x)\hat{f}(x)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_j(x) \gamma_i(x) C(||s^i - s^j||) + \sigma^2 - 2 \sum_{i=1}^{n} \gamma_i(x) C(||x - s^i||)$$

$$= \Gamma^T(x) C \Gamma(x) + \sigma^2 - 2 \Gamma^T(x) \bar{c}(x) \hspace{1cm} (21)$$

As a consequence, the solution of Eq. (20) is:

$$\Gamma(x) = C^{-1} \bar{c}(x) \hspace{1cm} (22)$$
In the ordinary Kriging method, the stationary model assumption gives the following constraint:

\[
\forall x \in S, \quad \Gamma^T(x)I = \sum_{i=1}^{n} \gamma_i(x) = 1 \quad (23)
\]

To enforce the unbiasedness constraint about the summation of the weighting coefficients, a Lagrange multiplier \(\lambda(x)\) is introduced. The original problem (20) with the constraint (23) is then rewritten as:

\[
\frac{\partial}{\partial \Gamma(x)} \text{MSE} \; \hat{f}(x) + \Gamma^T(x)I\lambda(x) = 0
\]

leading to

\[
\Gamma_+(x) = C_{+}^{-1} \; \bar{c}_+(x) \quad (25)
\]

where

\[
\Gamma_+(x) = \begin{pmatrix} \Gamma(x) \\ \lambda(x) \end{pmatrix}, \quad C_+ = \begin{pmatrix} C & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{c}_+(x) = \begin{pmatrix} \bar{c}(x) \\ 1 \end{pmatrix}
\]

The estimator of the function \(f(x)\) is then given by:

\[
\hat{f}(x) = \bar{c}^T(x) \; C_+^{-1} F(X_s) \quad (27)
\]

In ordinary Kriging, the most widely used covariogram function is the gaussian covariogram:

\[
C(||x - s_i||) = \sigma^2 \exp \left( - ||x - s_i|| \right) \quad (28)
\]

The linear system (25) is solved using the mathematical library of LAPACK (Linear Algebra PACKage) [24].
5 Numerical Optimization Results

The error maps constructed using the Kriging-based response surface are displayed in Figs. 6 to 8. Different error measures are considered, depending on the location and the norm. It is observed that the error admit several local extrema and are not convex. Therefore, simple gradient based optimization methods may be unable to obtain the best LES solution. Another interesting observation is that the minimum error is not obtain at the two locations with the same value of the parameters $C_S$ and $smu4$. This lead to the conclusion that optimizing the LES setup considering a restricted subset of the reference data may lead to a solution which is not globally fully satisfactory.

This last point is further illustrated plotting the Pareto front of the error for the two error norms (see Figs. 9 and 10). It is seen that minimizing the error at a single location can yield an error growth at the remaining one.

The values of the optimal parameters corresponding to the different cost functions are presented in Table. 2, and values of the cost functions in the different cases are reported in Table 3.

The global optimal solution is compared to experimental reference data in Figs. 11 and 12. The global optimization yield a restricted local improvement of the results, and some local increase of the error can be observed. The increase of the quality of the solution on the temperature field is much more striking than on the velocity field.
Figure 6: Ordinary Kriging for the plane X/D=1. Cost function: a) $F^{(1)}$, and b) $F^{(2)}$.

Figure 7: Ordinary Kriging for the plane X/D=8. Cost function: a) $F^{(1)}$, and b) $F^{(2)}$. 
Figure 8: Ordinary Kriging for the planes $X/D=1$ and $X/D=8$ ($L_1$ norm).

Cost function: a) $F^{(1)}$, and b) $F^{(2)}$.

Table 2: Kriging minima for the two different cost-functions

<table>
<thead>
<tr>
<th></th>
<th>Plane</th>
<th>Cs</th>
<th>smu4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F^{(1)}$</td>
<td>$X/D=1$</td>
<td>0.1465</td>
<td>0.01237</td>
</tr>
<tr>
<td></td>
<td>$X/D=8$</td>
<td>0.1622</td>
<td>0.00901</td>
</tr>
<tr>
<td></td>
<td>Global</td>
<td>0.1622</td>
<td>0.00901</td>
</tr>
<tr>
<td>$F^{(2)}$</td>
<td>$X/D=1$</td>
<td>0.1444</td>
<td>0.01213</td>
</tr>
<tr>
<td></td>
<td>$X/D=8$</td>
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<td>0.00901</td>
</tr>
<tr>
<td></td>
<td>Global</td>
<td>0.1449</td>
<td>0.01234</td>
</tr>
</tbody>
</table>
Figure 9: Pareto front for $F^{(1)}$ function.

Figure 10: Pareto front for $F^{(2)}$ function.
Figure 11: Optimized profiles in the X/D=1 and X/D=8 planes ($L_2$ norm) for the velocity.
Figure 12: Optimized profiles in the X/D=1 and X/D=8 planes ($L_2$ norm) for the cooling effectiveness.

<table>
<thead>
<tr>
<th></th>
<th>$X/D = 1$</th>
<th>$X/D = 8$</th>
<th>Global</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F^{(1)}$</td>
<td>$F^{(2)}$</td>
<td>$F^{(1)}$</td>
</tr>
<tr>
<td>Standard LES</td>
<td>0.322</td>
<td>$6.706.10^{-2}$</td>
<td>0.205</td>
</tr>
<tr>
<td>Optimized LES</td>
<td>0.310</td>
<td>$6.243.10^{-2}$</td>
<td>0.212</td>
</tr>
</tbody>
</table>

Table 3: Relatives errors in reference simulations with the experimental profiles (Cooling effectiveness $\eta$ and Axial velocity $U$)
6 Conclusion

A methodology for the analysis of LES result sensitivity was proposed. It is based on Kriging-based surface response, and requires the computation of a minimal set of LES solutions. The sensitivity of the results is observed to be very dependent on several parameters, such as the location and the physical variable under consideration. The construction of the surface response allows for the search of optimal LES solution. A LES solution is referred to as optimal if it corresponds to an absolute minimum of an error-based cost function. Different definition of the cost function have been considered. It is observed that optimal values of the computational setup exhibit very large variations, showing that the concept of ”best LES solution” is only a relative one in practical cases.

The complexity of the test flow induce large local variations in the sensitivity of the solution, a consequence being that the minimizing an error norm can lead to an increase of another error measure. Again, the concept of ”best LES solution” can be questioned.

References


